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Optimal limited authority for principal [☆]

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Abstract

This article studies a principal-agent problem where the only commitment for the uninformed principal is to restrict the set of decisions she makes following a report by the informed agent. We show that an ex ante optimal equilibrium for the principal corresponds to a finite partition of the state space, and each retained decision is ex post suboptimal for the principal, biased toward the agent's preference. Generally an optimal equilibrium does not maximize the number of decisions the principal can credibly retain. Compared to no commitment, limited authority improves the quality of communication from the agent. As a result, it can give the principal a higher expected payoff than delegating the decision to the agent.

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1. Introduction

A principal needs to elicit information from an agent in order to make decisions, but their inherent conflict of interest makes truthful communication difficult. When the principal cannot credibly give up her authority to make the final decision, the seminal article by Crawford and Sobel [11] (hereafter CS) shows that the principal's decisions suffer from the agent's incentive to distort his information in favor of his bias. When the principal can credibly delegate her decision-making authority, the agent uses his information efficiently but his decision is biased. The reality, however, often lies somewhere in between these two extremes: the principal may credibly give up some, but not all, aspects of her decision-making authority.

This article presents such a model of *limited authority*: *ex ante*, the principal can credibly rule out certain decisions as infeasible; but for the remaining decisions, she cannot commit to any particular decision rule *ex post* such as adopting the agent's recommendation without change. Real life examples of this type of limited authority abound. For instance, consider a typical university tenure system. The university (the principal) needs to decide on individual tenure cases based on recommendations from the department (the agent), whose interests are not perfectly aligned with those of the university. Despite the many possible decisions the university can make, such as increasing the pay without tenure or deferring the decision until a later date, the university is committed to only two decisions: up or out at the end of a probationary period. In the absence of an explicit tenure standard, the university is not committed to approving the department's decision. Our model offers an explanation of why the up-or-out rule is optimal in this limited authority environment.

Under limited authority, the principal needs to decide *ex ante* how much authority to retain *ex post*. On one hand, by retaining more decisions the principal can make better use of the agent's reported information. On the other hand, more retained decisions create a bigger credibility problem: the information content of the report is lower because the agent anticipates the principal's incentive to exploit it. We characterize the optimal limited authority using the same general framework as CS, where both the agent and the principal have payoff functions that are strictly concave in decision; both prefer a higher decision when the state is higher; and the agent has an upward bias in decision relative to the principal. We show that under the optimal limited authority, finitely many decisions are retained. The agent partitions the state space and makes a recommendation from the set of retained decisions for each partition element, and the principal always follows his recommendation.

The principal can never improve her expected payoff through randomization even though she only chooses from finitely many decisions. There is a general intuition behind this result. In a partitional equilibrium, when the agent has an upward bias, putting a marginally greater weight on the highest decision in any decision lottery makes the agent recommend the lottery for a set of higher states. This change does not affect the principal's credibility since the highest decision is used in the original lottery. But the principal benefits directly because, initially indifferent among the decisions in the lottery for a set of lower states, she strictly prefers the highest decision for the set of higher states. She also benefits indirectly because the decisions after the change are on average better aligned with the true state. As a result, if the principal uses a lottery in equilibrium, she is better off gradually shifting the weight from lower decisions to higher decisions in the lottery to induce this better alignment of states and decisions. This process continues until she uses only the highest decision—which remains an equilibrium—and receives a higher payoff.

A similar argument explains why the principal is strictly better off under the optimal limited authority than in any CS equilibrium. Starting from an informative CS equilibrium, by marginally increasing any decision, the principal induces the agent to make the recommendation for a set of higher states. Because all decisions in CS equilibrium are ex post optimal, this marginally higher decision remains credible and has no direct impact on the principal's payoff. She benefits indirectly, however, because this higher decision is better aligned with the true state.

To better understand the tradeoff for the principal under limited authority, in particular the properties of the retained decisions, we turn to the example with a uniformly distributed state and convex, symmetric loss functions, a slight generalization of the uniform-quadratic example commonly used in the literature. We fully characterize the principal's optimal limited authority in this case for any fixed number of retained decisions. In the optimal limited authority, all the retained decisions are above the principal's ex post optimal decisions—shifted in the direction of the agent's bias—given that she learns the partitional elements. Moreover, retained decisions are more evenly distributed under the optimal limited authority than the induced decisions in a CS equilibrium. Intuitively, the principal restricts the set of decisions she can choose from, which reduces the agent's incentive to distort. This increases the possible number of decisions that can be credibly retained, and decreases the distance between them.

A surprising feature of our optimal limited authority model is that the principal may choose not to maximize the number of decisions that she can credibly use, contrary to the predictions of both the cheap talk and delegation models. Instead, the principal must trade off the number of decisions she can use with the placement of these decisions. On one hand, if the agent believes that the principal simply rubber-stamps any recommendation, then adding a decision that is rarely recommended by the agent barely helps the principal. On the other hand, adding a decision that is close to another retained decision exacerbates the principal's credibility problem so that it can only be addressed by altering the placements of the decisions. If too many decisions are retained, the placement can become so extreme that some decisions are used with almost zero probability; and the actually used ones are, on average, less aligned with the true state than decisions that are optimally placed on their own. The ensuing reduction in the quality of decision-making outweighs the gain from the increased number of decisions, making the principal worse off. Consequently, the principal may prefer using a smaller number of decisions under limited authority.

The principal's expected payoff under optimal limited authority is higher than under no commitment but necessarily lower than under full commitment of the principal. Our main welfare comparisons, however, are between the two environments with limited commitment power of the principal: optimal limited authority and full delegation, under which the principal takes an arms-length approach and simply lets the agent decide. Dessein [12] shows that the principal prefers full delegation to any informative cheap talk equilibrium when the state is uniformly distributed. This is because communication becomes increasingly noisy as the state increases, and thus the loss of information under communication outweighs the loss of control under full delegation. We show that limited authority reduces the agent's incentives to distort, and hence the loss of information under communication. As a result, the principal's expected payoff is of the same magnitude as under full delegation when the bias is arbitrarily small and the number of retained decisions is arbitrarily great, and can be strictly higher when the bias is larger and more than one decision is retained. Moreover, for any value of the bias, limited authority performs better than full delegation if the principal is close to being risk neutral. This happens because a less risk-averse principal cares less about the residual loss of information under limited authority and more about the loss of control under full delegation.

This article is directly related to the literature on delegation initiated by Holmstrom [19], who shows that the optimal outcome under full commitment of the principal is achieved by restricting the set of decisions and delegating decision-making authority to the agent. Our article analyzes the environment in which the principal cannot delegate authority to the agent, but can restrict the set of decisions. Closely related are Dessein [12] and Marino [29], who study the optimal delegation problem where the principal can veto the agent's decision and replace it with some default decision; and Mylovanov [33], who instead assumes that the principal can choose the default decision ex ante. Less related to our work, Milgrom and Roberts [31] and Szalay [34] analyze how restricting the set of decisions affects influence activities and information acquisition respectively. Sections 2.3 and 5.3 discuss the related literature in greater detail.

To proceed, Section 2 sets up the limited authority model by adding to the CS model a first move in which the principal chooses the set of retained decisions. Section 2.3 provides more motivations for the limited authority assumption. Section 3 derives general properties of the optimal limited authority by characterizing it as a solution to a constrained maximization problem. Section 4 provides full characterization of the example with a uniformly distributed state and convex loss functions of both the principal and the agent. Section 5 compares the principal's welfare under optimal limited authority and other organizational forms. Section 6 discusses extensions of the model. All proofs can be found in Appendix A.

2. The model

2.1. Setup

This article analyzes the CS model with one modification. In CS the set of decisions is the real line while we assume that, ex ante, the principal can credibly restrict the set of decisions from which she can choose ex post. The model specified in this section is called a model of *limited authority* throughout the article.

Formally, there is an informed agent A (he) and an uninformed principal P (she). Payoffs of A and P , denoted by $u^A(y, \theta)$ and $u^P(y, \theta)$, are both functions of the decision y and the state of the world θ . The timing of the game is as follows:

1. P chooses a decision set Y , a compact subset of the real line.
2. A observes Y and privately learns θ , drawn from the interval $(0, 1]$ according to a positive probability density function $f(\theta)$.
3. A sends a cheap talk message m from the interval $[0, 1]$.
4. P receives m and makes a decision $y \in Y$.

All aspects of the game except for the true state are common knowledge. We make the CS assumptions on functions $u^A(y, \theta)$ and $u^P(y, \theta)$, which are maintained throughout the article:

Assumption 1. There exists a function u and a scalar $b > 0$ such that $u^A(y, \theta) = u(y, \theta, b)$ and $u^P(y, \theta) = u(y, \theta, 0)$. Moreover,

1. u is twice continuously differentiable in all variables.
2. $u_{yy}(y, \theta, \beta) < 0$ for all $y \in \mathbb{R}$, $\theta \in [0, 1]$, and $\beta \in [0, b]$.
3. $u_y(y(\theta, \beta), \theta, \beta) = 0$ for some function $y(\theta, \beta)$, and for all $\theta \in [0, 1]$ and $\beta \in [0, b]$.

4. $u_{y\theta}(y, \theta, \beta) > 0$ for all $y \in \mathbb{R}$, $\theta \in [0, 1]$, and $\beta \in [0, b]$.
5. $u_{y\beta}(y, \theta, \beta) > 0$ for all $y \in \mathbb{R}$, $\theta \in [0, 1]$, and $\beta \in [0, b]$.

Parts 2 and 3 imply that both A and P 's preferences are single-peaked. Parts 1–3 together imply that $y^i(\theta) \equiv \arg \max_{y \in \mathbb{R}} u^i(y, \theta)$ is well defined and continuous in θ for all $\theta \in [0, 1]$ and $i = A, P$. Part 4 is a sorting condition, which ensures that both $y^A(\theta)$ and $y^P(\theta)$ are increasing in θ for all $\theta \in [0, 1]$. Finally, part 5 guarantees that $y^P(\theta) < y^A(\theta)$ for all $\theta \in [0, 1]$.

This game turns out to be equivalent to the following *delegation game*. First, P chooses a delegation set Y , and then A chooses some y from Y , which P can approve or change to some other \tilde{y} in Y . The only formal difference is that in the delegation game A makes a recommendation y from Y , instead of sending a cheap talk message m from $[0, 1]$. The reduction in A 's strategy space turns out to be immaterial.¹ Observe that in both games, P cannot commit to a decision rule over Y contingent on the message or recommendation from A . Rather, she is free to choose any decision in Y ex post.

2.2. Solution concept and definitions

The solution concept we use is Perfect Bayesian Equilibria (hereafter PBE). A PBE is P 's choice of Y , A 's report strategy $\sigma : 2^{\mathbb{R}} \times (0, 1] \rightarrow \Delta[0, 1]$, P 's decision strategy $\rho : 2^{\mathbb{R}} \times [0, 1] \rightarrow \Delta \tilde{Y}$, and P 's belief $p : 2^{\mathbb{R}} \times [0, 1] \rightarrow \Delta(0, 1]$, such that strategies are optimal given the players' beliefs, and beliefs are derived from Bayes' rule whenever possible.² Formally, the equilibrium conditions are

$$(i) \quad Y \in \arg \max_{\tilde{Y} \subset \mathbb{R}} \int_{\tilde{Y} \times (0, 1] \times [0, 1]} u^P(y, \theta) \rho(y | \tilde{Y}, m) \sigma(m | \tilde{Y}, \theta) f(\theta) dy d\theta dm;$$

(ii) for all $\tilde{Y} \subset \mathbb{R}$, for all θ , and for \tilde{m} in the support of $\sigma(\cdot | \tilde{Y}, \theta)$,

$$\tilde{m} \in \arg \max_{m \in [0, 1]} \int_{\tilde{Y}} u^A(y, \theta) \rho(y | \tilde{Y}, m) dy;$$

(iii) for all $\tilde{Y} \subset \mathbb{R}$, for all $m \in [0, 1]$, and for any \tilde{y} in the support of $\rho(\cdot | \tilde{Y}, m)$,

$$\tilde{y} \in \arg \max_{y \in \tilde{Y}} \int_0^1 u^P(y, \theta) p(\theta | \tilde{Y}, m) d\theta; \quad \text{and}$$

(iv) for all $\tilde{Y} \subset \mathbb{R}$, for all θ , and for all m in the support of $\sigma(\cdot | \tilde{Y}, \tilde{\theta})$ for some $\tilde{\theta}$,

$$p(\theta | \tilde{Y}, m) = \frac{\sigma(m | \tilde{Y}, \theta) f(\theta)}{\int_0^1 \sigma(m | \tilde{Y}, \tilde{\theta}) f(\tilde{\theta}) d\tilde{\theta}}.$$

¹ Formally established as part of the proof of [Theorem 1](#) in the next section, this claim is obviously true if we restrict the attention to equilibria where P uses a pure strategy on the equilibrium path. The proof of [Theorem 1](#) establishes the claim allowing for the possibility of random decisions by P .

² A technical issue arises with the existence of the conditional distribution function, $p(\theta | Y, m)$, which can be bypassed using the notion of distributional strategies (see [Milgrom and Weber \[32\]](#)) and [Theorem 33.3](#) of [Billingsley \[7\]](#).

We adopt the following definitions. The decision y is *induced* by θ (or equivalently θ *induces* y) in a PBE if on the equilibrium path y is chosen by P with positive probability when the state is θ in this PBE, or

$$\int_{\{m: \rho(y|Y,m)>0\}} \sigma(m|Y, \theta) dm > 0.$$

The decision y is *induced in a PBE* if y is induced in at least one state. A PBE is *informative* if there are at least two induced decisions, and *uninformative* otherwise. The *uninformative decision* y^P is defined as $y^P \equiv \arg \max_{y \in \mathbb{R}} \int_0^1 u^P(y, \theta) f(\theta) d\theta$. Finally, a PBE is a *partition equilibrium* $(\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ if $\{\theta_i\}_{i=0}^n$ is a partition of $(0, 1]$, and $\{y_i\}_{i=1}^n = Y$ is a set of induced decisions where

$$\begin{aligned} 0 = \theta_0 < \theta_1 < \dots < \theta_n = 1, \\ y_1 < \dots < y_n, \end{aligned} \tag{1}$$

such that any $\theta \in (\theta_{i-1}, \theta_i]$ induces decision y_i for all $i = 1, \dots, n$. Condition (1) is called the *partition condition*. Clearly, a partition equilibrium can be supported as a PBE of the delegation game where the delegation set Y chosen by P on the equilibrium path has the following properties. First, it is *minimal*, in that each decision $y \in Y$ is induced; and second, it is *veto-free*, in that P chooses the same y chosen by A .

Two remarks are in order. First, all CS equilibria can be supported as a PBE in this framework. Indeed, consider any CS equilibrium. Let P choose all the decisions induced in this CS equilibrium. Because all these equilibrium decisions are incentive compatible for A and ex post optimal for P , neither player has an incentive to deviate. If P chooses a decision set different from those induced in the CS equilibrium, then A sends uninformative messages; and thus P makes the best decision in her chosen decision set based on her prior belief. Note that this observation also implies that a PBE always exists.

Second, similar to the CS model, for each PBE there exists an outcome-equivalent PBE in which all messages in $[0, 1]$ are sent on the equilibrium path. Therefore, we cannot refine the set of PBE using standard equilibrium refinements such as those of Cho and Kreps [10] which restrict out-of-equilibrium beliefs.³ This article mostly focuses on PBE that maximizes P 's expected payoff, which we refer to as the *optimal* PBE. Such a refinement is natural if P not only chooses Y at the first stage, but also announces the outcome she plans to implement with the chosen decision set Y .

2.3. Discussion of the model

We model a specific form of limited commitment: ex ante the principal can credibly restrict the set of decisions available to her ex post, but she cannot commit to any decision rule. We now elaborate on the settings in which this assumption is reasonable.

Our limited authority model describes a contracting environment in which the authority to make final (irreversible) decisions resides with the principal, but *only* these final decisions are

³ Some refinements for cheap talk games have been proposed in the literature but they do not generally select a unique equilibrium. A notable exception is due to Chen et al. [9], which selects the most informative equilibrium under some regularity conditions.

verifiable. The main innovation is to study a more primitive contracting environment than the full-commitment framework initiated by Holmstrom [19], while at the same time demonstrate what ‘simple’ contracts can achieve relative to the no-contracting, cheap talk framework of CS. In particular, in this model neither communication from the agent to the principal, such as reports on his information or recommendations to the principal, nor decision rights is verifiable.⁴ As in the incomplete contract literature initiated by Grossman and Hart [15] and Hart and Moore [16], communication and decision rights may be observable but not verifiable for various reasons. For example, it may be prohibitively costly for the agent to present physical evidence of his communication with the principal in the court. Similarly, to delegate formal authority to the agent, the principal may need to sell relevant productive assets to the agent, which may be impractical because the same assets are used by the principal for other purposes.

Our limited authority model is thus applicable to environments with this type of restriction on contractibility.⁵ In particular, our model sheds light on why some organizations have certain institutional constraints such as the tenure example mentioned in the introduction. Similarly, organizations may choose to impose technological constraints. For instance, in many organizations, managers make decisions using system-wide software packages, such as SAP ERP. This software is typically adjusted to the specific needs of each organization so that certain decisions are made unavailable, such as trading of some products at certain prices in a financial company.

3. General analysis

In this section we provide a general analysis of the optimal PBE in our limited authority model. We start by characterizing the optimal PBE as a solution to a constrained maximization problem in [Theorem 1](#). This is a useful result that we exploit further in the uniform-convex loss setup in [Section 4](#) to completely characterize the optimal PBE. Here we use it to establish the main result of the section, [Proposition 1](#), that the optimal PBE strictly improves P 's welfare relative to the most informative equilibrium of CS. Under further assumptions on the payoff functions u^A and u^P , [Proposition 2](#) provides a tight upper bound on A 's bias parameter b for P to benefit from limited authority relative to the CS model.

Our first result establishes the existence of optimal PBE under limited authority and characterizes its basic properties. In particular, it shows that the optimal PBE is a partition equilibrium with a finite number of induced decisions.

Theorem 1. *An optimal PBE exists and is a partition equilibrium with a finite number of elements. Moreover, among all partition equilibria $(\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ with a finite n , it maximizes $\sum_{i=1}^n \int_{\theta_{i-1}}^{\theta_i} u^P(y_i, \theta) f(\theta) d\theta$ subject to, for each $i = 2, \dots, n$,*

⁴ Allowing reports or recommendations by the agent to be verifiable would of course turn our model into an exercise in mechanism design without transfers; likewise, allowing the decision rights to be contractible would change our model into an optimal delegation problem. Both these problems have been extensively studied in the literature; see for example the more recent works by Kovac and Mylovanov [25] and Alonso and Matouschek [3].

⁵ Hart and Moore [17] impose a similar contractibility assumption. They assume that ex ante the parties can restrict the set of outcomes over which they bargain ex post. However, the parties cannot commit to any specific mechanism according to which the outcome from this restricted set is chosen ex post. Also, Hermalin et al. [18] propose a similar approach to model situations in which a contract has ambiguous provisions. That is, each contingency in a contract is associated with a set of outcomes from which the final outcome is chosen. In this context, the imperfect commitment assumption requires that the same set of possible outcomes should be associated with each contingency.

$$u^A(y_i, \theta_{i-1}) = u^A(y_{i-1}, \theta_{i-1}), \quad (2)$$

$$\int_{\theta_{i-1}}^{\theta_i} u^P(y_i, \theta) f(\theta) d\theta \geq \int_{\theta_{i-1}}^{\theta_i} u^P(y_{i-1}, \theta) f(\theta) d\theta. \quad (3)$$

Because P can freely choose any decision in her pre-specified decision set Y , a useful starting point is to invoke a revelation-principle type of argument to simplify the characterization of the decision set that P may choose in a PBE. We exploit the assumptions on the payoff functions u^A and u^P to show that any PBE can be supported as one in the delegation game, even though P may randomize over Y on and off the equilibrium path. This implies that we can restrict attention to PBE's in which P 's equilibrium decision set Y is minimal and veto-free, the latter properly defined to allow randomization by P . That is, P does not include any decision that she never uses and more importantly, she always follows the recommendation by A , including recommendations for a randomized decision.

The problem can still be complicated because of the lack of structure in terms of possible deviations of P : she can deviate to any mixed strategy over the set Y . To reduce the number of incentive constraints of P , we first show that any PBE has a partitional structure with a finite number of elements.⁶ Next, we note that P may have incentives to deviate only to the decisions adjacent to A 's recommendation because P 's payoff function is strictly concave. More interestingly, in any PBE in which P does not randomize, local downward incentive conditions suffice for all P 's incentive conditions. Intuitively, whenever A , who has an upward bias, is indifferent between two decisions, P must strictly prefer the lower decision.

We proceed to show the most interesting part of [Theorem 1](#): P never randomizes in an optimal PBE. The proof is involved because of the finiteness of decision set Y .⁷ We show that for any PBE with non-degenerate lotteries, P can increase her expected payoff by replacing each non-degenerate lottery with the higher decision in the lottery. Intuitively, P can at most be indifferent between two decisions in a non-degenerate lottery due to the strict concavity of her payoff. Since A is biased upward, putting a marginally greater weight on the higher decision in a non-degenerate lottery induces A to recommend the lottery for a set of higher states. However, since P is initially indifferent between the two decisions in the lottery for a set of lower states, she strictly prefers the higher decision now: this change directly makes her better off. Moreover, at the thresholds, A is indifferent between adjacent lotteries, so P strictly prefers the lower lotteries due to a downward bias relative to A . Therefore, as the thresholds move up, A switches to the lower lotteries and P benefits indirectly. Our proof exploits these direct and indirect benefits to P repeatedly, by gradually shifting the weight from lower decisions to higher decisions in all non-degenerate lotteries until she uses only the higher decision in each lottery. Finally, we show that this change does not affect P 's incentive conditions following A 's adjustments in the partitioning of the states, by using the fact that the higher decision is initially used in the lottery.

The above arguments allow us to reduce the problem of finding the optimal PBE to a constrained maximization problem where the set of feasible choices is all partition equilibria with a

⁶ The proof of the finiteness is quite standard, except that the distance between three rather than two adjacent induced lotteries is bounded away from zero.

⁷ In CS, P 's payoff function is strictly concave in a decision and the set of decisions is convex. Thus, upon receiving a message, P has a unique optimal decision.

finite number of elements that satisfy A 's indifference conditions (2) and P 's adjacent downward incentive conditions (3). This is summarized in Theorem 1, which also establishes that there is a solution to the maximization problem.

Clearly, P cannot do worse than in any CS equilibrium, as she can replicate any CS equilibrium outcome by restricting the set of decisions to those induced in the CS equilibrium. Our second result shows that P can, in fact, do strictly better.

Proposition 1. *P 's expected payoff is strictly higher in the optimal PBE than in any informative CS equilibrium.*

In a CS equilibrium, each induced decision is ex post optimal for P in that it maximizes her payoff over all possible decisions $y \in \mathbb{R}$ given P 's belief about the state after receiving A 's message. Therefore, P 's incentive conditions (3) are not binding in an informative CS equilibrium, and she can marginally increase any induced decision y_i without violating (3). As P increases y_i , by the Envelope theorem, her expected payoff is unaffected by the introduction of ex post inefficiencies, but is raised due to resulting increases in the partition thresholds θ_{i-1} and θ_i . For example, as θ_{i-1} increases to θ'_{i-1} , an upwardly biased A induces y_{i-1} instead of a higher decision y_i for states $\theta \in (\theta_{i-1}, \theta'_{i-1}]$, which increases P 's expected payoff.

The logic of Proposition 1 implies that P can strictly improve her expected payoff by restricting the set of decisions even when no informative CS equilibrium exists. More formally, suppose that an informative CS equilibrium exists whenever b is less than b^* , with two decisions y_1 and y_2 . Then there exists ε such that for all b less than $b^* + \varepsilon$, ε sufficiently small, P 's expected payoff is strictly higher in the optimal PBE than in any CS equilibrium. By Proposition 1, for b less than b^* , P can increase either y_1 or y_2 to achieve the desired PBE. By the continuity of u , these new induced decisions still constitute a PBE and P is strictly better off than in the uninformative CS equilibrium at $b = b^* + \varepsilon$.

Under additional assumptions on the function u , we can further strengthen Proposition 1. We show that P 's expected payoff is strictly higher in the optimal PBE than a babbling equilibrium if and only if delegation is valuable under full commitment. Adopting a definition from Alonso and Matouschek [3], we say that *delegation is valuable* if P can improve on the uninformed decision y^P by committing to letting A choose from some set of decisions.

Proposition 2. *Let $u^P(y, \theta) = -(y - y^P(\theta))^2$ and $u^A(\cdot, \theta)$ be symmetric around $y^A(\theta)$. P 's expected payoff is strictly higher in the optimal PBE than in any CS equilibrium if and only if delegation is valuable.*

The 'only if' part is immediate, because by Theorem 1 the optimal PBE is a partition equilibrium and any partition equilibrium can be implemented through delegation as the incentive conditions (3) for P are absent in delegation under full commitment. The proof of the 'if' part is based on a result due to Alonso and Matouschek [3]. They show that if delegation is valuable, then P can improve on implementing the uninformed decision y^P by letting A choose between exactly two decisions. We show that these two decisions satisfy P 's incentive condition (3), and thus can be induced in a PBE.

4. The uniform-convex loss example

This section focuses on a slight generalization of the leading example of CS.

Assumption 2. $f(\theta) = 1$ for $\theta \in (0, 1]$, $b < \frac{1}{2}$, and $u(y, \theta, \beta) = -l(|y - (\theta + \beta)|)$, where l is strictly convex with $l(0) = l'(0) = 0$.

Assumption 2 includes the widely used uniform-quadratic example of $l(z) = z^2$ as a special case.⁸ Clearly, Assumption 2 satisfies Assumption 1, so Theorem 1 and Proposition 1 hold.⁹ The uniform-convex example is particularly well-behaved to apply the constrained maximization program given in Theorem 1. This is because the combination of a uniform state distribution and symmetric payoff functions ensures that, given the partition of the state space, only the distance between each induced decision and the corresponding P 's ex post optimal decision affects P 's payoff, and hence her choice of optimal decision set.

In the constrained maximization problem given in Theorem 1, the number of induced decisions n is a choice variable. In order to tackle this problem and to characterize the optimal PBE in Proposition 5, we take the following approach. We begin Section 4.1 by establishing in Lemma 1 that in any optimal PBE each induced decision is higher than what is ex post optimal for P , conditional on P learning the corresponding partition element. This simplifies P 's incentive conditions (3), and allows us to find an upper bound on the number of induced decisions in the optimal PBE for any fixed b in Proposition 3. We call the PBE that achieves this upper bound the *maximal limited authority*. Next, in Lemma 2 we solve a hypothetical full-commitment problem of maximizing P 's expected payoff with any n induced decisions, subject only to the partition conditions (1) and A 's indifference conditions (2). Since the solution yields a higher expected payoff to P for a larger n but satisfies P 's incentive conditions (3) only if n is small, we obtain in Proposition 4 a lower bound on the number of induced decisions in the optimal PBE. We call the PBE that achieves this lower bound the *minimal limited authority*. In Section 4.2, for each fixed b and each n between the two bounds, we derive in Lemma 3 the optimal PBE conditional on n induced decisions. Finally, we compare the conditionally optimal PBE's in terms of P 's expected payoff, and select among them the one with the highest payoff for P in Proposition 5.

Our solution approach makes it easier to understand the main economic insight in this section. Namely, the maximal limited authority given in Proposition 3 is generally suboptimal for P because having the maximal number of induced decisions can lead to a greater ex post credibility problem, forcing P to use some of the decisions only rarely. Instead, it may be optimal for P to reduce the number of induced decisions, or to adopt the minimal limited authority constructed in Proposition 4 with relaxed incentive conditions. Doing so improves the overall alignment of the induced decisions with the true state, and hence P 's expected payoff. An additional benefit of the approach is that the characterizations of the maximal limited authority and the minimal limited authority are particularly simple and amenable to comparative statics analysis. They are used to provide various bounds on P 's expected payoff under the optimal limited authority in our welfare analysis of Section 5.

⁸ There is another uniform-quadratic example that has been analyzed in recent papers including Gordon [14] and Alonso et al. [1]. In this example, A has an outward rather than upward bias such that his payoff is given by $u^A(y, \theta) = -(y - b - c\theta)^2$ where $b < 0$ and $b + c > 1$. Intuitively, an outwardly biased A prefers extreme decisions when the state of the world is extreme. In the example with outwardly biased A , there exists an equilibrium with a countable number of induced decisions which eliminates an integer problem peculiar to the leading example of CS and simplifies the analysis.

⁹ For $b \geq \frac{1}{2}$, in the optimal PBE P makes the uninformative decision, which can be seen from the proof of Proposition 3. Thus we focus on the more interesting case $b < \frac{1}{2}$.

4.1. Maximal and minimal limited authority

From Theorem 1, an optimal PBE exists and it is a partition equilibrium that satisfies A 's indifference conditions (2) and P 's adjacent downward incentive conditions (3). In the present uniform-convex loss model, these conditions can be rewritten as: for all $i = 2, \dots, n$,

$$\theta_{i-1} + b - y_{i-1} = y_i - \theta_{i-1} - b; \tag{4}$$

$$|y_i - y_i^*| \leq |y_i^* - y_{i-1}|, \tag{5}$$

where $y_i^* = \frac{1}{2}(\theta_{i-1} + \theta_i)$ is P 's ex post optimal decision conditional on the interval $(\theta_{i-1}, \theta_i]$. We now show that each induced decision y_i is higher than the ex post optimal decision y_i^* .

Lemma 1. *In any optimal PBE with n induced decisions, $y_i > y_i^*$ for each $i = 1, \dots, n$.*

In contrast with the CS equilibrium in which all P 's decisions are ex post optimal, this lemma shows that P sacrifices the best use of A 's information to provide him with better ex ante incentives. Moreover, this sacrifice is in the direction of A 's bias. The first part of the proof of Lemma 1 establishes that if $y_i \leq y_i^*$ for some $i = 1, \dots, n$ in the optimal PBE, then P 's $(i + 1)$ -th incentive condition binds. Intuitively, if $y_i \leq y_i^*$ and the $(i + 1)$ -th condition is slack, P can obtain a greater expected payoff by marginally increasing y_i without affecting any incentive condition. Her payoff gain is clear: a higher y_i moves her closer to her ex post optimal decision given the same belief of the states; and the resulting increases in thresholds θ_{i-1} and θ_i mean that A now recommends y_i for a set of higher states, making P better off by the same argument as in Proposition 1.¹⁰ The second part of the proof uses the special structure of the uniform-convex loss model to show that $y_i \leq y_i^*$ is incompatible with binding P 's $(i + 1)$ -th incentive condition.

We now restate the constrained maximization problem of Theorem 1 for the uniform-convex loss setup by substituting out A 's indifference conditions (4). The choice variables are $\{y_i\}_{i=1}^n$ and n . By partition condition (1), $\theta_0 = 0$ and $\theta_n = 1$, so P 's objective function becomes

$$U^n = - \sum_{i=1}^n \int_{\theta_{i-1}}^{\theta_i} l(|\theta - y_i|) d\theta. \tag{6}$$

The constraints are, in addition to (4),

$$y_1 < \dots < y_n, \tag{7}$$

$$y_1 + y_2 > 2b, \tag{8}$$

$$y_{i+1} - y_{i-1} \geq 4b, \quad \text{for each } i = 2, \dots, n - 1, \tag{9}$$

$$y_n + y_{n-1} \leq 2(1 - b). \tag{10}$$

Conditions (7) are part of the partition condition (1). Condition (8) ensures that $\theta_1 > 0$; $\theta_{n-1} < 1$ is implied by condition (10); and $\theta_1 < \theta_2 < \dots < \theta_{n-2} < \theta_{n-1}$ follow from (7). Conditions (9) and (10) are equivalent to P 's incentive conditions (5) for $i = 2, \dots, n$ by Lemma 1.

¹⁰ This holds for the general model setup in Section 2, not just the uniform-convex loss model here. In the general model, this result implies that in an optimal PBE the highest decisions y_n and y_{n-1} satisfy $y_n > y_n^*$ and $y_{n-1} > y_{n-1}^*$, and that no two adjacent decisions y_i and y_{i+1} are below y_i^* and y_{i+1}^* respectively. Further, in the hypothetical problem of full commitment with a fixed number of decisions introduced later in this section, every decision y_i is strictly higher than y_i^* for similar reasons.

Condition (10) takes a different form because $\theta_n = 1$ by partition condition (1), instead of being determined by A 's indifference condition (4). Note that conditions (9) are absent when $n = 2$.

As the number of decisions n is a choice variable, we begin by finding an upper bound on the optimal number of decisions. Clearly, conditions (9) and (10) place constraints on the distance between decisions, and hence the maximum number of decisions induced in an optimal PBE. Our next result characterizes this number and, more importantly, shows that there always exists a PBE in which this maximum number of decisions is induced.

Proposition 3. *The number of decisions induced in an optimal PBE is strictly less than $1/(2b) + 1$. Conversely, there exists a PBE with n induced decisions for any natural $n < 1/(2b) + 1$.*

We establish the second part of Proposition 3 by a construction which we denote as $\bar{Y}^n = \{\bar{y}_i^n\}_{i=1}^n$ for all $n < 1/(2b) + 1$. For $n \geq 3$, this construction binds all P 's incentive conditions with symmetric and equidistant decisions.¹¹ That is, $\bar{Y}^n = \{\frac{1}{2} - (n + 1 - 2i)b\}_{i=1}^n$ for $3 \leq n < 1/(2b) + 1$. Among these PBE's, the one with the largest n is called *maximal limited authority*. It is instructive to compare the maximal limited authority with the most informative CS equilibrium. In a CS equilibrium, the distance between subsequent induced decisions grows at the rate $4b$, that is, $y_{i+1} - y_i = y_i - y_{i-1} + 4b$ for $i = 2, \dots, n - 1$. Hence the number of induced decisions n in any CS equilibrium has to satisfy $2n(n - 1)b < 1$. Under \bar{Y}^n , in contrast, the distance between two adjacent decisions is always $2b$. As a result, the number of induced decisions under the maximal limited authority is the largest natural n satisfying $2(n - 1)b < 1$, which is greater than that in the most informative CS equilibrium.

Although Proposition 3 is specific to the uniform-convex loss setup, there is a more general logic behind the result that more decisions can be induced under maximal limited authority than under the most informative CS equilibrium. In a CS equilibrium, each induced decision y_i is ex post optimal conditional on the corresponding interval $(\theta_{i-1}, \theta_i]$ because P has no commitment power. Instead, because P has some commitment power in our model, by Theorem 1, P 's incentive conditions require only that P prefers y_i to the adjacent lower decision y_{i-1} conditional on $(\theta_{i-1}, \theta_i]$. Thus the partitioning of the state space under limited authority need not be as rightward skewed as in a CS equilibrium.

We now construct a lower bound on the number of decisions that can be induced in the optimal PBE. Intuitively, for any given bias, in the optimal PBE with n induced decisions, incentive conditions (3) are not binding if n is sufficiently small. Thus, as n grows, P 's expected payoff increases until n reaches the lower bound at which incentive conditions start to bind. Formally, for each n , we consider the hypothetical problem of maximizing (6) subject to constraints (7), (8), and

$$y_{n-1} + y_n < 2(1 + b), \tag{11}$$

where we have dropped P 's incentive conditions (9) and (10), but added (11) to ensure that $\theta_{n-1} < 1$. The following lemma provides a characterization of the solution to this problem, which we denote as $\underline{Y}^n = \{\underline{y}_i^n\}_{i=1}^n$.

¹¹ When $n = 2$, the maximal limited authority coincides with the full-commitment solution introduced in Lemma 2, so P 's incentive condition does not bind. See the proof of the proposition in Appendix A.

Lemma 2. For any natural n , \underline{Y}^n is given by $y_i^n = \frac{1}{2} - (n + 1 - 2i)\underline{\delta}^n$, $i = 1, \dots, n$, where $\underline{\delta}^n > 0$ is uniquely determined by

$$2l(\underline{y}_1^n) = l(b + \underline{\delta}^n) + l(|b - \underline{\delta}^n|).$$

Note that $\underline{\delta}^n$ decreases with b for each n , and decreases with n for fixed b . Since we have dropped P 's incentive conditions, \underline{Y}^n gives P her highest expected payoff under *full commitment* with the restriction to a finite number n of decisions. To maximize P 's payoff given by (6), P places all decisions at equal distance and symmetric around $\frac{1}{2}$. The decisions need to be equidistant to make the partition of the state space uniform in that $\theta_i - \theta_{i-1} = 2\underline{\delta}^n$ for all $i = 2, \dots, n - 1$. The uniform partition in turn minimizes the loss of information, which can be loosely understood as the average residual uncertainty of the state of the world provided that P learns the partition elements (see Section 5.3 for more details when the payoff function is quadratic). The decisions \underline{y}_i^n are symmetric around $\frac{1}{2}$ because P is unbiased in that $y^P(\theta) = \theta$. As $n \rightarrow \infty$, the decisions become arbitrarily close to $[b, 1 - b]$, because $\underline{\delta}^n \rightarrow 0$ and $\underline{y}_1^n \rightarrow b$ as $n \rightarrow \infty$. In fact, the optimal decision set is equal to $[b, 1 - b]$ in the full-commitment model in which P can commit to not to change A 's recommendation without the restriction to a finite number of decisions (see the proof of Proposition 6).

A robust feature of models with full commitment is that more decisions can only improve A 's communication quality because P commits to not using the information A revealed strategically. Thus, not surprisingly, Lemma 2 implies that as n increases, P 's expected payoff \underline{U}^n in the hypothetical full-commitment problem strictly increases. An important implication is the following result.

Proposition 4. The number of decisions induced in an optimal PBE is at least 2; if 3 or more decisions are induced, then it is at least as large as $\max\{n \mid \underline{\delta}^n \geq b\}$.

To see why, note from Lemma 2 that, for fixed b , the distance $2\underline{\delta}^n$ between any two adjacent decisions in \underline{Y}^n decreases with n . Moreover, for any $n \geq 3$, P 's incentive conditions (9) are satisfied if and only if $\underline{\delta}^n \geq b$. P 's remaining incentive condition (10) is satisfied when $\underline{\delta}^n \geq b$ since $\underline{y}_1^n > 0$. Finally, \underline{Y}^2 is always incentive compatible for P for $b < \frac{1}{2}$ (see the proof of Proposition 3), implying that at least two decisions can be induced in an optimal PBE.

We refer to \underline{Y}^n as the *minimal limited authority* in this uniform-convex loss model when n achieves the lower bound from Proposition 4. By construction, P 's incentive conditions (9) are dropped in deriving each \underline{Y}^n . Thus, the minimal limited authority has the largest number of induced decisions in any PBE in which P 's incentive conditions are all slack.

4.2. Optimal limited authority

Propositions 3 and 4 respectively identify the upper and the lower bounds of the number of induced decisions for any fixed b in an optimal PBE. We now characterize the optimal PBE conditional on each number of induced decisions n lying between the two bounds. To do so, we first vary n and characterize the interval of values of b for which n falls between the upper and the lower bounds. We then solve the problem of maximizing P 's expected payoff (6) by choosing a set of n decisions, subject to all constraints (7)–(10). The solution to this problem, $Y^n = \{y_i^n\}_{i=1}^n$, is called the *n-optimal limited authority* for any value of b in the interval since it is conditional on a fixed n .

Fix any $n \geq 2$. By Proposition 3, Y^n is defined only if $b < \bar{b}^n$ where

$$\bar{b}^n \equiv \frac{1}{2(n-1)}. \tag{12}$$

Note that $\bar{b}^2 = \frac{1}{2}$. By Proposition 4, for each $n \geq 3$, \underline{Y}^n satisfies P 's incentive conditions (9) and (10) if $b \leq \underline{\delta}^n$. Since $\underline{\delta}^n$ is implicitly a decreasing function of b given in Lemma 2, $b \leq \underline{\delta}^n$ is equivalent to $b \leq \underline{b}^n$, where \underline{b}^n is uniquely determined by setting $\underline{\delta}^n$ to b :

$$2l(1/2 - (n-1)\underline{b}^n) = l(2\underline{b}^n). \tag{13}$$

By Lemma 2, \underline{Y}^2 is always incentive compatible for P , so we write $\underline{b}^2 = \bar{b}^2 = \frac{1}{2}$. Moreover, $\underline{b}^n < \bar{b}^n$ for each $n \geq 3$, and both bounds are decreasing in n : the smaller A 's bias is, the more decisions can be induced. It follows that Y^n is defined for $\underline{b}^{n+1} < b < \bar{b}^n$. Within this interval, if $b \in (\underline{b}^{n+1}, \underline{b}^n]$, P 's incentive conditions (3) are satisfied under \underline{Y}^n , and thus $Y^n = \underline{Y}^n$. But if $b \in (\underline{b}^n, \bar{b}^n)$, we show next that the optimal decisions in Y^n may no longer be evenly-spaced even though Y^n remains symmetric around $\frac{1}{2}$ with every other decision exactly $4b$ apart. In particular, Y^n coincides with \bar{Y}^n if and only if n is odd.

Lemma 3. For any $n \geq 3$, the n -optimal limited authority Y^n is defined for $(\underline{b}^{n+1}, \bar{b}^n)$. Specifically, $Y^n = \underline{Y}^n$ for $b \in (\underline{b}^{n+1}, \underline{b}^n]$; $Y^n = \bar{Y}^n$ for $b \in (\underline{b}^n, \bar{b}^n)$ and n odd; and Y^n is otherwise given by $y_i^n = \frac{1}{2} - (n-2i)b - \delta_1^n$ for odd i , and $y_i^n = \frac{1}{2} - (n+2-2i)b + \delta_1^n$ for even i , where $\delta_1^n < b$ is determined by

$$2l(y_1^n) = l(b + \delta_1^n) + l(b - \delta_1^n) - \frac{n-2}{2} [l(3b - \delta_1^n) - l(b + \delta_1^n)]. \tag{14}$$

Unlike minimal and maximal limited authority, n -optimal limited authority depends on whether n is odd or even for $b > \underline{b}^n$. When n is odd, the decisions are all equidistant at $2b$. But when n is even, the decisions are equidistant in an alternating manner, with $y_{i+1}^n - y_i^n$ equal for all odd i and for all even i respectively (but strictly smaller than $2b$ for odd i). Intuitively, because P 's incentive conditions require only every other decision to be $4b$ apart, P can ‘fine-tune’ the placements of the decisions to increase her expected payoff in the case of even n . In particular, if decisions $\{y_i^n\}_{i=1}^n$ were all $2b$ apart and symmetric around $\frac{1}{2}$, then without violating any incentive conditions, P could increase her payoff by increasing y_i^n for all odd i and decreasing it for even i by the same amount.

The crucial feature of the n -optimal limited authority for $b > \underline{b}^n$ is that P 's incentive conditions are all binding. That is, whenever Y^n differs from \underline{Y}^n , P is indifferent between implementing each recommended decision y_i^n and replacing it with the adjacent lower decision y_{i-1}^n for each $i = 2, \dots, n-1$. Otherwise, if some, but not all incentive conditions (9) were binding, it would be possible to modify the decisions to make them more equidistant and thus increase P 's payoff. For example, if $y_{i+1}^n - y_{i-1}^n > 4b$ for some i , then we could increase y_{i-1} or decrease y_{i+1} without violating any incentive condition of P .

An important implication of binding P 's incentive conditions is that P 's expected payoff may not increase in the number of induced decisions. Instead, P must trade off the number of decisions with their placement: one more decision is useless if it is rarely used. In fact, it can be shown that the smallest threshold θ_1 approaches 0 as b approaches \bar{b}^n . In other words, A almost never recommends y_1^n and P almost never uses it. Essentially, P only uses the remaining $n-1$ decisions, which are placed asymmetrically and thus suboptimal to Y^{n-1} by Lemma 3.

We now characterize the optimal PBE by choosing the optimal n among all feasible n -optimal limited authority for each fixed b . Note that for all $n \geq 3$, from (12) and (13) we have

$$\underline{b}^n < \bar{b}^{n+1} < \underline{b}^{n-1}.$$

That is, the upper bound and the lower bound on the number of induced decisions in any optimal PBE differ either by 2 when $b \in (\underline{b}^n, \bar{b}^{n+1})$, or by 1 when $b \in (\bar{b}^{n+1}, \underline{b}^{n-1})$.¹² As a result of the above inequalities, we can restrict the search for the optimal PBE in the interval $[\underline{b}^n, \underline{b}^{n-1})$ to three conditionally optimal decision sets: Y^{n-1} , Y^n and Y^{n+1} . Note that $Y^{n-1} = \underline{Y}^{n-1}$ by Lemma 3 for any b in this interval, so that P 's incentive conditions are not binding under Y^{n-1} . To emphasize this feature, we write \underline{Y}^{n-1} instead of Y^{n-1} in the ensuing analysis.

The general idea behind the comparison among \underline{Y}^{n-1} , Y^n and Y^{n+1} is as follows. First, as mentioned above, at b just below the cutoff value \bar{b}^{n+1} , P strictly prefers Y^n to Y^{n+1} because the additional decision in Y^{n+1} does nothing to improve her expected payoff, but distorts the quality of her decisions. Second, at $b = \underline{b}^n$, the optimal decision sets under full commitment and limited authority are identical: $\underline{Y}^n = Y^n$. P 's expected payoff jumps down discontinuously at \underline{b}^n if decisions change from \underline{Y}^n to \underline{Y}^{n-1} . In contrast, Y^n changes continuously with b at \underline{b}^n . Consequently P is strictly better off with Y^n than with \underline{Y}^{n-1} if b is sufficiently close to and greater than \underline{b}^n .

To go beyond these comparisons, we specialize to the case of power loss functions. This case, standard in the existing literature, also allows for a sharper characterization of the optimal PBE and facilitates comparative statics analysis in Section 5.

Proposition 5. *Suppose that $l(z) = (|z|)^q$ with $q > 1$. For n sufficiently large, there exists $b^{n,n-1} \in (\underline{b}^n, \underline{b}^{n-1})$ such that the induced decisions in the optimal PBE are given by Y^n for all $b \in [\underline{b}^n, b^{n,n-1})$, and by \underline{Y}^{n-1} for all $b \in [b^{n,n-1}, \underline{b}^{n-1})$.*

As b increases in the interval $(\underline{b}^n, \underline{b}^{n-1})$, P 's expected payoff decreases under each of \underline{Y}^{n-1} , Y^n and Y^{n+1} . The proof of Proposition 5 uses the assumption of power loss function to rank the rate of decrease for the three sets of decisions for any fixed n . In particular, we show that P 's expected payoff $U^n(b)$ under Y^n decreases slower than her expected payoff $U^{n+1}(b)$ under Y^{n+1} . Next, we show that $U^n(b)$ can cross P 's expected payoff $\underline{U}^{n-1}(b)$ under \underline{Y}^{n-1} only from above. Moreover, if n is sufficiently large, P strictly prefers \underline{Y}^{n-1} to Y^n at \underline{b}^{n-1} .¹³ Shifting the indices forward by 1 and noting that $Y^n = \underline{Y}^n$ at \underline{b}^n , we have that P strictly prefers Y^n to Y^{n+1} . This evaluation then allows us to establish the proposition.

Proposition 5 makes it clear that the optimal limited authority does not generally coincide with the maximal limited authority. This non-optimality is reflected in two different ways. First,

¹² To see why the above inequalities hold, note that the function $g(n, b) \equiv 2l(1/2 - (n-1)b) - l(2b)$ is decreasing in b for $b \in (0, \bar{b}^n)$ and is equal to 0 at \underline{b}^n for all n . The first inequality above holds because $g(n, \bar{b}^{n+1}) = 2l(1/(2n)) - l(1/n) < 0$ for any convex loss function l ; the second inequality holds because $g(n-1, \bar{b}^{n+1}) = l(1/n) > 0$.

¹³ More precisely, P strictly prefers \underline{Y}^{n-1} to Y^n at \underline{b}^{n-1} if n is odd, but the opposite is true if n is even and small and if q is large. What happens when n is even is that, starting from placing all n decisions at equal distance of $2\underline{b}^{n-1}$ and symmetric around $\frac{1}{2}$, by 'fine tuning' the placements of the decisions according to Lemma 3 P can obtain a strictly higher payoff than under \underline{Y}^{n-1} if n is small and q is large, which is precisely when such fine-tuning matters greatly to P 's payoff. In this case, the induced decisions in the optimal PBE is given by Y^n for all $b \in [\underline{b}^n, \underline{b}^{n-1})$. Conversely, the fine-tuning matters little if n is large and/or q is small. An example is Corollary 1 below, where $q = 2$ implies that P strictly prefers \underline{Y}^{n-1} to Y^n at \underline{b}^{n-1} for all n .

when b falls in $[\underline{b}^n, \bar{b}^{n+1})$, Y^{n+1} is available but never optimal for n sufficiently large. Second, when b falls in $(\underline{b}^{n,n-1}, \underline{b}^{n-1})$, Y^n is available but P strictly prefers \underline{Y}^{n-1} for n sufficiently large. As already mentioned, the incentive conditions of P are slack under \underline{Y}^{n-1} , but are binding under Y^n . Intuitively, in an optimal PBE, P retains fewer decisions to relax the incentive conditions due to limited authority.

The result that under limited authority P does not always maximize the number of induced decisions contrasts strongly with the standard models where P has either full commitment or no commitment. Recall from [Theorem 1](#), however, that we restrict the search for the optimal PBE under limited authority to decision sets that are minimal and veto-free. The minimality requires that each decision is induced in some states, precluding the standard reasoning that adding a decision cannot make P worse off. In our model of limited authority, similar to models with full commitment, a larger number of induced decisions tends to allow A to make better use of his private information, resulting in a smaller loss of information that would benefit P . But the similarity stops here: each additional decision also presents P with a credibility problem, which can only be addressed by altering the placements of the decisions. When such placements become too extreme, the induced decisions become on average less aligned with the true state. These distortions from avoiding the credibility problem may be so great that P prefers a PBE with a smaller number of induced decisions but a higher quality of decision making.

We conclude this section by pointing out that in the familiar case of $l(z) = z^2$, [Proposition 5](#) can be further strengthened. In fact, the characterization of the set of induced decisions in the optimal PBE holds for each $n \geq 3$, not just for sufficiently large n . Furthermore, we can further establish that the cutoff value $b^{n,n-1}$ for each $n \geq 3$ is strictly greater than \bar{b}^{n+1} . Thus, the number of induced decisions under the maximal and optimal limited authority is the same for $b \in [\bar{b}^{n+1}, b^{n-1,n})$, and differs by 1 for b in the interval $[\underline{b}^n, \bar{b}^{n+1})$ and in the interval $[b^{n-1,n}, \underline{b}^{n-1})$. In the last case, the optimal limited authority and the minimum limited authority coincide.

Corollary 1. *Suppose that $l(z) = z^2$. For each $n \geq 3$, there exists $b^{n,n-1} \in (\bar{b}^{n+1}, \underline{b}^{n-1})$ such that the induced decisions in the optimal PBE are given by Y^n for all $b \in [\underline{b}^n, b^{n,n-1})$, and by \underline{Y}^{n-1} for all $b \in [b^{n,n-1}, \underline{b}^{n-1})$.*

5. Welfare analysis

This section focuses on P 's ex ante expected payoff under optimal limited authority. Throughout this section, we say that one outcome is *better* than another if ex ante, P strictly prefers the former outcome. In [Section 5.1](#), we compare quantitatively our model of limited commitment with the two benchmark models where P has full commitment and no commitment. By full commitment or *optimal delegation*, we mean the best outcome that P achieves when she commits to a mapping from A 's messages to decisions; and by no commitment or *cheap talk*, we mean the most informative CS equilibrium outcome. [Section 5.2](#) carries out the main comparison between two limited commitment models, limited authority and *full delegation*, where A 's preferred decision is chosen for all states. In both these subsections, we maintain [Assumption 2](#) and mostly restrict to power loss functions, although some of the results hold for the general model of [Section 3](#). Finally, [Section 5.3](#) presents further welfare analysis for the special case of the quadratic loss function.

5.1. Comparison with optimal delegation and cheap talk

To understand the quantitative comparison of P 's payoff when she has different commitment power, observe that for $b \geq \frac{1}{2}$, optimal delegation, limited authority, and cheap talk implement the same outcome, the uninformed decision $\frac{1}{2}$, and thus are equivalent. For $b < \frac{1}{2}$, P 's expected payoff continuously decreases with A 's bias for any given commitment power. In this case, optimal delegation is better than limited authority which in turn is better than cheap talk. [Proposition 6](#) quantifies these comparisons when A 's bias is small.

Proposition 6. *Suppose that $l(z) = (|z|)^q$ with $q > 1$. For b arbitrarily small, P 's expected payoff is $-(2b)^q / (q + 1) + o(b^q)$ under limited authority, $-b^q + o(b^q)$ under optimal delegation, and $-(2b)^{q/2} 2 / ((q + 1)(q + 2)) + o(b^{q/2})$ under cheap talk.*

When A 's bias is small, P 's expected payoff is of the same order of A 's bias under optimal delegation and limited authority, and is much lower under cheap talk. Intuitively, under limited authority, induced decisions are approximately $2b$ apart, so there are approximately $1/(2b)$ induced decisions, and P 's expected payoff (6) is approximately

$$-\frac{1}{2b} \int_0^{2b} \theta^q d\theta = -\frac{(2b)^q}{q + 1}.$$

The optimal delegation set is $[b, 1 - b]$ as suggested by [Lemma 2](#) with large n , so P incurs a payoff loss of b^q due to A 's bias. Under cheap talk, however, the partition is coarse such that the distance between adjacent decisions grows at the approximate rate $4b$, leading to a much larger loss (infinitely larger when b approaches 0) compared to the other models. Not only more decisions can be induced under both optimal delegation and limited authority, these decisions are also more evenly spaced and thus better aligned with the true state.

5.2. Comparison with full delegation

In our limited authority model, ex ante P can credibly exclude some decisions but she cannot credibly promise to let A decide among the retained decisions. Under full delegation, on one hand, P 's commitment power is decreased because she cannot exclude any decision; but on the other hand her commitment power is increased because now she commits herself to letting A decide. As argued by [Dessein \[12\]](#), P can, and does fully delegate her decision-making authority to a better informed A in many real life examples. The welfare comparison between limited authority and full delegation helps us understand whether it is more important for an organization to limit the authority of decision-makers or to pass the authority to those who hold the most relevant information.

[Dessein \[12\]](#) shows that under the assumption that the state is uniformly distributed, full delegation is better than cheap talk as long as A 's bias is not so large that cheap talk is uninformative. In fact, full delegation and optimal delegation give P the same expected payoff if b is sufficiently small. The following result follows immediately from [Proposition 6](#).

Corollary 2. *Suppose that $l(z) = (|z|)^q$ with $q > 1$. Full delegation is better than limited authority if b is arbitrarily small.*

In contrast to Dessein [12], however, for any power loss function, there exists b at which limited authority is informative and is better than full delegation. Indeed, limited authority is informative so long as $b < \frac{1}{2}$. As b approaches $\frac{1}{2}$, P 's expected payoff approaches $-2^{-q}/(q+1)$, which is her payoff under limited authority from making the uninformed decision of $\frac{1}{2}$. This is clearly higher than -2^{-q} , her payoff under full delegation.

Perhaps more surprisingly, for any number of induced decisions, limited authority can be better than full delegation when the power of the loss function is sufficiently close to 1. To illustrate, we use our maximal limited authority construction \bar{Y}^n characterized in Proposition 3 for any $n \geq 3$. Recall from Section 4 that $\bar{b}^{n+1} = 1/(2n)$ is the largest b such that $n+1$ decisions can be induced under maximal limited authority, thus \bar{Y}^n exists at $b = \bar{b}^{n+1}$. The advantage of using \bar{Y}^n instead of optimal limited authority is its simplicity: all the decisions are $2b$ apart, with $\bar{\theta}_i^n = \bar{y}_i^n$ from A 's indifference condition. We can then show that P 's expected payoff (6) under \bar{Y}^n is strictly higher than under full delegation if and only if

$$n < \frac{2^q - 1}{2^q - (q + 1)}.$$

Observe that the right-hand side of the above inequality is decreasing in q for $q \in (1, 2)$, becomes arbitrarily large as q approaches 1, and equals 3 at $q = 2$. Thus, for any $n \geq 3$, there exists \bar{q}^n such that \bar{Y}^n is better than full delegation at \bar{b}^{n+1} if and only if $q < \bar{q}^n$. Further, $\bar{q}^n \in (1, 2]$ decreases with n and converges to 1 as n becomes arbitrarily large. This result shows that if P is sufficiently close to being risk neutral, optimal limited authority can be better than full delegation, no matter how small A 's bias is or how informative the communication under limited authority is.

Little can be established in general about comparative statics with respect to b for fixed q .¹⁴ The comparative statics with respect to q , however, is general.

Proposition 7. *Suppose that $l(z) = (|z|)^q$ with $q > 1$. For any $b \in (0, 1/2)$, there exists $q(b)$ such that limited authority is better than full delegation if and only if $q < q(b)$.*

Intuitively, under full delegation, P incurs a deterministic loss of b^q , but under limited authority, P incurs a stochastic loss of $(|y - \theta|)^q$ where $y = y_i$ when θ lies in the interval $(\theta_{i-1}, \theta_i]$. Thus it is easy to see that at the extremes, limited authority is better than full delegation for any b if P is risk neutral (q is arbitrarily close to 1), and the reverse holds if P is infinitely risk averse (q is arbitrarily large). In fact, Proposition 7 shows that there exists a cutoff value of q such that P is indifferent. For all q below this cutoff, l becomes less convex and P becomes less risk averse, so she is more willing to take a stochastic loss if the partition equilibrium $(\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ is fixed. But since the partition equilibrium is optimally chosen by P , limited authority becomes even more attractive than full delegation as q decreases.

¹⁴ In particular, it is not true that limited authority is better than full delegation if and only if b is strictly greater than a cutoff. As can be seen from Fig. 1 in Section 5.3, under the quadratic loss function, P 's expected payoff is the same under limited authority and full delegation at multiple values of b in the interval $(0, 1/2)$. Therefore, for q slightly less than 2, the continuity of P 's expected payoff in q and Proposition 7 imply that limited authority is better than full delegation if and only if b is close to $1/6$ or greater than 0.197.

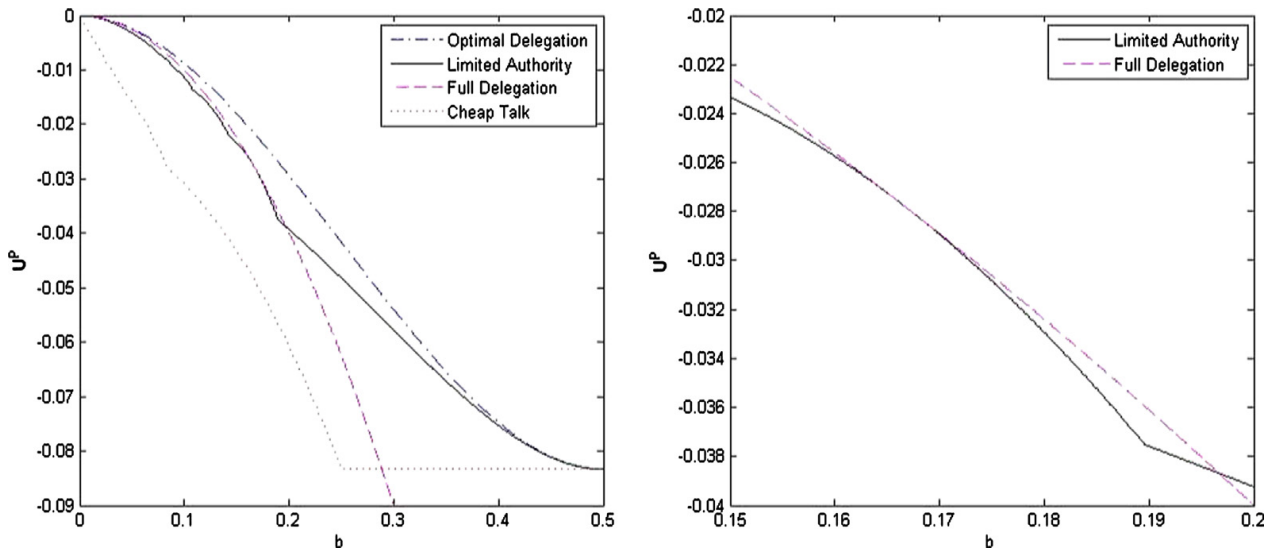


Fig. 1. P 's expected payoff.

5.3. Further comparisons under quadratic loss

We now turn to the familiar quadratic loss function example used widely in the communication and delegation literature. Because of its simplicity, we can explicitly calculate optimal limited authority from the characterization of Corollary 1. This enables us to make the welfare comparisons in the previous subsections exact.

Fig. 1 illustrates P 's expected payoff for all possible values of A 's bias $b \in (0, \frac{1}{2})$ under the four models we have studied. Clearly, for the whole range of A 's bias, P 's expected payoffs are considerably lower under cheap talk than under limited authority, full delegation and optimal delegation. In particular, P 's payoff under limited authority is almost as high as her payoff under optimal delegation even though only finite decisions can be induced under limited authority. Moreover, full delegation and limited authority give P the same payoff more than once, and limited authority dominates when b is greater than approximately 0.197.

The key feature of limited authority is that P improves her payoff by giving up some control ex post for better information ex ante. One natural question to ask is what happens to A 's payoff. Under quadratic loss, we can answer this question by decomposing P and A 's expected payoffs as the sum of the loss of information and the loss of control:

$$\begin{aligned}
 U^P &= -\mathbb{E}[(y - \theta)^2] = -\underbrace{\mathbb{E}[(y_m - \mathbb{E}[\theta|m])^2]}_{P\text{'s Loss of Control}} - \underbrace{\mathbb{E}[\text{Var}(\theta|m)]}_{\text{Loss of Information}}, \\
 U^A &= -\mathbb{E}[(y - (\theta + b))^2] = -\underbrace{\mathbb{E}[(y_m - \mathbb{E}[\theta|m] - b)^2]}_{A\text{'s Loss of Control}} - \underbrace{\mathbb{E}[\text{Var}(\theta|m)]}_{\text{Loss of Information}},
 \end{aligned}$$

where y_m , $\mathbb{E}[\theta|m]$, and $\text{Var}(\theta|m)$ are respectively the decision taken, the expectation, and the variance of the state θ given a message m (under P 's beliefs). The loss of information is defined as the expected conditional variance of the state given P 's belief at the time of decision making. Therefore, the loss of information captures the residual uncertainty that P has after communication takes place, which is the same for P and A . The loss of control for P and A is defined as the expected losses from making decision y_m instead of their ex post optimal decisions given the message m .

Ex ante, both P and A prefer limited authority to cheap talk. To see this, note that from the above decomposition we have

$$U^A = U^P + 2b(y_m - \mathbb{E}[\theta|m]) - b^2.$$

By [Proposition 2](#), U^P is higher under limited authority than under cheap talk. Since $y_m > \mathbb{E}[\theta|m]$ under limited authority by [Lemma 1](#) while $y_m = \mathbb{E}[\theta|m]$ under cheap talk, U^A is also higher under limited authority than under cheap talk. Intuitively, P is better off under limited authority compared to cheap talk, because her loss of information decreases more than the increase in her loss of control; and A is also better off because P cedes part of the control to A by choosing higher than ex post optimal decisions.

Many existing papers have analyzed extensions of the CS model that improve communication quality and hence P 's welfare. The assumption of quadratic loss makes it particularly simple to compare our limited authority model with these existing models. To do so, we introduce the following synthesis. Start with the CS cheap talk model, in which Nature draws the state of the world $\theta \in \Theta$, A privately observes θ and sends a message $m \in M$ to P , and P makes a decision $y \in Y$. Imagine a fourth non-strategic player who takes some input and returns a possibly stochastic output according to some pre-specified mapping. Each possible way that a fourth player is introduced into the game corresponds to a different *organizational form*. First, the fourth player may replace an existing player, P or A . Under *delegation*, the fourth player replaces P : A sends a message m to the fourth player instead of to P , and the fourth player then makes a decision y . This includes both full delegation, in which the fourth player is just A , and optimal delegation, in which the fourth player is optimally designed by P . Under *persuasion*, the fourth player replaces A : the fourth player observes the state θ and sends a message m to P . Second, the fourth player may be an impartial mediator who acts on state, message or decision. Under *informational control*, the fourth player privately observes the state θ and sends a private signal to A , which becomes his sole information in the ensuing cheap talk game. Under *noisy talk*, the fourth player receives a message m from A and sends a perturbed message to P . Finally, in our limited authority model, P receives a message from A and then recommends a decision y to the fourth player who makes a decision.¹⁵

Using the quadratic loss function, we can compare P and A 's expected payoffs for a small bias b under all organizational forms by calculating the first term in the Taylor expansion (see [Table 1](#)).¹⁶

¹⁵ Holmstrom [[19](#)], Melumad and Shibano [[30](#)], Alonso and Matouschek [[3](#)], and Kovac and Mylovonov [[25](#)] study optimal delegation, while Dessein [[12](#)] studies full delegation. Kamenica and Gentzkow [[22](#)] analyze optimal persuasion that maximizes A 's expected payoff, while full information transmission maximizes P 's expected payoff. Ivanov [[21](#)] studies optimal informational control. Goltsman et al. [[13](#)] study optimal message mediation and show that it is equivalent to optimal noisy talk studied by Blume et al. [[8](#)]. There are also papers that can be viewed as a combination of the introduced organizational forms in that more than one non-strategic player is added. For example, Anesi and Seidmann [[5](#)] study optimal delegation in the uniform-quadratic example with a finite number of states. This can be viewed as a combination of delegation and informational control where A only observes a partitioned element that contains the underlying continuous state. A complementary set of papers such as Ivanov [[20](#)], Li [[28](#)] and Ambrus et al. [[4](#)] study communication with strategic message mediators.

¹⁶ We believe that these results hold more generally with a caveat that each row of [Table 1](#) is multiplied by some constant. In particular, we expect them to hold if P 's and A 's payoff functions satisfy [Assumption 2](#) and $l''(0) > 0$. Intuitively, as the bias goes to zero, the distance between any two subsequent decisions also goes to zero. Therefore, we can approximate the loss functions by quadratic functions, and also approximate the state distribution function by a piecewise uniform distribution.

Table 1
P and *A*'s expected payoffs under all organizational forms for a small bias.

	Persuasion	Informational control	Optimal delegation	Full delegation	Limited authority	Noisy talk	Cheap talk
U^P	0	$-\frac{1}{3}b^2$	$-b^2$	$-b^2$	$-\frac{4}{3}b^2$	$-\frac{1}{3}b$	$-\frac{1}{3}b$
U^A	$-b^2$	$-\frac{4}{3}b^2$	$-\frac{8}{3}b^3$	0	$-\frac{1}{3}b^2$	$-\frac{1}{3}b$	$-\frac{1}{3}b$

To understand these payoff comparisons, we again decompose the payoffs of *P* and *A* into the loss of information and the loss of control. In terms of the loss of control, in all organizational forms, either *P* or *A* has essentially no loss of control, and thus the other party has a loss of control equal to b^2 . Under delegation and limited authority, *P* has commitment power and effectively delegates authority to *A* to improve information transmission, and her loss of control b^2 is simply due to *A*'s bias. Under remaining organizational forms, *P* makes an ex post optimal decision, and thus she has no loss of control. Regarding the loss of information, there is essentially no loss of information under delegation and persuasion because the state is almost fully revealed. The loss of information is approximately $\frac{1}{3}b^2$ under informational control and limited authority because induced decisions are approximately $2b$ apart from each other.¹⁷ Under cheap talk and noisy talk, however, the partition is coarse such that the distance between adjacent decisions grows at the approximate rate $4b$, leading to a much larger loss of information of approximately $\frac{1}{3}b$. Combining these two parts lead to the payoff comparisons in Table 1.

6. Concluding remarks

Our model of limited authority aims to explore and understand the environment in which the principal has some degree of commitment power, but not all. We now discuss how the optimal equilibrium may be affected by different assumptions about the communication process and contracting environment (see Kolotilin [24] for more detailed expositions) as well as some thoughts for further research.

Because only finitely many decisions can be induced in any optimal equilibrium under limited authority, one may wonder whether the principal benefits from a finite decision set per se, that is, when the principal's decision space is discrete. It can be shown that in the uniform-convex setup in Section 4, the equilibria in the CS model with a discrete set of decisions are similar to equilibria in the CS model with a continuous set of decisions, but with a modified smaller bias. Therefore, by discretizing the set of available decisions, the principal can effectively decrease the agent's bias and improve communication.¹⁸ In fact, under the optimal limited authority given in Proposition 5, the agent's effective bias disappears so that a uniform partition becomes feasible.

One possible critique of the limited authority model is that parties can renegotiate, after the agent's recommendation, to a decision not in the pre-specified set if both prefer to do so. To address this issue, we can strengthen our solution concept to optimal renegotiation-proof

¹⁷ This connection between limited authority and informational control is due to the limited commitment power of both *A* and *P*. Under limited authority, *P* makes the decision space discrete to relax her incentive conditions, whereas under informational control, *P* makes the state space discrete to relax *A*'s incentive conditions.

¹⁸ This result resembles that of Alonso and Matouschek [2] who show that the principal's commitment power reduces the agent's 'effective' bias.

equilibrium. In our model the scope for renegotiation is limited for two reasons. First, because transfers between the parties are not allowed, both the principal and the agent must prefer the renegotiated decision to the optimal equilibrium decision. Second, in any PBE there is unresolved uncertainty about the state of the world after the agent's recommendation. Hence the principal does not know to which decision, if any, she should renegotiate. In the uniform-quadratic setup, we can show that a renegotiation-proof equilibrium exists, and has a maximal number of induced decisions characterized by [Proposition 3](#).

Another critique is that if the principal can credibly restrict the set of decisions ex ante, she may also be able to commit to transfers contingent on decisions. Following the literature on communication and delegation, we have ruled out such transfers in the present limited authority model.¹⁹ Note that with transfers, the principal can do at least as well as in the limited authority model by committing to no transfers for decisions in the pre-specified set and very large transfers for decisions outside the set. In fact, with transfers, the principal can achieve the first best outcome, which maximizes the sum of the principal and agent's expected payoffs if there are no monetary frictions. Not all decision rules, however, can be supported with transfers. For example, the principal's and the agent's optimal decision rules are not achievable. Further, if there are frictions, such as when the agent is protected by limited liability or if the principal can only 'burn' money, then there is a tradeoff between making efficient decisions and leaving a quasi-rent to the agent. Due to this tradeoff, there is incomplete information revelation for high states of the world, even though full information revelation is feasible.²⁰

In the current model, the principal never vetoes the agent's recommendation in equilibrium. A further topic of research is to extend our model to allow veto to happen in equilibrium. One way to model this is to imagine that there is some small, exogenous probability that the principal can observe the true state after hearing the agent's recommendation, and may consequently desire to change her decision (still within the pre-specified decision set) given this information. In this case, it is without loss of generality to restrict to equilibria in which the principal follows the agent's recommendation when she does not learn the state and otherwise makes a choice independent of the recommendation. Thus, any equilibrium characterized in [Theorem 1](#) remains feasible, and further, the principal can do better by adding any decision that is chosen with zero probability in equilibrium so long as her incentive conditions (when she does not learn the state) are unaffected. The interesting question is how the principal optimally modifies the decisions that are used with positive probabilities when she does not learn the state, in order to retain more decisions that she will use when she does. Answering this question can further our understanding of the principal's tradeoff between maintaining the flexibility of responding to new information and establishing the credibility of letting the agent best use his private information.

¹⁹ Monetary transfers may be explicitly ruled out by law, or implicitly ruled out if the parties involved are very risk averse with respect to money. The assumption that there are no transfers is strong: even though explicit transfers between parties may be ruled out, the parties can effectively 'burn' money, which generally improves their welfare. See, for example, Austen-Smith and Banks [\[6\]](#).

²⁰ This result is analogous to that of Krishna and Morgan [\[26\]](#) who consider a communication game in which the principal can commit to transfers contingent on messages and the agent is protected by limited liability. Kartik [\[23\]](#) obtains a somewhat similar result. He shows that in a model of communication with lying costs, there is pooling for high states of the world.

Appendix A. Proofs

Proof of Theorem 1. We prove the theorem through a series of claims.

First, we establish that it is without loss of generality to restrict attention to PBE's in which all decisions in P 's equilibrium choice Y are induced, each message from A is a *recommendation* of some probability distribution over Y , and no recommendation is vetoed by P . This is a version of the revelation principle adapted to our setting. Fix any PBE and the subgame after P has made the equilibrium choice Y . We refer to any response by P to a message m from A as a *lottery*, and a particular choice from Y as a *degenerate lottery*. We say that two PBE's are *outcome equivalent* if they both result in the same (random) mapping from states to decisions on the equilibrium path.

Claim 1. Consider a PBE with P 's equilibrium choice Y . There exists an outcome-equivalent PBE with P 's equilibrium choice $\tilde{Y} \subseteq Y$, where \tilde{Y} is the union of the supports of all induced lotteries and for any induced lottery there is a unique y in its support chosen by A as a message.

Proof. Fix any PBE and the subgame after P has chosen the equilibrium Y . Since $u^P(\cdot, \theta)$ is strictly concave, there are at most two decisions y and y' in Y that are optimal given the equilibrium belief of P conditional on any m . Thus, a non-degenerate lottery has exactly two decisions. Moreover, if y and y' in Y satisfying $y < y'$ are in the support of some lottery, then $(y, y') \cap Y = \emptyset$; otherwise, strict concavity of $u^P(\cdot, \theta)$ implies that the lottery would not be optimal for P . Finally, no two induced lotteries have the same support. Otherwise, if $y, y' \in Y$ with $y < y'$ are in the common support of two distinct lotteries induced after A chooses m and m' respectively, then one of them, say the lottery following m' , first order stochastically dominates the other. Since $u_{y\beta} > 0$, P being indifferent between y and y' given the belief conditional on m implies that A strictly prefers y' to y given the same belief. Thus, there are states in which A is supposed to choose m but would find it profitable to deviate to m' to induce the lottery following m' , a contradiction. By the same argument, if $y, y' \in Y$ with $y < y'$ are the support of some induced lottery, y' is not induced as a degenerate lottery.

Let \tilde{Y} be the union of the supports of all induced lotteries following Y . We construct an outcome-equivalent PBE where P chooses \tilde{Y} instead of Y on the equilibrium path and A 's message space is restricted to P 's choice of set of decisions on and off the equilibrium path. For any choice of P that is not \tilde{Y} , including Y , let the continuation in the new PBE be such that A chooses the lowest decision in the set chosen by P regardless of realized θ and P chooses a decision that is optimal in the set given her prior belief. It remains to specify the continuation equilibrium in the new PBE following \tilde{Y} that is outcome equivalent to the continuation equilibrium in the original PBE following Y . For each degenerate lottery $y \in Y$ induced in the continuation equilibrium following Y after A chooses some message m , let A choose y in the subgame following \tilde{Y} and let P 's belief be the same as in the original PBE conditional on m ; and for each non-degenerate lottery where P randomizes between y and y' with $y < y'$ following Y after A chooses some message m' , let A choose y' in the subgame following \tilde{Y} and let P 's belief be the same as in the original PBE conditional on m' . All equilibrium conditions are satisfied in the new PBE following \tilde{Y} as they are a subset of the equilibrium conditions in the original PBE following Y . Further, by construction \tilde{Y} is part of the new PBE, because Y is part of the original PBE, and the equilibrium payoff for P is greater than or equal to the payoff from choosing y^P . \square

Second, we show that in any PBE the number of induced lotteries is finite. Denote $\{y, y'; w\}$ as a lottery induced in some continuation game after P has chosen Y , with P choosing y with

probability $w \in (0, 1)$ and $y' \geq y$ with probability $1 - w$. We adopt the convention that a degenerate lottery is represented by $y' = y$. The proof of Claim 1 implies that any two distinct lotteries $\{y_1, y'_1; w_1\}$ and $\{y_2, y'_2; w_2\}$ can be ordered, with the first lower than the latter, such that $y_1 \leq y'_1 \leq y_2 \leq y'_2$, with at least one strict inequality and $y'_1 = y_2$ implying that $y'_2 > y_2$.

Claim 2. *The number of decisions induced in any PBE is finite.*

Proof. Fix some PBE and the subgame after P has chosen the equilibrium Y . Let $\{y_i, y'_i; w_i\}$, $i = 1, 2, 3$, be three distinct induced lotteries, in increasing order. Since both $\{y_2, y'_2; w_2\}$ and $\{y_3, y'_3; w_3\}$ are induced, there is a state $\hat{\theta}$ such that

$$w_2 u^A(y_2, \hat{\theta}) + (1 - w_2) u^A(y'_2, \hat{\theta}) = w_3 u^A(y_3, \hat{\theta}) + (1 - w_3) u^A(y'_3, \hat{\theta}).$$

Since $u^A(\cdot, \hat{\theta})$ is strictly concave, $y^A(\hat{\theta}) \in (y_2, y'_2)$. Further, since $u^A_{y\theta} > 0$, the lottery $\{y_2, y'_2; w_2\}$ is not induced for any $\theta > \hat{\theta}$, as

$$\begin{aligned} & w_3(u^A(y_3, \theta) - u^A(y_3, \hat{\theta})) + (1 - w_3)(u^A(y'_3, \theta) - u^A(y'_3, \hat{\theta})) \\ & \geq u^A(y_3, \theta) - u^A(y_3, \hat{\theta}) \\ & \geq u^A(y'_2, \theta) - u^A(y'_2, \hat{\theta}) \\ & \geq w_2(u^A(y_2, \theta) - u^A(y_2, \hat{\theta})) + (1 - w_2)(u^A(y'_2, \theta) - u^A(y'_2, \hat{\theta})), \end{aligned}$$

with at least one inequality being strict. This implies that $\{y_2, y'_2; w_2\}$ can only be induced if the state θ is smaller than $\hat{\theta}$. As a result, we have $y^P(\hat{\theta}) > y_1$; otherwise, since $u^A_{y\theta} > 0$, a similar argument as above would imply that P prefers the lottery $\{y_1, y'_1; w_1\}$ to $\{y_2, y'_2; w_2\}$ for all $\theta < \hat{\theta}$ but then $\{y_2, y'_2; w_2\}$ would never be induced. It then follows that $y_1 < y^P(\hat{\theta}) < y^A(\hat{\theta}) < y'_3$. Since $y^P(\theta) < y^A(\theta)$ for all $\theta \in [0, 1]$ and are both continuous, there exists $\varepsilon > 0$ such that $y^A(\theta) - y^P(\theta) \geq \varepsilon$ for all $\theta \in [0, 1]$. There can be at most one induced decision greater than $y^P(1)$ and one lower than $y^P(0)$. The claim then follows immediately. \square

By the first two claims, for any PBE, it is without loss of generality to assume that the equilibrium Y has a finite number of decisions, and each decision $y \in Y$ is induced either in a degenerate lottery or in a lottery with another decision $y' \in Y$. Denote the induced lotteries as $\{y_i, y'_i; w_i\}$, $i = 1, \dots, n$, in increasing order. Since $u^A_{y\theta} > 0$, there is a partition $\{\theta_i\}_{i=0}^n$ of the state space $[0, 1]$, with $\theta_0 = 0$ and $\theta_n = 1$, such that each $\{y_i, y'_i; w_i\}$, $i = 1, \dots, n$, is induced in state $\theta \in (\theta_{i-1}, \theta_i]$. The necessary equilibrium conditions are A 's indifference conditions: for each partition threshold θ_i , $i = 1, \dots, n - 1$,

$$w_i u^A(y_i, \theta_i) + (1 - w_i) u^A(y'_i, \theta_i) = w_{i+1} u^A(y_{i+1}, \theta_i) + (1 - w_{i+1}) u^A(y'_{i+1}, \theta_i); \quad (15)$$

and P 's incentive condition for each lottery $\{y_i, y'_i; w_i\}$, $i = 1, \dots, n$,

$$\int_{\theta_{i-1}}^{\theta_i} (w_i u^P(y_i, \theta) + (1 - w_i) u^P(y'_i, \theta)) f(\theta) d\theta \geq \int_{\theta_{i-1}}^{\theta_i} u^P(\tilde{y}, \theta) f(\theta) d\theta \quad (16)$$

for $\tilde{y} = y_j, y'_j$ and all $j = 1, \dots, n$. If in addition,

$$\sum_{i=1}^n \int_{\theta_{i-1}}^{\theta_i} (w_i u^P(y_i, \theta) + (1 - w_i) u^P(y'_i, \theta)) f(\theta) d\theta \geq \int_0^1 u^P(y^P, \theta) f(\theta) d\theta$$

so that P 's expected payoff is greater than that from making an uninformed decision, then the above necessary conditions are also sufficient for PBE.

Third, we simplify the incentive conditions for P .

Claim 3. *In any PBE, P 's incentive conditions (16) are all slack except for $\tilde{y} = y'_{i-1}, y_i, y'_i, y_{i+1}$. Further, if $y_i = y'_i$ for all i , then P 's incentive conditions except for (3) are all slack.*

Proof. We first argue that adjacent incentive conditions are sufficient for all incentive conditions. Consider all P 's incentive conditions for $\{y_i, y'_i; w_i\}$. Since P prefers $\{y_i, y'_i; w_i\}$ to y'_{i-1} conditional on $(\theta_{i-1}, \theta_i]$, the most preferred decision conditional on the interval is higher than y'_{i-1} . By the strict concavity of u^P , P strictly prefers y'_{i-1} , and hence $\{y_i, y'_i; w_i\}$, to all decisions lower than y'_{i-1} conditional on $(\theta_{i-1}, \theta_i]$. By the same argument, P strictly prefers $\{y_i, y'_i; w_i\}$ to all decisions higher than y_{i+1} conditional on $(\theta_{i-1}, \theta_i]$.

Next, we argue that the adjacent upward incentive condition is satisfied if all induced lotteries are degenerate. To see this, note that in any partition equilibrium A 's indifference conditions (2) hold. Since $u_{y\beta} > 0$, A being indifferent between y_i and y_{i+1} in state θ_i implies that P strictly prefers y_i to y_{i+1} in the same state. By $u_{y\theta}^P > 0$, P then prefers y_i to y_{i+1} for all $\theta < \theta_i$, and in particular, for any $\theta \in (\theta_{i-1}, \theta_i]$. □

Fourth, we show that if an optimal PBE exists, then on the equilibrium path, P never randomizes over the set of decisions.

Claim 4. *For each PBE in which lotteries are induced, there exists another PBE in which only degenerate lotteries are induced and P obtains a higher expected payoff.*

Proof. Fix some PBE with induced lotteries $\{y_i, y'_i; w_i\}$, $i = 1, \dots, n$, in increasing order. We prove this claim in two steps. First, we show that there is another PBE in which P 's equilibrium choice of Y contains only y'_i , $i = 1, \dots, n$, and each decision y'_i is induced in a degenerate lottery. Each new threshold $\hat{\theta}_i$ is given by (15) where w_i and w_{i+1} are set to 0. Since $y_i \leq y'_i \leq y_{i+1} \leq y'_{i+1}$, the concavity of $u^A(\cdot, \hat{\theta}_i)$ and A 's indifference condition at $\hat{\theta}_i$ between y'_i and y'_{i+1} imply that $u^A(y_i, \hat{\theta}_i) \leq u^A(y'_i, \hat{\theta}_i)$ and $u^A(y_{i+1}, \hat{\theta}_i) \geq u^A(y'_{i+1}, \hat{\theta}_i)$. Then, since $u_{y\theta}^A > 0$, using the implicit function theorem applied to (15) gives that the solution to (15) decreases in w_i and w_{i+1} , which implies that each new threshold $\hat{\theta}_i$ is higher than the original threshold θ_i . The distribution function of the state θ conditional on $[\hat{\theta}_{i-1}, \hat{\theta}_i]$, given by $(F(\theta) - F(\hat{\theta}_{i-1})) / (F(\hat{\theta}_i) - F(\hat{\theta}_{i-1}))$, first order stochastically dominates the distribution function of the state θ conditional on $[\theta_{i-1}, \theta_i]$, because it is decreasing in $\hat{\theta}_{i-1}$ and $\hat{\theta}_i$. Since the difference $u^P(y'_i, \theta) - u^P(y'_{i-1}, \theta)$ is increasing in θ by the assumption of $u_{y\theta}^P > 0$, P prefers y'_i to y'_{i-1} conditional on $[\hat{\theta}_{i-1}, \hat{\theta}_i]$, because in the original PBE, P prefers y'_i to y'_{i-1} conditional on $[\theta_{i-1}, \theta_i]$. Since the downward incentive conditions are satisfied, Claim 3 implies that we indeed constructed a new PBE.

Second, we show that P obtains a higher expected payoff in the new PBE than in the original PBE by transforming the original PBE into the new PBE in such a way that P 's expected payoff continuously increases. We continuously decrease each lottery weight \tilde{w}_i from w_i to 0, one lottery at a time starting at $i = 1$ and ending at $i = n$, while increasing thresholds $\tilde{\theta}_i$ and $\tilde{\theta}_{i-1}$

to always satisfy A 's indifference conditions (15). Note that all other partition thresholds are unchanged when we continuously decrease \tilde{w}_i alone. The partial derivative of P 's expected payoff with respect to \tilde{w}_i is given by

$$\int_{\tilde{\theta}_{i-1}}^{\tilde{\theta}_i} (u^P(y_i, \theta) - u^P(y'_i, \theta)) f(\theta) d\theta,$$

which is negative because P prefers y'_i to y_i conditional on $(\tilde{\theta}_{i-1}, \tilde{\theta}_i]$ (recall that P is indifferent between y'_i and y_i conditional on $(\theta_{i-1}, \theta_i]$, so the argument from the previous paragraph applies). Thus, as we decrease \tilde{w}_i continuously, the direct effect on P 's expected payoff is positive. The partial derivative of P 's expected payoff with respect to $\tilde{\theta}_i$ is equal to $f(\tilde{\theta}_i)$ multiplied by

$$\begin{aligned} & \tilde{w}_i u^P(y_i, \tilde{\theta}_i) + (1 - \tilde{w}_i) u^P(y'_i, \tilde{\theta}_i) - (w_{i+1} u^P(y_{i+1}, \tilde{\theta}_i) + (1 - w_{i+1}) u^P(y'_{i+1}, \tilde{\theta}_i)) \\ &= \tilde{w}_i (w_{i+1} (u^P(y_i, \tilde{\theta}_i) - u^P(y_{i+1}, \tilde{\theta}_i)) + (1 - w_{i+1}) (u^P(y_i, \tilde{\theta}_i) - u^P(y'_{i+1}, \tilde{\theta}_i))) \\ & \quad + (1 - \tilde{w}_i) (w_{i+1} (u^P(y'_i, \tilde{\theta}_i) - u^P(y_{i+1}, \tilde{\theta}_i)) \\ & \quad + (1 - w_{i+1}) (u^P(y'_i, \tilde{\theta}_i) - u^P(y'_{i+1}, \tilde{\theta}_i))) \\ &> \tilde{w}_i (w_{i+1} (u^A(y_i, \tilde{\theta}_i) - u^A(y_{i+1}, \tilde{\theta}_i)) + (1 - w_{i+1}) (u^A(y_i, \tilde{\theta}_i) - u^A(y'_{i+1}, \tilde{\theta}_i))) \\ & \quad + (1 - \tilde{w}_i) (w_{i+1} (u^A(y'_i, \tilde{\theta}_i) - u^A(y_{i+1}, \tilde{\theta}_i)) \\ & \quad + (1 - w_{i+1}) (u^A(y'_i, \tilde{\theta}_i) - u^A(y'_{i+1}, \tilde{\theta}_i))) \\ &= \tilde{w}_i u^A(y_i, \tilde{\theta}_i) + (1 - \tilde{w}_i) u^A(y'_i, \tilde{\theta}_i) - (w_{i+1} u^A(y_{i+1}, \tilde{\theta}_i) + (1 - w_{i+1}) u^A(y'_{i+1}, \tilde{\theta}_i)) \\ &= 0, \end{aligned}$$

where the inequality follows from $u_{y\beta} > 0$, and the last equality follows from A 's indifference condition between $\{y_i, y'_i; \tilde{w}_i\}$ and $\{y_{i+1}, y'_{i+1}; w_{i+1}\}$ in state $\tilde{\theta}_i$. Because we replace one lottery at a time starting at $i = 1$, the lottery $\{y_{i-1}, y'_{i-1}; w_{i-1}\}$ must be degenerate. By construction $\tilde{w}_{i-1} = 0$ when we decrease \tilde{w}_i , so analogously the partial derivative of P 's expected payoff with respect to $\tilde{\theta}_{i-1}$ is equal to $f(\tilde{\theta}_{i-1})$ multiplied by

$$\begin{aligned} & u^P(y'_{i-1}, \tilde{\theta}_{i-1}) - (\tilde{w}_i u^P(y_i, \tilde{\theta}_{i-1}) + (1 - \tilde{w}_i) u^P(y'_i, \tilde{\theta}_{i-1})) \\ &= \tilde{w}_i (u^P(y'_{i-1}, \tilde{\theta}_{i-1}) - u^P(y_i, \tilde{\theta}_{i-1})) + (1 - \tilde{w}_i) (u^P(y'_{i-1}, \tilde{\theta}_{i-1}) - u^P(y'_i, \tilde{\theta}_{i-1})) \\ &> \tilde{w}_i (u^A(y'_{i-1}, \tilde{\theta}_{i-1}) - u^A(y_i, \tilde{\theta}_{i-1})) + (1 - \tilde{w}_i) (u^A(y'_{i-1}, \tilde{\theta}_{i-1}) - u^A(y'_i, \tilde{\theta}_{i-1})) \\ &= u^A(y'_{i-1}, \tilde{\theta}_{i-1}) - (\tilde{w}_i u^A(y_i, \tilde{\theta}_{i-1}) + (1 - \tilde{w}_i) u^A(y'_i, \tilde{\theta}_{i-1})) \\ &= 0. \end{aligned}$$

Thus, as we decrease \tilde{w}_i continuously, the indirect effects of increased $\tilde{\theta}_{i-1}$ and $\tilde{\theta}_i$ on P 's expected payoff are also positive. Finally, if we suppose that at least one induced lottery in the original PBE is non-degenerate, then the direct effect will be strictly positive, which implies that P 's expected payoff is strictly higher in the new PBE. \square

Fifth and last, we show that an optimal PBE exists. Combining the above claims, we have already established that an optimal PBE, if one exists, is a solution to the constrained maximization problem where the objective is P 's expected payoff and the feasible choices are all partition equilibria with a finite number of elements.

Claim 5. *An optimal PBE exists.*

Proof. Let us consider a relaxed problem in which strict inequalities of the partition condition (1) are replaced with weak inequalities. By Claim 2, the number of induced decisions n is uniformly bounded. Thus, the relaxed problem is a constrained maximization problem with finitely many variables. We claim that there exists \bar{y} such that we can impose $|y_i| \leq \bar{y}$ for all $i = 1, \dots, n$ without affecting the maximization problem. This follows because there can be at most one induced decision above $y^P(1)$ and one induced decision below $y^P(0)$, and further, there is at least one induced decision in $[y^P(0), y^P(1)]$. To see this, define $g_1(y_2, \theta_1)$ as y_1 that solves $u^A(y_1, \theta_1) = u^A(y_2, \theta_1)$ and $g_n(y_{n-1}, \theta_{n-1})$ as y_n that solves $u^A(y_{n-1}, \theta_{n-1}) = u^A(y_n, \theta_{n-1})$. The functions g_1 and g_n are decreasing in the first argument and increasing in the second argument which implies that $y_1 \geq g_1(y^P(1), 0)$ and $y_n \leq g_n(y^P(0), 1)$. Therefore, $|y_i| \leq \max\{|g_1(y^P(1), 0)|, |g_n(y^P(0), 1)|\} \equiv \bar{y}$ for all i .

The constraints, $|y_i| \leq \bar{y}$, together with a finite number of constraints (2) and (3) determine the compact set for variables $\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n$ over which the continuous function $\sum_{i=1}^n \int_{\theta_{i-1}}^{\theta_i} u^P(y_i, \theta) f(\theta) d\theta$ is maximized. Clearly, there exists a solution to this relaxed problem. Finally, we need to show that the value of the relaxed problem is achievable with strict inequalities (1), which will prove the existence of an optimal PBE. If some of θ_i or y_i coincide, we can take the maximal subset $\{\tilde{\theta}_i\}_{i=0}^{\tilde{n}} \subset \{\theta_i\}_{i=0}^n$ and a corresponding subset of induced decisions $\{\tilde{y}_i\}_{i=1}^{\tilde{n}} \subset \{y_i\}_{i=1}^n$ such that all $\tilde{\theta}_i$ and \tilde{y}_i are distinct. These $\{\tilde{\theta}_i\}_{i=0}^{\tilde{n}}$ and $\{\tilde{y}_i\}_{i=1}^{\tilde{n}}$ will satisfy (2)–(3) and strict inequalities of the partition condition (1). Moreover, this modification does not change P 's expected payoff. \square

This concludes the proof of Theorem 1. \square

Proof of Proposition 1. Consider a CS equilibrium $(\{\theta_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ with $n \geq 2$. We prove that for any sufficiently small ε , there exists a PBE with P 's equilibrium choice $\{\tilde{y}_i\}_{i=1}^n \equiv \{y_1, \dots, y_j + \varepsilon, \dots, y_n\}$, and the corresponding partition $\{\tilde{\theta}_i\}_{i=0}^n \equiv \{\theta_0, \dots, \theta_{j-1}(\varepsilon), \theta_j(\varepsilon), \dots, \theta_n\}$. Moreover, we prove that P 's expected payoff in this PBE is strictly higher than in the CS equilibrium. By the implicit function theorem applied to A 's indifference condition (2), $\theta_{j-1}(\varepsilon)$ and $\theta_j(\varepsilon)$ are continuous functions in a neighborhood of $\varepsilon = 0$ with

$$\left. \frac{d\theta_{j-1}(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \frac{-u_y^A(y_j, \theta_{j-1})}{u_\theta^A(y_j, \theta_{j-1}) - u_\theta^A(y_{j-1}, \theta_{j-1})} \quad \text{for } j \neq 1,$$

$$\left. \frac{d\theta_j(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \frac{u_y^A(y_j, \theta_j)}{u_\theta^A(y_{j+1}, \theta_j) - u_\theta^A(y_j, \theta_j)} \quad \text{for } j \neq n.$$

For $j = 1$ and $j = n$, we have $\left. \frac{d\theta_0(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \left. \frac{d\theta_n(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = 0$ because $\theta_0(\varepsilon) = 1 - \theta_n(\varepsilon) = 0$.

In the CS equilibrium, $\int_{\theta_{i-1}}^{\theta_i} u^P(y_i, \theta) f(\theta) d\theta > \int_{\theta_{i-1}}^{\theta_i} u^P(y_{i-1}, \theta) f(\theta) d\theta$ for all i because y_i is ex post optimal in that $y_i = \arg \max_{y \in \mathbb{R}} \int_{\theta_{i-1}}^{\theta_i} u^P(y, \theta) f(\theta) d\theta$. Therefore, incentive conditions (3), $\int_{\tilde{\theta}_{i-1}}^{\tilde{\theta}_i} u^P(\tilde{y}_i, \theta) f(\theta) d\theta > \int_{\tilde{\theta}_{i-1}}^{\tilde{\theta}_i} u^P(\tilde{y}_{i-1}, \theta) f(\theta) d\theta$, hold for all i because functions $u^P(y, \theta), \theta_{j-1}(\varepsilon), \theta_j(\varepsilon)$ are continuous, and ε is sufficiently small.

The derivative of P 's expected payoff with respect to ε at $\varepsilon = 0$ is given by

$$\begin{aligned} & (u^P(y_{j-1}, \theta_{j-1}) - u^P(y_j, \theta_{j-1})) f(\theta_{j-1}) \frac{d\theta_{j-1}(\varepsilon)}{d\varepsilon} \Big|_{\varepsilon=0} \\ & + (u^P(y_j, \theta_j) - u^P(y_{j+1}, \theta_j)) f(\theta_j) \frac{d\theta_j(\varepsilon)}{d\varepsilon} \Big|_{\varepsilon=0} + \int_{\theta_{j-1}}^{\theta_j} u_y^P(y_j, \theta) f(\theta) d\theta. \end{aligned}$$

The last term in the above expression is 0 because y_j is ex post optimal. The second term is positive for $j \neq n$ because $u^P(y_j, \theta_j) - u^P(y_{j+1}, \theta_j) > 0$ holds by (2) and $u_{y\beta} > 0$; and $\frac{d\theta_j(\varepsilon)}{d\varepsilon} \Big|_{\varepsilon=0} > 0$ holds by $u_y^A(y_j, \theta_j) > 0$ and $u_\theta^A(y_{j+1}, \theta_j) - u_\theta^A(y_j, \theta_j) > 0$. More specifically, $u_y^A(y_j, \theta_j) > 0$ holds by (2) and $u_{yy}^A > 0$, whereas $u_\theta^A(y_{j+1}, \theta_j) - u_\theta^A(y_j, \theta_j) > 0$ holds by $u_{y\theta}^A > 0$. Analogously, the first term is positive for $j \neq 1$. To sum up, the above expression is positive for all j . \square

Proof of Proposition 2. For the ‘only if’ part, recall from Proposition 1 that any optimal PBE is a partition equilibrium. The claim then follows immediately, because any partition equilibrium can be implemented by delegation under full commitment.

For the ‘if’ part, we use a result obtained by Alonso and Matouschek [3]. They show that if delegation is valuable then there exists $\theta^* \in (0, 1)$ such that $S(\theta^*) < 0 < T(\theta^*)$, where $T(\theta^*) \equiv \int_0^{\theta^*} (y^A(\theta^*) - y^P(\theta)) f(\theta) d\theta$ and $S(\theta^*) \equiv \int_{\theta^*}^1 (y^A(\theta^*) - y^P(\theta)) f(\theta) d\theta$. The decision rule $\tilde{y}(\theta)$ given by

$$\tilde{y}(\theta) = \begin{cases} y^A(\theta^*) + T(\theta^*) - S(\theta^*) & \text{if } \theta > \theta^*, \\ y^A(\theta^*) + S(\theta^*) - T(\theta^*) & \text{if } \theta \leq \theta^*, \end{cases}$$

satisfies A 's indifference condition (2) at θ^* because A 's payoff function is symmetric around $y^A(\theta^*)$. Moreover, the difference in P 's expected payoff under $\tilde{y}(\theta)$ and the uninformative decision y^P is equal to $-4T(\theta^*)S(\theta^*)$, and is positive. Finally, conditional on $\theta > \theta^*$, P prefers $y^A(\theta^*) + T(\theta^*) - S(\theta^*)$ to $y^A(\theta^*) + S(\theta^*) - T(\theta^*)$, because the difference in her expected payoff under the two decisions is $-4(T(\theta^*) - S(\theta^*))S(\theta^*)$, which is positive. Thus, $\tilde{y}(\theta)$ can be supported as a PBE. \square

Proof of Lemma 1. Suppose that in the optimal PBE, $y_i \leq y_i^*$ for some $i = 2, \dots, n$. We first show by contradiction that P 's $(i + 1)$ -th incentive condition binds. Suppose not. Consider marginally increasing y_i , keeping all other decisions unchanged. We know from the proof of Proposition 1 that θ_{i-1} and θ_i both increase, with all other thresholds unaffected. By the concavity of $u^P(\cdot, \theta)$, P 's i -th incentive condition is slack and hence unaffected because $y_{i-1} < y_i \leq y_i^*$. Similarly, her $(i - 1)$ -th incentive condition remains satisfied because θ_{i-1} increases as y_i increases. The proof of Proposition 1 has already established that P 's expected payoff is increased when either θ_i or θ_{i-1} increases. Her expected payoff is further increased because y_i moves closer to her ex post optimal decision y_i^* on $(\theta_i, \theta_{i+1}]$. A contradiction.

The above result immediately implies that $y_n > y_n^*$. For each $i = 2, \dots, n - 2$, note that P 's $(i + 1)$ -th incentive condition binding implies that $y_i < y_{i+1}^* < y_{i+1}$, so we can rewrite it as $y_{i+1} + y_i = \theta_{i+1} + \theta_i$. Using (4) for θ_i and θ_{i+1} , we then have

$$y_{i+2} - y_i = 4b.$$

However, if $y_i \leq y_i^*$, from A 's indifference conditions (4) we would have

$$y_{i+1} - y_i \geq 4b + (y_i - y_{i-1}) > 4b,$$

which contradicts P 's binding $(i + 1)$ -th incentive condition. This establishes the lemma for $i = 2, \dots, n - 2$.

Next, we show that $y_{n-1} > y_{n-1}^*$. Suppose not. Consider marginally increasing y_{n-1} and decreasing y_n in such a way that θ_{n-1} remains unchanged. Then P 's n -th incentive condition is unaffected. However, this increases P 's expected payoff because by assumption $y_{n-1} \leq y_{n-1}^*$, $y_n > y_n^*$, and because θ_{n-2} increases as a result of increasing y_{n-1} . A contradiction.

Finally, it can be verified using the proof of Proposition 5 that $y_1 > y_1^*$ at the optimal PBE. Note that this result is not needed for rewriting P 's incentive conditions (5). \square

Proof of Proposition 3. Adding up condition (9) for $i = 2, \dots, n - 1$, we have

$$y_n + y_{n-1} - (y_1 + y_2) \geq 4b(n - 2).$$

Also, using conditions (8) and (10), we have $2b(n - 1) < 1$, or $n < 1/(2b) + 1$.

For the converse, let n be a natural number strictly less than $1/(2b) + 1$. By definition of N , $1/(2N) \leq b < 1/(2(N - 1))$. Note that $N \geq 2$ because $b < \frac{1}{2}$ by assumption.

If $n = 1$, then there exists a babbling equilibrium with the induced decision $\bar{y}_1^1 = \frac{1}{2}$.

If $n = 2$, consider the 'full-commitment' problem of choosing two decisions y_1 and y_2 with $0 \leq y_1 \leq y_2 \leq 1$ that maximizes P 's expected payoff

$$\bar{U}^2 = - \int_0^{\theta_1} l(|y_1 - \theta|) d\theta - \int_{\theta_1}^1 l(|y_2 - \theta|) d\theta,$$

subject only to A 's indifference condition $\theta_1 = \frac{1}{2}(y_1 + y_2) - b$. The first order conditions with respect to y_1 and y_2 are

$$\frac{\partial \bar{U}^2}{\partial y_1} = \frac{1}{2} [l(y_2 - \theta_1) + l(|y_1 - \theta_1|)] - l(y_1) = 0;$$

$$\frac{\partial \bar{U}^2}{\partial y_2} = -\frac{1}{2} [l(y_2 - \theta_1) + l(|y_1 - \theta_1|)] + l(1 - y_2) = 0.$$

The above conditions imply that $y_1 = 1 - y_2$. It is straightforward to verify that the second order condition is satisfied. The above first order conditions become identical, and we can rewrite it as

$$2l(1/2 - \delta) = l(b + \delta) + l(|b - \delta|),$$

where $\delta = \frac{1}{2}(y_2 - y_1)$. By the convexity of l , the right-hand side is increasing in δ , so there is a unique $\underline{\delta}^2 \in (0, 1/2)$ satisfying the above condition. The solution to the full-commitment problem is then given by $\bar{y}_1^2 = \frac{1}{2} - \underline{\delta}^2$ and $\bar{y}_2^2 = \frac{1}{2} + \underline{\delta}^2$, with $\theta_1 = \frac{1}{2} - b$. Note that $b < \frac{1}{2}$ implies that $\theta_1 > 0$. The incentive condition of P is satisfied at this solution, because $\bar{y}_1^2 + \bar{y}_2^2 = 1 < 1 + \theta_1$. We can thus take the solution to the full-commitment problem to be a limited authority PBE. Since the solution has $\bar{y}_2^2 - \bar{y}_1^2 = 2\underline{\delta}^2 > 0$, it gives P a strictly higher payoff than making the uninformed decision of $\frac{1}{2}$.

Finally, for any $n \geq 3$, consider the set of n decisions \bar{Y}^n given by $\bar{y}_i^n = \frac{1}{2} - b(n + 1 - 2i)$ for each $i = 1, \dots, n$. Then, by A 's indifference condition, $\theta_i = \bar{y}_i^n$ for each $i = 1, \dots, n - 1$. It

is straightforward to verify that conditions (8), (9) and (10) are all satisfied. The expected payoff for P under this construction is given by

$$\bar{U}^n = -2 \int_0^{\bar{y}_1^n} l(\bar{y}_1^n - \theta) d\theta - (n - 1) \int_{\bar{y}_1^n}^{\bar{y}_2^n} l(\bar{y}_2^n - \theta) d\theta. \tag{17}$$

Since $\bar{y}_2^n = \bar{y}_1^{n-2}$ by construction, it is straightforward to show that $\bar{U}^n > \bar{U}^{n-2}$: the difference is given by

$$\bar{U}^n - \bar{U}^{n-2} = 2 \int_0^{\bar{y}_1^n} [l(\bar{y}_2^n - \theta) - l(\bar{y}_1^n - \theta)] d\theta,$$

which is positive.

The above argument immediately implies that the expected payoff to P under the above construction with n decisions is greater than making the uninformed decision of $\frac{1}{2}$ for all $n \geq 3$ and odd. To complete the proof of the proposition, we only need to show that the above construction \bar{Y}^4 for $n = 4$ is better than making the uninformed decision of $\frac{1}{2}$ for P . (This step is necessary because the payoff formula (17) does not apply to the case of $n = 2$.) It is straightforward to show that

$$\begin{aligned} & \int_0^1 l(1/2 - \theta) d\theta - \sum_{i=1}^4 \int_{\theta_{i-1}}^{\theta_i} l(\bar{y}_i - \theta) d\theta \\ & > \left[\int_{\bar{y}_3}^{\bar{y}_3+b} l(\theta - 1/2) d\theta - \int_{\bar{y}_2}^{1/2} l(\bar{y}_3 - \theta) d\theta \right] + \left[\int_{1/2}^{\bar{y}_3} (l(\theta - 1/2) - l(\bar{y}_3 - \theta)) d\theta \right] \\ & \quad + \left[\int_{\bar{y}_2}^{1/2} l(1/2 - \theta) d\theta - \int_{\bar{y}_3}^{\bar{y}_3+b} l(\theta - \bar{y}_3) d\theta \right] \\ & = 0, \end{aligned}$$

where the first line follows because $\frac{1}{2}$ is a more extreme decision than the corresponding decisions \bar{y}_1 , \bar{y}_2 and \bar{y}_4 outside the interval $[\bar{y}_2, \bar{y}_3 + b]$, and the second line follows because each term in the bracket is zero. (One integral that appears in $\bar{U}(4)$ is $\int_{\bar{y}_3}^{\bar{y}_4} l(\bar{y}_4 - \theta) d\theta$, which is equal to $\int_{\bar{y}_3}^{\bar{y}_4} l(\theta - \bar{y}_3) d\theta$ by a change of variables; the first part of the latter integral, from \bar{y}_3 to $\bar{y}_3 + b$, is the integral that appears in the last bracket.) \square

Proof of Lemma 2. For $n = 1$, it is trivially true that $\underline{y}_1^1 = \frac{1}{2}$. For $n = 2$, \underline{Y}^2 is derived in the proof of Proposition 3. Fix any $n \geq 3$. Arguments similar to the proof of Claim 5 in Theorem 1 can show that \underline{Y}^n exists. We will guess and verify later that conditions (7), (8), and (11) are not binding at \underline{Y}^n . Denote $\delta_i = \frac{1}{2}(y_{i+1} - y_i)$.

The derivative of P 's expected payoff with respect to y_i is

$$\begin{aligned} \frac{\partial U^n}{\partial y_i} &= \frac{1}{2} [l(y_{i+1} - \theta_i) - l(y_i - \theta_{i-1})] - \frac{1}{2} [l(|y_{i-1} - \theta_{i-1}|) - l(|y_i - \theta_i|)] \\ &= \frac{1}{2} [l(\delta_i + b) + l(|\delta_i - b|)] - \frac{1}{2} [l(\delta_{i-1} + b) + l(|\delta_{i-1} - b|)] \end{aligned} \tag{18}$$

where we have used A 's indifference conditions. Since l is convex, the function $l(b + \delta) + l(|b - \delta|)$ is increasing in δ regardless of $\delta \geq b$ or $\delta < b$. As a result, $\partial U^n / \partial y_i$ has the same sign as $\delta_i - \delta_{i-1}$, implying that $\delta_i = \delta_{i-1}$ at the optimal decisions \underline{Y}^n .

Thus, the optimal decisions satisfy $y_i - y_{i-1} = 2\delta > 0$ for all $i = 2, \dots, n$, so the optimum is interior. From A 's indifference conditions, we have $\theta_i - \theta_{i-1} = 2\delta$ for all $i = 2, \dots, n - 1$. Since the state is uniformly distributed, we can rewrite P 's expected payoff as

$$U^n = - \int_0^{\theta_1} l(|y_1 - \theta|) d\theta - (n - 2) \int_{\theta_1}^{\theta_2} l(|y_2 - \theta|) d\theta - \int_{\theta_{n-1}}^1 l(|y_n - \theta|) d\theta. \tag{19}$$

Now we differentiate (19) with respect to y_1 and y_n . From the two first order conditions we immediately have $l(y_1) = l(1 - y_n)$, and thus $y_1 = 1 - y_n = \frac{1}{2} - (n - 1)\delta$. The two conditions then become identical, and are given by

$$\frac{\partial U^n}{\partial y_1} = \frac{1}{2} [l(b + \delta) + l(|b - \delta|)] - l(1/2 - (n - 1)\delta) = 0. \tag{20}$$

We claim that there exists a unique $\underline{\delta}^n \in (0, 1/(2(n - 1)))$ that solves (20). Since l is convex, $\partial U^n / \partial y_1$ is strictly increasing in δ regardless of whether $\delta \geq b$ or $\delta < b$, so there can be at most one value of δ that solves (20). At $\delta = 0$, we have $\partial U^n / \partial y_1 < 0$ because $b < \frac{1}{2}$; and at $\delta = 1/(2(n - 1))$, we have $\partial U^n / \partial y_1 > 0$. Thus, a unique $\underline{\delta}^n \in (0, 1/(2(n - 1)))$ exists that solves (20). Condition (20) is a necessary condition for $\underline{\delta}^n$ to be optimal. Since there exists a unique solution $\underline{\delta}^n$, (20) is also sufficient.

To complete the derivation of \underline{Y}^n , we verify that the dropped constraints are satisfied. Condition (7) is satisfied because $\underline{\delta}^n > 0$. Condition (8) is equivalent to $\underline{y}_1^n > b - \underline{\delta}^n$. This is satisfied if $\underline{\delta}^n \geq b$ since $\underline{\delta}^n < 1/(2(n - 1))$ implies that $\underline{y}_1^n > 0$; it also holds if $\underline{\delta}^n < b$, because in that case it is implied by (20). Finally, condition (11) is satisfied because $\underline{y}_n^n = 1 - \underline{y}_1^n$, $\underline{y}_{n-1}^n < \underline{y}_n^n$, and $\underline{y}_1^n > 0$. \square

Proof of Lemma 3. The lemma follows immediately from the three claims below.

Claim 6. P 's incentive conditions (9) bind at Y^n for $b \in [\underline{b}^n, \bar{b}^n)$.

Proof. To get a contradiction, without loss of generality suppose that there exists i , $i = 2, \dots, n - 2$, such that $y_{i+2} - y_i = 4b$ and $y_{i+1} - y_{i-1} > 4b$ at Y^n . As in the proof of Lemma 2, denote $\delta_i = \frac{1}{2}(y_{i+1} - y_i)$. Below we derive a contradiction by changing some decision y_k slightly in such a way that all conditions (7)–(10) are still satisfied. We show that this change increases P 's expected payoff by condition (18), using the fact that the effect of the change has the same sign as $\delta_k - \delta_{k-1}$.

To begin, note that each y_k appears in at most two incentive conditions of P , the $(k - 1)$ -th and the $(k + 1)$ -th of (9). If $\delta_i \geq \delta_{i-1}$, which implies $\delta_{i+1} < \delta_i$ because by assumption $\delta_{i+1} + \delta_i = 2b < \delta_i + \delta_{i-1}$, then we can decrease y_{i+1} slightly. If $\delta_i < \delta_{i-1}$, then there are two cases. If $i = 2$ or $y_i - y_{i-2} > 4b$, then we can decrease y_i slightly; otherwise, if $y_i - y_{i-2} = 4b$, which implies that $\delta_{i-1} > \delta_{i-2}$ because by assumption $\delta_{i-1} > \delta_i$ and $\delta_{i-1} + \delta_i > 2b$, we can increase y_{i-1} slightly. \square

Claim 7. For any $n \geq 3$ and odd, and $b \in (\underline{b}^n, \bar{b}^n)$, Y^n is given by $y_i^n = \frac{1}{2} - (n + 1 - 2i)b$ for all $i = 1, \dots, n$.

Proof. By Claim 6, $y_{i+2} - y_i = 4b$ for all $i = 1, \dots, n - 2$. Then, $y_i = y_1 + 2(i - 1)b$ for i odd, and $y_i = y_2 + 2(i - 2)b$ for i even. Further, $\theta_i - \theta_{i-1} = 2b$ for all $i = 2, \dots, n - 1$. Using the assumption that the state is uniformly distributed, we can rewrite P 's expected payoff (6) as

$$\begin{aligned}
 U^n = & - \int_0^{\theta_1} l(|y_1 - \theta|) d\theta - \frac{n-1}{2} \int_{\theta_1}^{\theta_2} l(|y_2 - \theta|) d\theta \\
 & - \frac{n-3}{2} \int_{\theta_1}^{\theta_2} l(|y_1 + 2b - \theta|) d\theta - \int_{\theta_{n-1}}^1 l(|y_n - \theta|) d\theta.
 \end{aligned} \tag{21}$$

The first order conditions with respect to y_1 and y_2 are

$$\begin{aligned}
 \frac{\partial U^n}{\partial y_1} &= -l(y_1) + l(1 - y_n) - \frac{n-1}{4} [l(3b - \delta_1) - l(b + \delta_1)] = 0; \\
 \frac{\partial U^n}{\partial y_2} &= \frac{n-1}{4} [l(3b - \delta_1) - l(b + \delta_1)] = 0
 \end{aligned}$$

where $\delta_1 = \frac{1}{2}(y_2 - y_1)$. It follows immediately that $y_1 = 1 - y_n$ and $\delta_1 = b$. Furthermore, it is straightforward to verify that the second order condition with respect to y_1 and y_2 are satisfied at $y_1 = 1 - y_n$ and $\delta_1 = b$. Finally, (7) is satisfied because $\delta_1 \in (0, 2b)$, and (8) and (10) are equivalent to $2(n - 1)b < 1$, and thus are satisfied because $b < \bar{b}^n$. \square

Claim 8. For any $n \geq 2$ and even, and $b \in (\underline{b}^n, \bar{b}^n)$, Y^n is given by $y_i^n = \frac{1}{2} - (n - 2i)b - \delta_1$ for odd i , and $y_i^n = \frac{1}{2} - (n + 2 - 2i)b + \delta_1$ for even i , where $\delta_1 < b$ is uniquely determined by (14).

Proof. Similar to Claim 7, we can rewrite P 's expected payoff (6) as

$$\begin{aligned}
 U^n = & - \int_0^{\theta_1} l(|y_1 - \theta|) d\theta - \frac{n-2}{2} \int_{\theta_1}^{\theta_2} l(|y_2 - \theta|) d\theta \\
 & - \frac{n-2}{2} \int_{\theta_1}^{\theta_2} l(|y_1 + 2b - \theta|) d\theta - \int_{\theta_{n-1}}^1 l(|y_n - \theta|) d\theta.
 \end{aligned} \tag{22}$$

The first order conditions with respect to y_1 and y_2 are

$$\begin{aligned}
 -l(y_1) + \frac{1}{2} [l(b + \delta_1) + l(|b - \delta_1|)] - \frac{n-2}{4} [l(3b - \delta_1) - l(b + \delta_1)] &= 0; \\
 l(1 - y_n) - \frac{1}{2} [l(b + \delta_1) + l(|b - \delta_1|)] + \frac{n-2}{4} [l(3b - \delta_1) - l(b + \delta_1)] &= 0
 \end{aligned}$$

where $\delta_1 = \frac{1}{2}(y_2 - y_1)$. It follows immediately that $y_1 = 1 - y_n$ and δ_1 satisfies (14). Furthermore, we can easily verify that the second order condition with respect to y_1 and y_2 are satisfied. Finally, (7) is satisfied because $\delta_1 \in (0, 2b)$, and (8) and (10) are equivalent to $2(n - 1)b < 1$, and thus are satisfied because $b < \bar{b}^n$.

To see that there is a unique $\delta_1^n \in (0, b)$ that satisfies (14), note that since $b > \underline{b}^n$, the left-hand side of (14) is strictly smaller than the right-hand side at $\delta_1 = b$. As δ_1 decreases, the left-hand

side of (14) increases while the right-hand side decreases because l is convex. At $\delta_1 = 0$, the left-hand side of (14) is strictly greater than the right-hand side because $b < \bar{b}^n$. It follows that there exists a unique $\delta_1^n \in (0, b)$ that satisfies condition (14). \square

This concludes the proof of Lemma 3. \square

Proof of Proposition 5. First we establish a series of claims.

Claim 9. For any l that satisfies Assumption 2, and for each $n \geq 3$, $dU^n(b)/db > dU^{n+1}(b)/db$ for all $b \in (\underline{b}^n, \bar{b}^{n+1})$.

Proof. First, we compute $dU^n(b)/db$. For n odd, from (21), using the Envelope Theorem we have

$$\frac{dU^n(b)}{db} = -2(n-1)[l(2b) - l(y_1^n)]. \tag{23}$$

For n even, from (22) and the Envelope Theorem, we have

$$\frac{dU^n(b)}{db} = -(n-2)l(3b - \delta_1^n) - nl(b + \delta_1^n) + 2(n-1)l(y_1^n). \tag{24}$$

Note that (24) becomes (23) when δ_1^n is set to b .

Now, suppose that n is odd. Using the first order condition (14) for $n + 1$ we can rewrite (24) for $n + 1$ as

$$\frac{dU^{n+1}(b)}{db} = -(n-1)[l(3b - \delta_1^{n+1}) + l(b + \delta_1^{n+1})] - 2l(b + \delta_1^{n+1}) + 2nl(y_1^{n+1}).$$

Since l is convex, we have

$$l(3b - \delta_1^{n+1}) + l(b + \delta_1^{n+1}) > 2l(2b).$$

Further, (14) implies that $l(b + \delta_1^{n+1}) > l(y_1^{n+1})$. Thus,

$$\frac{dU^{n+1}(b)}{db} < -2(n-1)[l(2b) - l(y_1^{n+1})].$$

The claim then follows from $y_1^{n+1} < \frac{1}{2} - (n-1)b$ and (23).

Finally, suppose that n is even. Using the first order condition (14), we can rewrite (24) as

$$\frac{dU^n(b)}{db} = -(n+1)[l(b + \delta_1^n) + l(b - \delta_1^n) - 2l(y_1^n)] - (n-1)[l(b + \delta_1^n) - l(b - \delta_1^n)].$$

Since $\delta_1^n < b$, from (14) we have

$$l(b + \delta_1^n) + l(b - \delta_1^n) - 2l(y_1^n) < l(2b) - 2l(1/2 - (n-1)b).$$

Thus

$$\frac{dU^n(b)}{db} > -(n+1)[l(2b) - 2l(1/2 - (n-1)b)] - (n-1)[l(b + \delta_1^n) - l(b - \delta_1^n)].$$

The claim then follows from $\delta_1^n < b$ and (23) for $n + 1$. \square

Claim 10. Suppose that $l(z) = (|z|)^q$ with $q > 1$. For each $n \geq 3$, if $\underline{U}^{n-1}(b) = U^n(b)$ at some $b \in (\underline{b}^n, \underline{b}^{n-1})$, then $d\underline{U}^{n-1}(b)/db > dU^n(b)/db$.

Proof. First, we show that $\underline{y}_1^{n-1} > y_1^n$ for all $b \in (\underline{b}^n, \underline{b}^{n-1})$. From Lemma 2, we have that \underline{y}_1^n is decreasing in n for fixed b , regardless of whether $\underline{\delta}^n \geq b$ or $\underline{\delta}^n < b$. Thus, for n odd,

$$\underline{y}_1^{n-1} > \underline{y}_1^n = \frac{1}{2} - (n-1)\underline{\delta}^n > \frac{1}{2} - (n-1)b = y_1^n,$$

where the second inequality follows because $\underline{\delta}^n < b$ for $b > \underline{b}^n$. For n even, we can also show that $\underline{y}_1^n > y_1^n$ so that $\underline{y}_1^{n-1} > y_1^n$. To see this, suppose instead that $y_1^n \geq \underline{y}_1^n$. This implies that $\delta_1^n < \underline{\delta}^n$ because $\delta_1^n < b$. Then,

$$l(b + \delta_1^n) + l(b - \delta_1^n) - 2l(y_1^n) < l(b + \underline{\delta}^n) + l(b - \underline{\delta}^n) - 2l(\underline{y}_1^n) = 0$$

from the first order condition (20) in Lemma 2, which contradicts the first order condition (14) in Lemma 3.

Second, using the assumption of $l(z) = (|z|)^q$, we show that for all n , odd or even, and under both Y^n and \underline{Y}^n , P 's expected payoff $U(b)$ satisfies

$$\frac{dU(b)}{db} = \frac{1}{b} [(q+1)U(b) + (y_1)^q], \tag{25}$$

where $y_1 = y_1^n$ and $y_1 = \underline{y}_1^n$ respectively, which then establishes the claim. Under Y^n , the above identity linking $dU^n(b)/db$ to $U^n(b)$ immediately follows from (23) for n odd and (24) for n even. Under \underline{Y}^n , we need to compute $d\underline{U}^n/db$. For $b < \underline{b}^n$, from condition (13) we have $\underline{\delta}^n$ given in Lemma 2 is strictly greater than b . From (19), using the Envelope Theorem we have

$$\frac{d\underline{U}^n(b)}{db} = -(n-1)[l(\underline{\delta}^n + b) - l(\underline{\delta}^n - b)].$$

It is easy to verify (25) using $l(z) = (|z|)^q$. \square

Claim 11. Suppose that $l(z) = (|z|)^q$ with $q > 1$. Then, $\underline{U}^{n-1}(\underline{b}^{n-1}) > U^n(\underline{b}^{n-1})$ for any n sufficiently large.

Proof. From Lemma 2 and condition (13), at \underline{b}^{n-1} all $n-1$ decisions in \underline{Y}^{n-1} are $2\underline{b}^{n-1}$ apart, that is, $\underline{\delta}$ given in Lemma 2 is equal to \underline{b}^{n-1} . Further, from A 's indifference conditions we have $\theta_i = \underline{y}_i^{n-1}$ for all $i = 1, \dots, n-2$. We distinguish two cases.

First, suppose that n is odd. By Lemma 3, all n decisions in Y^n are also $2\underline{b}^{n-1}$ apart, with $\theta_i = y_i^n$ for all $i = 1, \dots, n-1$. Note that $\underline{y}_1^{n-1} - y_1^n = \underline{b}^{n-1}$. Using (19) and (21), we can show that the difference between P 's expected payoff \underline{U}^{n-1} under \underline{Y}^{n-1} and U^n under Y^n at \underline{b}^{n-1} is given by

$$\begin{aligned} \underline{U}^{n-1}(\underline{b}^{n-1}) - U^n(\underline{b}^{n-1}) &= \int_0^{2\underline{b}^{n-1}} l(\theta) d\theta - 2 \int_{\underline{y}_1^{n-1} - \underline{b}^{n-1}}^{\underline{y}_1^{n-1}} l(\theta) d\theta \\ &> \int_0^{2\underline{b}^{n-1}} l(\theta) d\theta - 2 \int_0^{\underline{y}_1^{n-1}} l(\theta) d\theta. \end{aligned}$$

Using the assumption of $l(z) = (|z|)^q$ with $q > 1$, we can explicitly compute \underline{y}_1^{n-1} in terms of \underline{b}^{n-1} from (13), which does not depend on n directly, and use it to show that the last expression above is strictly positive for any n . Note that this part of the proof holds for all $n \geq 3$ odd.

Second, suppose that n is even. In this case, under Y^n we have $\theta_{i+1} - \theta_i = 2\underline{b}^{n-1}$ and $\underline{y}_1^{n-1} - y_1^n = \delta_1^n$ at \underline{b}^{n-1} . As in the case of odd n , using (19) and (22) we can show that the difference between P 's expected payoff \underline{U}^{n-1} under \underline{Y}^{n-1} , and U^n under Y^n at \underline{b}^{n-1} is

$$\begin{aligned} & \underline{U}^{n-1}(\underline{b}^{n-1}) - U^n(\underline{b}^{n-1}) \\ &= \int_{\underline{b}^{n-1} - \delta_1^n}^{\underline{b}^{n-1} + \delta_1^n} l(\theta) d\theta - 2 \int_{y_1^n}^{y_1^n + \delta_1^n} l(\theta) d\theta + \frac{n-2}{2} \left[\int_{2\underline{b}^{n-1}}^{3\underline{b}^{n-1} - \delta_1^n} l(\theta) d\theta - \int_{\underline{b}^{n-1} + \delta_1^n}^{2\underline{b}^{n-1}} l(\theta) d\theta \right]. \end{aligned}$$

Under the assumption of $l(z) = (|z|)^q$ with $q > 1$, we have

$$\begin{aligned} & \int_{2\underline{b}^{n-1}}^{3\underline{b}^{n-1} - \delta_1^n} l(\theta) d\theta - \int_{\underline{b}^{n-1} + \delta_1^n}^{2\underline{b}^{n-1}} l(\theta) d\theta \\ &= \frac{1}{q+1} \left[(\underline{b}^{n-1} - \delta_1^n) \left((3\underline{b}^{n-1} - \delta_1^n)^q - (\underline{b}^{n-1} + \delta_1^n)^q \right) \right. \\ & \quad \left. + 2\underline{b}^{n-1} \left((3\underline{b}^{n-1} - \delta_1^n)^q + (\underline{b}^{n-1} + \delta_1^n)^q - 2(2\underline{b}^{n-1})^q \right) \right]. \end{aligned}$$

Using conditions (13) and (14), we can rewrite the above payoff difference as

$$\begin{aligned} & \underline{U}^{n-1}(\underline{b}^{n-1}) - U^n(\underline{b}^{n-1}) \\ &= \frac{1}{q+1} \left[2\underline{b}^{n-1} (\underline{b}^{n-1} + \delta_1^n)^q - \underline{y}_1^{n-1} (2\underline{b}^{n-1})^q + 2(y_1^n)^q (y_1^n - (\underline{b}^{n-1} - \delta_1^n)) \right. \\ & \quad \left. + (n-2)\underline{b}^{n-1} \left((3\underline{b}^{n-1} - \delta_1^n)^q + (\underline{b}^{n-1} + \delta_1^n)^q - 2(2\underline{b}^{n-1})^q \right) \right] \\ &> \frac{1}{q+1} \left[2\underline{b}^{n-1} (\underline{b}^{n-1} + \delta_1^n)^q - \underline{y}_1^{n-1} (2\underline{b}^{n-1})^q \right]. \end{aligned}$$

Note that the right-hand side of the above inequality does not depend on n directly. Using the assumption of $l(z) = (|z|)^q$ with $q > 1$, we can explicitly compute \underline{y}_1^{n-1} in terms of \underline{b}^{n-1} from (13) and use it to show that the last expression above is strictly positive if $\delta_1^n / \underline{b}^{n-1}$ is sufficiently close to 1. The desired claim then follows from the observation that under the assumption of $l(z) = (|z|)^q$ with $q > 1$, because $y_1^n = \underline{y}_1^{n-1} - \delta_1^n$ and \underline{y}_1^{n-1} does not depend on n directly, condition (14) implies that $\delta_1^n / \underline{b}^{n-1}$ is strictly increasing in n and approaches 1 when n becomes arbitrarily large. \square

Now, observe that $Y^n = \underline{Y}^n$ at $b = \underline{b}^n$, and recall from Lemma 2 that $\underline{U}^n(b) > \underline{U}^{n-1}(b)$. Then, from Claim 10 and Claim 11, we have that for any n sufficiently large, there exists $b^{n,n-1}$ such that $\underline{U}^{n-1}(b) < U^n(b)$ for $b \in (\underline{b}^n, b^{n,n-1})$ and $\underline{U}^{n-1}(b) > U^n(b)$ for $b \in (b^{n,n-1}, \underline{b}^{n-1}]$. Next, for any n sufficiently large, since Claim 11 implies that $U^n(\underline{b}^n) = \underline{U}^n(\underline{b}^n) > U^{n+1}(\underline{b}^n)$, it follows from Claim 9 that $U^n(b) > U^{n+1}(b)$ for all $b \in [\underline{b}^n, \bar{b}^{n+1}]$. This concludes the proof of Proposition 5. \square

Proof of Corollary 1. In the proof of Proposition 5, under the assumption of $l(z) = z^2$, Claim 11 holds for any $n \geq 3$. From Claim 11 in the proof of Proposition 5 we have that $\underline{U}^{n-1}(\underline{b}^{n-1}) > U^n(\underline{b}^{n-1})$ for any $n \geq 3$ odd. For n even, using the assumption of $l(z) = z^2$ we can explicitly compute δ_1^n and use it to show that $\underline{U}^{n-1}(\underline{b}^{n-1}) > U^n(\underline{b}^{n-1})$ for any n . Further, we can easily show that for any $n \geq 3$, even or odd, $\bar{U}^n(\bar{b}^{n+1}) > \underline{U}^{n-1}(\bar{b}^{n+1})$ where \bar{U}^n is P 's expected payoff under $\bar{Y}^n = \{\frac{1}{2} - (n+1-2i)b\}_{i=1}^n$. Since $U^n \geq \bar{U}^n$ by the definition of Y^n , from Claim 10 we have $b^{n,n-1} > \bar{b}^{n+1}$. \square

Proof of Proposition 6. First, consider optimal limited authority. By Proposition 5, for large n , P 's expected payoff (6) is bounded from below by \underline{U}^{n-1} and from above by \underline{U}^n , where n satisfies $b \in [\underline{b}^n, \underline{b}^{n-1}]$. Since $\underline{b}^n \in (\bar{b}^{n+2}, \bar{b}^{n+1})$, we have

$$\frac{1}{2b} - 1 < n < \frac{1}{2b}.$$

Since l is convex and increasing, $\underline{\delta}^{n-1} \in (b, b(n-1)/(n-2))$ and $\underline{\delta}^n \in (b(n-2)/(n-1), b]$. To see this, note that if $\underline{\delta}^{n-1} \geq b(n-1)/(n-2)$, then

$$2l(1/2 - (n-1)b) \geq 2l(1/2 - (n-2)\underline{\delta}^{n-1}) = l(b + \underline{\delta}^{n-1}) + l(\underline{\delta}^{n-1} - b) > l(2b),$$

contradicting $b \geq \underline{b}^n$. Similarly, if $\underline{\delta}^n \leq b(n-2)/(n-1)$, then

$$2l(1/2 - (n-2)b) \leq 2l(1/2 - (n-1)\underline{\delta}^n) = l(b + \underline{\delta}^n) + l(b - \underline{\delta}^n) < l(2b),$$

contradicting $b < \underline{b}^{n-1}$. Further,

$$b < \underline{y}_1^n < \underline{y}_1^{n-1} < \frac{1}{2} - (n-1)b < 2b.$$

To sum up, for $k = n, n-1$, $\underline{y}_1^k = O(b)$, $b + \underline{\delta}^k = 2b + o(b)$, $|b - \underline{\delta}^k| = o(b)$, and $k = 1/(2b) + o(1/b)$. Thus, P 's expected payoff under \underline{Y}^k is

$$\begin{aligned} \underline{U}^k &= -\frac{1}{q+1} (2(\underline{y}_1^k)^{q+1} + (k-1)((b + \underline{\delta}^k)^{q+1} + \text{sign}(\underline{\delta}^k - b) \cdot (|\underline{\delta}^k - b|)^{q+1})) \\ &= -\frac{2^q}{q+1} b^q + o(b^q). \end{aligned}$$

Second, we claim that the optimal delegation set is $[b, 1-b]$, so P 's expected payoff is

$$-(1-2b)b^q - 2 \int_0^b \theta^q d\theta = -b^q + o(b^q).$$

If the optimal delegation set is not an interval, then there exists $x \in [0, 1]$ and $\xi > 0$ such that states $\theta \in (\max\{0, x - \xi\}, x)$ induce $x + b - \xi$, and states $\theta \in (x, \min\{x + \xi, 1\})$ induce $x + b + \xi$ as Melumad and Shibano [30] show. If $x - \xi < 0$, then we can modify the delegation set such that states $\theta \in (0, x + \varepsilon)$ induce $x + b - \xi + 2\varepsilon$ and states $\theta \in (x + \varepsilon, \min\{x + \xi, 1\})$ induce $x + b + \xi$. This modification increases P 's expected payoff because its derivative with respect to ε at $\varepsilon = 0$ is

$$l(b + \xi) + l(b - \xi) - 2l(b + x - \xi) > l(b + \xi) + l(b - \xi) - 2l(b) > 0$$

by convexity of l . If $x - \xi \geq 0$, then we can do the same modification with the exception that states $\theta \in (x - \xi, x - \xi + 2\varepsilon)$ induce $\theta + b$. Similarly, this modification increases P 's expected payoff because its derivative with respect to ε at $\varepsilon = 0$ is

$$l(b + \xi) + l(b - \xi) - 2l(b) > 0.$$

Knowing that the optimal delegation set is an interval, it is easy to show that it must be $[b, 1 - b]$ by maximizing P 's expected payoff with respect to the endpoints of the interval.

Third, CS show that under cheap talk, there are

$$N(b) = \left\lfloor -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{b}} \right\rfloor$$

induced decisions with

$$\theta_i - \theta_{i-1} = \frac{1}{N(b)} + 2b(2i - N(b) - 1),$$

so P 's expected payoff is

$$-\sum_{i=1}^{N(b)} \int_{\theta_{i-1}}^{\theta_i} \left(\theta - \frac{\theta_i + \theta_{i-1}}{2} \right)^q d\theta = -\sum_{i=1}^{N(b)} \frac{(\theta_i - \theta_{i-1})^{q+1}}{(q+1)2^q}.$$

Note that the sum $\sum_{i=1}^{N(b)} g(i)$ can be bounded above by $\int_1^{N(b)+1} g(i) di$ and below by $\int_0^{N(b)} g(i) di$ if g is increasing in i . Since, $N(b) = 1/\sqrt{2b} + O(1)$ and $\theta_i - \theta_{i-1}$ is increasing in i , we have that P 's expected payoff under cheap talk is

$$-(2b)^{\frac{q}{2}} \frac{2}{(q+1)(q+2)} + o(b^{\frac{q}{2}}). \quad \square$$

Proof of Proposition 7. By Proposition 5, for any fixed $q > 1$ and $b \in (0, 1/2)$, P 's expected payoff (6) under limited authority is bounded from below by \underline{U}^{n-1} and from above by \underline{U}^{n+1} , where $n \geq 3$ satisfies $b \in [\underline{b}^n, \underline{b}^{n-1})$. Recall that \underline{Y}^2 satisfies P 's incentive condition for all $b < \frac{1}{2}$, so we set $\underline{b}^2 = \frac{1}{2}$. P 's expected payoff under full delegation is simply $-b^q$.

Consider first the limit case of $q = 1$. Using Lemma 2, we can easily verify that $\underline{b}^n = 1/(2n)$ for each $n \geq 3$. Fix any $n \geq 4$ and any $b \in [\underline{b}^n, \underline{b}^{n-1})$. By Lemma 2, $\underline{\delta}^{n-1} = 1/(2(n-1))$, and thus from Eq. (19) we have

$$\underline{U}^{n-1}(b) = -\frac{1}{4(n-1)} - (n-2)b^2.$$

It follows immediately that $\underline{U}^{n-1}(b) > -b$. For any $b \in [\underline{b}^3, 1/2)$, \underline{Y}^2 is given in the proof of Proposition 3. Using $q = 1$, we can easily verify that for $b \in [\underline{b}^3, 1/4)$, $\underline{\delta}^2 = \frac{1}{4}$ and

$$\underline{U}^2(b) = -\frac{1}{8} - b^2;$$

and for $b \in [1/4, 1/2)$, $\underline{\delta}^2 = \frac{1}{2} - b$ and

$$\underline{U}^2(b) = -b + b^2.$$

It can be easily verified that $\underline{U}^2(b) > -b$ for all $b \in [\underline{b}^3, 1/2)$. By continuity, for any $b \in (0, 1/2)$, limited authority is better than full delegation when q is sufficiently close to 1.

Next, for any $q > 1$ and $b \in (\underline{b}^n, \underline{b}^{n-1}]$, from Lemma 2 we have $\underline{\delta}^{n+1} \in (b(n-2)/n, b)$. To see this, note that $\underline{\delta}^{n+1} < b$ follows because $b > \underline{b}^{n+1}$; and if $\underline{\delta}^{n+1} \leq b(n-2)/n$, we would have

$$2l(1/2 - (n-2)b) \leq 2l(1/2 - n\underline{\delta}^{n+1}) = l(b + \underline{\delta}^{n+1}) + l(b - \underline{\delta}^{n+1}) < l(2b),$$

contradicting that $b \leq \underline{b}^{n-1}$. Then, from Eq. (19) we have

$$\underline{U}^{n+1}(b) = -\frac{1}{q+1} [2(\underline{y}_1^{n+1})^{q+1} + n((b + \underline{\delta}^{n+1})^{q+1} - (b - \underline{\delta}^{n+1})^{q+1})].$$

Since $\underline{\delta}^{n+1} > b(n-2)/n$, we have

$$\underline{U}^{n+1}(b) < -\frac{nb^{q+1}}{q+1} \left[\left(\frac{2(n-1)}{n} \right)^{q+1} - \left(\frac{2}{n} \right)^{q+1} \right].$$

Thus, for sufficiently large q , we have $\underline{U}^{n+1}(b) < -b^q$. Thus, for any $b \in (0, 1/2)$, full delegation is better than limited authority when q is sufficiently large.

Finally, we show that P 's expected payoff difference under the optimal limited authority and under full delegation is single-crossing in q for any $b \in (0, 1/2)$. Fix any $q > 1$. Under the optimal limited authority $Y = \{y_i^n\}_{i=1}^n$, the expected payoff of P can be written as $-\mathbb{E}[z^q]$ where the random variable z is given by $\sum_{i=1}^n |y_i^n - \theta| \cdot \mathbf{1}_{(\theta_{i-1}, \theta_i]}(\theta)$ where $\mathbf{1}_{(\theta_{i-1}, \theta_i]}(\theta)$ is the indicator function of the subset $(\theta_{i-1}, \theta_i]$ and θ is uniformly distributed on $(0, 1]$. Suppose that $\mathbb{E}[z^q] \leq b^q$ so that limited authority is weakly better than full delegation. For any $q' < q$, P 's expected payoff under the corresponding optimal limited authority is weakly greater than $-\mathbb{E}[z^{q'}]$, because $Y = \{y_i^n\}_{i=1}^n$ is weakly suboptimal when the power of the loss function is q' . Then,

$$\mathbb{E}[z^{q'}] = \mathbb{E}[(z^q)^{q'/q}] < (\mathbb{E}[z^q])^{q'/q} \leq (b^q)^{q'/q} = b^{q'},$$

where the first inequality holds by the strong version of Jensen's inequality because $q' < q$ and z is not a degenerate lottery, and the second holds by assumption. Thus, optimal limited authority is better than full delegation at $q' < q$. \square

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