Unobserved Mechanism Design:
Equal Priority Auctions

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Abstract

We study the impact for mechanism design of the possibility that some participants are unobservant of the rules associated with the trading mechanism but are otherwise rational. Since “deviations” by the mechanism designer are not observed by these participants the nature of the “equilibrium” of the design game changes, as do equilibrium mechanisms. We study the symmetric, regular case of the independent private value auction environment, and show how to characterize an interesting equilibrium outcome for the game by optimizing over reduced form direct mechanisms. This gives rise to a surprisingly simple mechanism that we call an equal priority auction. Observant bidders with intermediate valuations receive offers with the same probability as unobservant bidders, even though observant buyers will accept the offers for sure, while unobservant bidders might not.

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1 Introduction

There is an acronym that floats around the internet - TL;DR - that explains why no one reads your email messages. It means “too long, didn’t read.” The long translation we adapt in this paper is “there is no point in reading your message, because I already know everything that’s relevant to me.” We refer to this as “rationally unobservant.” It sounds like an oxymoron, but is meant to capture the idea that one can be unobservant and rational at the same time.

The message of this paper is that this kind of behavior can impact trading mechanisms. We aren’t the first to notice that traders are sometimes unobservant. The marketing literature has documented this behavior of buyers when they make purchase decisions. The simplest trading mechanism of all is a price commitment. Dickson and Sawyer [1990] asked buyers in supermarkets about their price knowledge as they were shopping. Even when the item being placed in their basket had been specially marked down and heavily advertised, 25% of consumers did not even realize the good was on special. Marketing has a problem when prices can’t influence buyer behavior because buyers may be unobservant.

We are interested in more than prices; we want to know how trading mechanisms are impacted by buyers who are possibly unobservant but are always rational. We consider what is probably the best understood trading problem of all - the independent private value auction. Having rationally unobservant buyers who don’t observe the seller’s commitment turns the constrained optimization problem of mechanism design into a game of imperfect information where deviations by the seller may not be observed by some buyers. We show that under a plausible restriction - unobservant buyers convey no information to sellers - the equilibrium outcome is characterized by something we call an equal priority auction.

The equal priority auction treats observant bidders - those who observe the trading mechanism - with intermediate valuations in exactly the same way as unobservant bidders. When the auction attempts to trade with them, it makes a take-it-or-leave-it price offer that is independent of any messages they may have sent. When bidders have very high or very low valuations, the seller treats messages as bids. If the seller decides to sell to one of these bidders, she will make an offer equal to what can be thought of as the second highest bid she has received - much as she would in a standard auction.
One appealing feature of the independent private value auction problem for mechanism design is that finding the revenue maximizing mechanism can be reduced to a problem of solving a maximization problem with a single parameter - the reserve price. The equilibrium mechanism with rationally unobservant bidders can be found by solving a problem with four parameters - a reserve price, two cutoff valuations that define observant bidders treated equally as unobservant bidders, and a price offer to them. This is a harder problem, but still computationally tractable. The numerical solutions we have found in simple environments suggest that fixed price trading is quite common. In fact, it is easy to see without any calculation that if every bidder is equally likely to be observant or unobservant, the trade will occur at a fixed price (with no auctions) more than half the time. This may be an explanation for why auctions aren’t particularly common in many trading platforms. Even on trading platforms on which auctions are used, such as eBay, there are “buy it now” options where trading takes place at a fixed price.\footnote{The environment on eBay doesn’t fit our model exactly because bidders arrive randomly. Buy it now options disappear on eBay once a bidder submits a bid, which starts the auction.}

Equal priority auction is an indirect mechanism that we use to replicate the equilibrium outcome in the unobserved mechanism design game where unobservant bidders babble. Observant bidders with intermediate valuations and unobserved bidders are treated the same, but of course observant bidders know the auction rule while unobservant ones don’t. If there is a hidden link on a website that gives access to the auction, then observant bidders are free to click on the link but unobservant ones wouldn’t know how to do it. It doesn’t matter to the seller or any bidder whether or not observant bidders with intermediate valuations click on the link to participate in the auction. This has an important implication. An econometrician who tries to recover distributions of valuations based on bids placed in the auction would get biased estimates, because intermediate valuations might be missing from the bidding data. Our model of unobserved mechanisms is admittedly crude, and we only look at babbling equilibria here, but equal priority auctions make empirically relevant points about the way rationally unobservant agents impact trading mechanisms.
1.1 Heuristic

Consider the symmetric, regular case of the independent private value auction environment. The formal analysis in this paper is based on two arguments. The first is that in an environment with unobservant bidders, standard auction mechanisms can’t be supported as equilibrium even though the seller would much prefer to use them. The fault lies with the seller who can’t resist the temptation of exploiting rationally unobservant bidders.

To see why, suppose the seller wants to use a second price auction with optimal reserve. This means that observant bidders read the auction rules, as they might on eBay, then realize they should bid their valuations. Unobservant bidders don’t read the rules, so they only anticipate a second price auction. Acting on their expectations, they also bid their valuations.

What makes this break down is the fact that if the seller changes the auction rules, the unobservant won’t realize it, and will continue to bid their valuations no matter what the seller does. A simple deviation can extract the surplus of the unobservant. For example, the seller can ask bidders to attach a coupon code to their bid. The coupon code isn’t secret, it is plainly visible in the description of the bidding rules. A bidder who reads the new rules will see the coupon code and attach it to their bid. A bidder who doesn’t read won’t add the code. The new mechanism commits to a second price auction for bids submitted with a code, but treats bids with no code attached as if it were a first price auction. In other words, if the highest bid is submitted by an unobservant bidder, the seller will commit to make them an offer equal to their bid, instead of offering them the second highest bid. This breaks the equilibrium, because it should be expected by rational unobservant bidders.

The second argument involves how the seller should respond. The seller will want to sell to the unobservant bidders when observant bidders have low valuations. So the natural idea would be to have an auction, then if bids are too low, make an offer to the unobservant. But observant bidders don’t have to bid. They can pretend to be unobservant. Since they are observant, they know when the seller will make an offer to the unobservant and what that offer will be. To prevent observant bidders from pretending to be unobservant, the seller has to keep the offer to unobservant bidders higher than she would like it to be.
The seller then faces a trade off - keep the offer high and fully separate the observant from the unobservant, or lower the take it or leave it offer and allow some of the observant bidders to pool with the unobservant. We show that the latter is what happens in equilibrium, which is where the equal priority phrase comes from in our title. Intuitively, pooling sacrifices auction revenue from observant bidders, but this is minuscule when the seller is fully separating them from unobservant bidders. The revenue gain from lowering the offer to unobservant bidders then makes pooling profitable. Pooling happens at intermediate valuations, because it is too costly to include in the equal priority pool observant bidders with high valuations, and unnecessary to include those with low valuations.

1.2 Literature

As mentioned above, the idea that consumers might not notice prices is an old one in the marketing literature. The approach had been used earlier in economics, as in, say Butters [1977], in which buyers randomly observe price offers in a competitive environment. In that literature, firms advertise prices which some buyers see, while others do not. These papers considered the same problem that we do, which is how this unobservant buyers would affect the prices that firms offer. The difference is that we are interested in mechanisms, not prices.

The presence of unobservant buyers provides type dependent outside options to observant buyers. This is the basic problem in the literature on competing mechanisms. One example is the paper by McAfee [1993]. His model had buyers whose outside option involved waiting until next period to purchase in a competing auction market just like the one in the current period. He imposed large market assumptions to ensure that the value of these outside options was independent of the reserve price that any seller in the existing market chose. In our paper, the value of this outside option depends on the nature of the mechanism the seller chooses for the observant. This makes it resemble later papers on competing mechanisms in terms of outside options, like Virag [2010] who studies finite competing auction models where a seller who raises her reserve price increases congestion in other auctions, or Hendricks and Wiseman [2020] who study the same problem in a sequential auction environment.

\footnote{See also Varian [1980], or Stahl [1994]. In Varian [1980], unobservant buyers are loyal to a specific seller, observant buyers are just interested in the lowest price.}
With buyers potentially unobservant of the selling mechanism but nonetheless having rational expectations, the seller’s commitment power is limited. There is an extensive literature on limited commitment (for example Bester and Strausz [2001], Kolotilin et al. [2013], Liu et al. [2019], or Skreta [2015]). To our knowledge, our model is the first to study commitment with respect to a subset of traders involved in the same transaction. A recent paper by Akbarpour and Li [2020] provides another model of limited commitment. They assume that each individual buyer only observes the part of the seller’s commitment in relation to the buyer’s own report, and impose a “credibility” constraint that the seller does not wish to secretly alter other parts of the commitment. The logic we described above explaining why the second price auction can’t survive as an equilibrium is used in a similar way in their paper. The difference between their approach and ours is that they assume the credibility constraint applies to all buyers and describe mechanisms that are immune to this constraint. Here we assume that credibility is an issue only for some buyers and find equilibrium mechanisms.

Our observant buyers can “prove” they are observant in the same sense as Porath et al. [2014]. The main difference is that they assume that the social choice function is known by all the players, while in our model the driving force is the presence of buyers who are unobservant of the seller’s mechanism. They also assume players have complete information about the state, but in our model only buyers know their own valuations.

Finally, our observant buyers can pretend they are unobservant but not the other way around. The one-sidedness of this incentive condition is similar to Denekere and Severinov [2006], who study an optimal non linear pricing problem with a fraction of consumers constrained to reporting their valuations truthfully. As in our paper, a “password” mechanism separates “honest” consumers from “strategic” consumers who can misrepresent their valuations costlessly. The main difference is that we start with a standard independent private value auction problem rather than a non linear pricing problem. More importantly, our unobservant buyers are uncommunicative in the class of equilibria we focus on, but they are rational rather than behavioral or face prohibitive communication cost.

\(^3\)See also Sher and Vohra [2015]. They use graph theory to study a more general non linear pricing problem with voluntary provision of hard evidence.
2 Unobserved Mechanism Design Game

There are \( n \) potential buyers of a single homogeneous good. Each buyer \( i \) has a privately known valuation \( v_i \) that is independently drawn from the interval \([0, 1]\). We assume that all valuations are distributed according to some distribution \( F \) with strictly positive and continuously differentiable density \( f \). Buyer \( i \)'s payoff when they buy at price \( p \) is given by \( v_i - p \). Following the standard auction literature, we define

\[
\phi(w) = w - \frac{1 - F(w)}{f(w)}
\]

as the virtual valuation function.

Whether or not a buyer is observant of how a mechanism works is private information. So buyer \( i \)'s type is given by the pair \((v_i, \tau_i)\), where \( v_i \in [0, 1] \) and \( \tau_i \in \{\epsilon, \mu\} \), where \( \epsilon \) means observant, and \( \mu \) means unobservant. Each buyer has type \( \mu \) with probability \( \alpha \in (0, 1) \) that is independent of their valuation or the types of other buyers.

The seller’s reservation value is zero, so the profit from selling at price \( p \) is just \( p \). Define

\[
\pi(p) = (1 - F(p))p.
\]

This is the seller’s the revenue function from a take-it-or-leave-it offer \( p \) to a single buyer.

There is a common message space \( M \) used by all buyers to communicate with the seller. A message from buyer \( i \) will be denoted \( b_i \). We make no assumptions on \( M \) itself except that it is rich enough to embed the product of the set of buyer valuations and the interval \([0, 1]\). The latter is meant to allow the seller to randomize over mechanisms; this will become clear later.\(^4\) The message space \( M \) is common knowledge among the seller and all buyers, observant or unobservant.

After processing all the buyers’ messages, the seller’s mechanism makes an offer to one of the buyers.\(^5\) This offer can be rejected. When it is, there is simply no trade at all.

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\(^4\) In practice, \( M \) could incorporate coupons or browsing history, as well as bids.

\(^5\) In the standard mechanism design paradigm, a mechanism produces an allocation according to a mapping from messages. Representing the output of a mechanism as an offer instead of an allocation has no implications. This is no longer the case in our unobserved mechanism problem. See section 5 for comments on modeling the output of an unobserved mechanism as an algorithm.
Our assumption is that it is common knowledge that all buyers understand and believe a take-it-or-leave-it price commitment. This makes for a cleaner analysis of the issues we are interested in. The equilibrium mechanism we describe will be part of an equilibrium even if multiple offers are allowed after the first one is rejected.

Formally, a mechanism $\gamma$ for the seller is a collection $\{\mathcal{M}, (p_i, q_i)_{i=1}^n\}$, where $\mathcal{M}$ is the common message space, and $q_i$ is a mapping from a profile of messages $(b_1, \ldots, b_n)$ to a probability with which a take-it-or-leave-it offer $p_i(b_1, \ldots, b_n)$ is made to buyer $i$. The mapping $q_i$ must satisfy

$$\sum_{i=1}^n q_i(b_1, \ldots, b_n) \leq 1$$

for every profile of messages. Let $\Gamma$ be the set of feasible mechanisms.\(^6\)

**Definition 1** The imperfect information game $G(\alpha)$ is an extensive form game in which the seller first commits to some $\gamma \in \Gamma$, observant buyers send messages to the seller that depend on $\gamma$, and unobservant buyers send messages that are independent of $\gamma$. Allocations and final payoffs are determined by the mechanism $\gamma$, the realized messages, and the acceptance decisions of any buyer who receives an offer. The parameter $\alpha$ gives the common belief with which each buyer and the seller believe that each of the buyers is unobservant.

A strategy rule in $G(\alpha)$ for buyer $i$ is a function $\sigma_i : [0, 1] \times \{\epsilon, \mu\} \times \Gamma \to \Delta(\mathcal{M})$ that specifies what message the buyer will send for each of the valuations conditional on whatever the buyer knows about the seller’s mechanism.\(^7\) Since an unobservant buyer never sees the mechanism the seller commits to, we have the informational constraint

$$\sigma_i(v_i, \mu, \gamma) = \sigma_i(v_i, \mu, \gamma') = \sigma_i(v_i, \mu)$$

for all $\gamma$ and $\gamma'$. We retain this assumption throughout the paper.

We need to allow the seller to mix among mechanisms in $\Gamma$. In our setup, observant buyers can pretend to be unobservant, but not conversely. Allowing the seller to mix means

\(^6\)We have thus restricted each mechanism $\gamma$ in $\Gamma$ to have each $p_i$ mapping a profile of messages to a single price, instead of a distribution of prices. This can be easily dropped without affecting the formalism; for the equilibria we construct in the paper this restriction is without loss.

\(^7\)For simplicity we consider only equilibria where buyers use pure strategies. It is straightforward to extend the following conditions to allow for mixing by buyers.
that the seller can potentially identify observant buyers by, for example, providing a coupon code that must be submitted with a bid. But if the unobservant guess the coupon code, they will submit informative bids. So to prevent the unobservant buyers from guessing this password, it has to be random. There is nothing secret about this password, it is freely available to anyone who takes the time to read the rules of the mechanism.

The important reason for allowing the seller to mix is that in this game pure strategy (for the seller) equilibrium typically won’t exist. If such equilibrium did exist, unobservant buyers would guess the coupon code. But since the unobservant don’t actually see the coupon, the seller won’t be able to resist the temptation of exploiting their informative bids. This is just because of the fact that strategies are common knowledge in any Nash based equilibrium. Another way of saying is that unobservant buyers have “rational expectations.”

Formally, refer to the seller’s mixture as \( \psi \in \Delta (\Gamma) \). Let \( R (\gamma, (\sigma_i (\cdot, \cdot, \gamma)))_{i=1}^n \) be the expected revenue for the seller from mechanism \( \gamma \) when an unobservant buyer \( i \) uses strategy \( \sigma_i (\cdot, \mu) \) an observant buyer \( i \) uses \( \sigma_i (\cdot, \epsilon, \gamma) \). This is given by

\[
E_{v,\tau} \left[ \sum_{i=1}^n q_i (\sigma(v, \tau, \gamma)) p_i (\sigma(v, \tau, \gamma)) \mathbb{1}_{v_i \geq p_i (\sigma(v, \tau, \gamma))} \right].
\]

A perfect Bayesian equilibrium for this game is a mixture \( \psi \) for the seller, and strategy rules \( (\sigma_i (v_i, \tau_i, \gamma))_{i=1}^n \), for buyers, that satisfy the usual conditions:\(^8\) for each buyer \( i \) with \( \tau_i = \epsilon \),

\[
E_{v-i,\tau-i} \left[ q_i (\sigma_i (v_i, \epsilon, \gamma), \sigma_{-i} (v_{-i}, \tau_{-i}, \gamma)) \cdot \max \{ v_i - p_i (\sigma_i (v_i, \epsilon, \gamma), \sigma_{-i} (v_{-i}, \tau_{-i}, \gamma)), 0 \} \right] \\
\geq E_{v-i,\tau-i} \left[ q_i (b', \sigma_{-i} (v_{-i}, \tau_{-i}, \gamma)) \cdot \max \{ v_i - p_i (b', \sigma_{-i} (v_{-i}, \tau_{-i}, \gamma)), 0 \} \right] \tag{1}
\]

for all \( v_i \in [0, 1], b' \in \mathcal{M} \) and \( \gamma \in \text{supp}(\psi) \) (the support of \( \psi \)); for each buyer \( i \) with \( \tau_i = \mu \),

\[
E_{v-i,\tau-i,\gamma \in \text{supp}(\psi)} \left[ q_i (\sigma_i (v_i, \mu), \sigma_{-i} (v_{-i}, \tau_{-i}, \gamma)) \cdot \max \{ v_i - p_i (\sigma_i (v_i, \mu), \sigma_{-i} (v_{-i}, \tau_{-i}, \gamma)), 0 \} \right] \\
\geq E_{v-i,\tau-i,\gamma \in \text{supp}(\psi)} \left[ q_i (b', \sigma_{-i} (v_{-i}, \tau_{-i}, \gamma)) \cdot \max \{ v_i - p_i (b', \sigma_{-i} (v_{-i}, \tau_{-i}, \gamma)), 0 \} \right] \tag{2}
\]

\(^8\)In the conditions below, the max operation appears when taking expectations because a mechanism generates an offer instead of an outcome.
for all \( v_i \in [0,1] \), and \( b_i \in \mathcal{M} \); and for the seller,

\[
R(\gamma_i, (\sigma_i(\cdot, \cdot, \gamma_i))_{i=1}^n) \geq R(\gamma'_i, (\sigma_i(\cdot, \cdot, \gamma'_i))_{i=1}^n)
\]

for all \( \gamma \in \text{supp}(\psi) \), and \( \gamma' \in \Gamma \).

We will focus on symmetric perfect Bayesian equilibria of the game \( \mathcal{G}(\alpha) \). Symmetry here requires that the set of feasible mechanisms \( \Gamma \) contains only symmetric mechanisms that treat any two buyers who send the same message in the same way. This restricts not only the mechanisms in the equilibrium mix \( \psi \), but also deviations by the seller. Correspondingly, symmetry also requires all buyers, in particular observant buyers, to use the same strategy in response to any mechanism \( \gamma \), regardless of whether it is on or off the equilibrium path.

## 2.1 Relationship to standard mechanism design

Our unobserved mechanism design game is a kind of informed principal problem. The unusual part about it is that the seller has more information about the mechanism she is using than about the product itself. Like the informed principal problem, in our unobserved mechanism design game, equilibrium strategies of the seller and buyers must constitute a fixed point. In its simplest form, unobservant buyers’ expectations about the relationship between the messages they send and the offers they get must coincide with the actual relationship once the seller best replies to those expectations. As a result, our game potentially has many equilibrium outcomes. The source of multiplicity comes from different information contents of messages by unobservant buyers.

A babbling equilibrium is one example. So long as the seller’s mechanism has a way of distinguishing messages from observant and unobservant buyers, no unobservant buyer will send an informative message because he believes the seller’s mechanism won’t respond to it. Since the seller thinks unobservant buyers are babbling, there is no reason for their mechanism to respond to these messages.

There may exist equilibrium outcomes in which unobservant buyers with very low valuations will separate from higher valuation buyers by saying they aren’t interested in an offer. To support this, some of the unobservant who believe they will never accept a seller offer
must nonetheless act as if they might accept an offer. Messages from unobservant buyers are informative of their valuations, but they expect at most one “serious” offer. These outcomes can be constructed as straightforward extensions of what we describe below, but the challenge is to show that the seller will commit not to make offers to unobservant buyers who say they are uninterested for incentive reasons, even when there is no profitable alternative.

There may also exist equilibria in which unobservant buyers send even richer information with their messages, which are effectively cheap talk. All of these alternative equilibria where unobservant buyers convey information to the seller exhibit the same logic that we describe below. For example, informative cheap talk messages allow the seller to make offers to the buyers whose messages say they are most likely to accept them. This effect tends to raise the seller’s expected revenues. Yet to maintain incentive compatibility the seller has to make fairly attractive offers so that the observant are incentivized to reveal that they are observant. This makes it hard to determine whether equilibria with more information conveyed by unobservant buyers actually benefit the seller.

A natural approach to find an equilibrium outcome as a fixed point is to adapt the revelation principle. The usual composition of the outcome function and the strategies can be used to create reduced form mechanisms. Yet not all reduced form mechanisms that look incentive compatible correspond to equilibrium outcomes because unobservant buyers can only use a restricted set of communication strategies. In addition, equilibrium outcomes in which unobservant buyers learn how to participate in a reduced form mechanism can’t correspond to equilibrium outcomes because the seller may be able to deviate without being observed by unobservant buyers.

So we adapt the revelation principle by conditioning the search for equilibrium outcomes as fixed points on a particular communication strategy of unobservant buyers. Given this strategy, the seller solves an optimal mechanism design problem where the objective function includes both revenues from unobservant and observant buyers, but incentive constraints are imposed only on observant buyers. These constraints require observant buyers to report

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9To be precise, for any threshold valuation sufficiently close to 0, we can construct an equal priority mechanism in which observant buyers with intermediate valuations are pooled together with unobservant buyers with valuations above the threshold (who tell the seller that they are interested in an offer). Further, the equal priority mechanism is optimal conditional on the seller not making an offer to unobservant buyers with valuations below the threshold.
their valuations truthfully and not to have strict incentives to pretend to be unobservant by adopting the latter’s communication strategy. There are no incentive constraints on unobservant buyers at this stage, even if the communication strategy is informative of their valuations and is exploited by the seller. If an optimal mechanism of this kind can be characterized, then one may hope to find a fixed point by showing that the communication strategy of unobservant buyers is indeed a best response to the optimal mechanism. This last step also relies on randomization by the seller in implementing the optimal mechanism to prevent unobservant buyers from participating in the mechanism in the same way as observant buyers.

There are at least three difficulties with the above approach. We have to be able to characterize optimal mechanisms for a given communication strategy of unobservant buyers; finding a fixed point in terms of a particular communication strategies within a class can be hard; it is not clear what class of communication strategies should be considered. In this paper, we are able to sidestep these difficulties by restricting to a special class of equilibrium outcomes of the unobserved mechanism design game, one where unobservant buyers babble. Babbling makes the fixed point nature of the approach irrelevant. We leave it to future research to characterize equilibrium outcomes with informative communication by unobservant buyers.

3 Direct Mechanisms

We are going to define a special kind of direct mechanism that we can use to characterize an important class of equilibrium outcomes. In particular, it will allow us to characterize equilibrium in which the seller uses a symmetric mechanism, observant buyers use the same strategy, and unobservant buyers use uninformative messages (babble). As in a standard mechanism design problem, observant buyers participate in a direct mechanism by choosing what valuations to report; and since unobservant buyers babble, formally they “keep silent.”

First we introduce notation to define symmetric mechanisms. In what follows the notation \( m \) always means the number of unobservant buyers. We reorder \( n \) buyers such that the first \( n - m \) of them are observant; the orders among the observant and among the unobservant

11
are arbitrary. For each \( v = (v_1, \ldots, v_n) \in [0,1]^n \) and each \( i = 1, \ldots, n-m \), let

\[
\rho^i_m(v) = (v_i, v_2, \ldots, v_{i-1}, v_1, v_{i+1}, \ldots, v_{n-m}, v_{n-m+1}, \ldots, v_n);
\]

that is, \( \rho^i_m(v) \) switches the positions of \( v_1 \) and \( v_i \). The permutation is introduced to simplify notation as we need to keep track of both the identity of a buyer and the number of unobservant buyers. Now we have

**Definition 2** A direct mechanism \( \delta \) is \( \{(q^\tau_m(v), p^\tau_m(v))_{m=0}^{n-1}, (q^\mu_m(v), p^\mu_m(v))_{m=1}^n\} \) where, for each \( m \), \( q^\tau_m(v), p^\tau_m(v) : [0, 1]^n \rightarrow [0, 1] \), \( \tau = \epsilon, \mu \), satisfy

- \( (q^\tau_m(v), p^\tau_m(v)), \tau = \epsilon, \mu, \) are invariant to \( (v_{n-m+1}, \ldots, v_n) \);
- \( (q^\epsilon_m(v), p^\epsilon_m(v)) \) are invariant to permutations of \((v_2, \ldots, v_{n-m})\), and \( (q^\mu_m(v), p^\mu_m(v)) \) are invariant to permutations of \((v_1, \ldots, v_{n-m})\);
- for all \( v \),

\[
\sum_{i=1}^{n-m} q^\epsilon_m(\rho^i_m(v)) + mq^\mu_m(v) \leq 1. \tag{4}
\]

The function \( q^\mu_m(v) \) gives the probability with which an offer \( p^\mu_m(v) \) is made to an unobservant buyer given that there are \( m \) unobservant buyers and the profile of valuations is \( v = (v_1, \ldots, v_n) \). The function \( q^\epsilon_m(v) \) gives the probability with which an offer \( p^\epsilon_m(v) \) is made to buyer 1 given that the profile of valuations is \( v = (v_1, \ldots, v_n) \). Since unobservant buyers keep silent, we require the allocation and the offer functions of both the observant and the unobservant buyers to be independent of the valuations of the latter. Symmetry requires the allocation and the offer functions of unobservant buyers to be invariant to permutations of the valuation profile of the observant, and the allocation and the offer functions of each observant buyer to be invariant to permutations of the valuation profile of the other observant buyers. Since \( \rho^i_m(v) \) switches the positions of the first element of \( v \) and its \( i \)-th element, the sum \( \sum_{i=1}^{n-m} q^\epsilon_m(\rho^i_m(v)) \) gives the probability that the offer is made to one of the first \( n-m \) elements of \( v \). Then (4) ensures that when the observant buyers have valuations given by the first \( n-m \) valuations in \( v \), the total probability with which the object is offered to one of them and to one of the unobservant buyers is less than or equal to 1.
The probability with which an observant buyer whose valuation is $w$ receives an offer when there are $m$ unobservant, and the expected price, are

$$Q^e_m(w) = \mathbb{E}_v [q^e_m(v)|v_1 = w]; \quad P^e_m(w) = \mathbb{E}_v [q^e_m(v)p^e_m(v)|v_1 = w].$$

These expressions implicitly assume that an observant buyer accepts the offer he receives with probability one. There is no max operator for observant buyers. This assumption is justified because observant buyers know the mechanism.

For each $m = 0, \ldots, n - 1$, let $B(m; n - 1, \alpha)$ be the probability that there are $m$ unobservant buyers among the $n - 1$ others. This probability is given by

$$B(m; n - 1, \alpha) = \binom{n - 1}{m} (1 - \alpha)^{n-1-m} \alpha^m.$$

Now by taking expectations over $m$ we have the usual reduced form functions:

$$Q^e(w) = \sum_{m=0}^{n-1} B(m; n - 1, \alpha) Q^e_m(w); \quad P^e(w) = \sum_{m=0}^{n-1} B(m; n - 1, \alpha) P^e_m(w).$$

The expected payoff to an observant buyer with valuation $w$ is therefore

$$U^e(w) = wQ^e(w) - P^e(w).$$

The expected payoff to an unobservant buyer with valuation $w$ is

$$U^\mu(w) = \sum_{m=0}^{n-1} B(m; n - 1, \alpha) \mathbb{E}_v [q^\mu_{m+1}(v) \max \{w - p^\mu_{m+1}(v), 0\}].$$

The max operator now appears because an offer to an unobservant buyer may be rejected. The above also gives the deviation payoff for an observant buyer who pretends to be unobservant, who knows the offers to the unobservant generated by a direct mechanism. The definition below gives the conditions for incentive compatibility that take into consideration this possible deviation, as well as the standard deviation of misreporting their valuations.\textsuperscript{10}

\textsuperscript{10}Necessary and sufficient conditions for incentive compatibility for observant buyers with respect to valua-
Definition 3 A direct mechanism $\delta$ is incentive compatible if

$$U^\epsilon (w) = \int_0^w Q^\epsilon (x) \, dx,$$

(5)

$Q^\epsilon (\cdot)$ is non-decreasing, and $U^\epsilon (w) \geq U^\mu (w)$ for every $w$.

Using standard arguments and properties of the binomial distribution, we can show that in any incentive compatible mechanism the seller’s revenue from observant buyers is

$$n(1 - \alpha) \int_0^1 Q^\epsilon (w) \phi(w)f(w) \, dw.$$

(6)

The seller’s revenue from unobservant buyers is given by

$$\sum_{m=1}^n B(m; n, \alpha) \mathbb{E}_v [mq^\mu_m (v) \pi (p^\mu_m (v))].$$

The seller’s total revenue $R(\delta)$ from $\delta$ is the sum of the above expressions.

The following result provides a two-way relationship between an “optimal” direct mechanism and the outcome of a symmetric equilibrium of the unobserved mechanism design game with babbling by unobservant buyers. This can be viewed as an adapted version of the revelation principle.

Theorem 1 For any symmetric equilibrium of the game $G (\alpha)$ where unobservant buyers babble, there is an incentive compatible direct mechanism $\delta^*$ that achieves the equilibrium expected revenue for the seller, and $R(\delta^*) \geq R(\delta)$ for every incentive compatible direct mechanism $\delta$. Conversely, any incentive compatible direct mechanism $\delta^*$ that maximizes $R(\delta)$ can be used to construct an equilibrium in the Bayesian game $G (\alpha)$.

Our argument for going from an equilibrium outcome to an incentive compatible direct mechanism begins in the same way as in the standard revelation principle. From a symmetric equilibrium of unobservable mechanism game $G (\alpha)$, we define a direct mechanism. The assumption that unobservant buyers babble in the equilibrium is used because the direct
mechanism does not allow allocations or offers to depend on the valuations of the unobservant. The equilibrium condition (1) for observant buyers becomes the incentive compatibility condition in the direct mechanism for truthful reporting of the valuation and a participation condition with type dependent outside option from pretending to be unobservant. The equilibrium condition (3) for the seller requires the direct mechanism to be revenue maximizing among all direct mechanisms. Otherwise, so long as the message space $M$ is rich enough, the seller could deviate to a more profitable direct mechanism, given that unobservant buyers do not observe any deviation while observant buyers do.

In the other direction, from an optimal direct mechanism, we construct a symmetric equilibrium of the unobserved mechanism design game $G(\alpha)$ where unobservant buyers babble. Consider an indirect mechanism that embeds the optimal direct mechanism with a random password with the support of $[0, 1]$. The message space $M$ is the product of the support of valuation and $[0, 1]$, and buyers are all asked to report their valuation and the realized password. Observant buyers are identified as those who match the realized password, and their valuation reports are accepted as truthful, while valuation reports from buyers who don’t match the password are ignored. The equilibrium condition (1) for observant buyers is satisfied because the direct mechanism is incentive compatible. In equilibrium unobservant buyers babble because their reports are ignored, and so the equilibrium condition (2) is satisfied. Given that unobservant buyers babble, the equilibrium condition (3) for the seller is satisfied, because the direct mechanism is optimal, and because it is optimal to commit to ignore buyers who can’t match the realized password.

4 Equal Priority Mechanisms

Our main result is that for valuation distributions such that $\pi(\cdot)$ is concave, the outcome of a symmetric equilibrium of the game $G(\alpha)$ where unobservant buyers babble corresponds to an optimal “equal priority mechanism.” We’ll establish the main result in two parts. First we’ll describe the set of equal priority mechanisms, and then the one that gives the seller the highest expected revenue. Later we’ll show how to verify that the seller cannot do strictly better among all direct mechanisms.
An equal priority mechanism is fully characterized by four numbers, a “reserve price” \( r \), a take-it-or-leave-it offer \( t \), and the upper and lower bound \( v_+ \) and \( v_- \) of an interval of buyer valuations, satisfying \( r \leq v_- \leq v_+ \). Unobservant buyers keep silent. Let \( m \) be the number of buyers who keep silent, and \( k \) be the number of reported valuations in the interval \([v_-, v_+]\). The mechanism treats the \( m \) unobservant buyers and the \( k \) observant buyers with the same allocation priority; we refer to them as “equal priority pool.” Priorities of observant buyers with reported valuations above \( v_+ \) and those with valuations below \( v_- \) are equal to the valuations themselves, with the former all higher and the latter all lower than the \( m + k \) buyers in the equal priority pool. In words, the allocation and offers in an equal priority mechanism are determined in the following way:

- When the highest reported valuation is less than \( r \): no offer is made to the buyer; for each \( m \geq 1 \), with probability \( 1/m \) an offer \( t \) is made to each unobservant buyer.

- When the highest reported valuation is between \( r \) and \( v_- \): if \( m = 0 \), the buyer is made an offer equal to the maximum of the second highest reported valuation and \( r \); if \( m \geq 1 \), no offer is made to the buyer, and instead with probability \( 1/m \), an offer \( t \) is made to each unobservant buyer.

- When the highest reported valuation is between \( v_- \) and \( v_+ \): if \( m + k = 1 \), the buyer is made an offer equal to the maximum of the second highest reported valuation and \( r \); if \( m + k \geq 2 \), with probability \( 1/(m+k) \) an offer \( v_- \) is made to each observant buyer with reported valuation in the interval \([v_-, v_+]\) and an offer \( t \) to each unobservant buyer.

- When the highest reported valuation is above \( v_+ \): the buyer is made an offer equal to the second highest reported valuation if it is above \( v_+ \); if the second highest reported valuation is in \([v_-, v_+]\) or \( m \geq 1 \), an offer \((v_- + (m+k)v_+)/(m+k+1)\) is made to the buyer; if it is below \( v_- \) and \( m = 0 \), the buyer is made an offer equal to the maximum of the second highest reported valuation and \( r \).

In terms of the offer rule, an equal priority mechanism \( \{r, v_-, v_+; t\} \) is a second-price auction with a reserve price \( r \) for observant buyers, combined with a take-it-or-leave-it offer \( t \) to unobservant buyers. However, we have an equal priority pool consisting of observant
buyers with valuations between \( v_- \) and \( v_+ \) and unobservant buyers. As a result, the second price, or the offer made to the buyer with the highest reported valuation, is the maximum of \( r \) and the second highest reported valuation, only if the second highest reported valuation is outside \( [v_- , v_+] \), and only if there are no unobservant buyers when the second highest reported valuation is lower than \( v_- \).

Formally, using the notation of direct mechanisms introduced in section 3, we can represent an equal priority mechanism \( \{ r, v_- , v_+ ; t \} \) as follows. Suppose that \( v_1 \) is the highest reported valuation, and \( v_2 \) be the second highest reported valuation. The collection of functions \( \{(q_m^e(v), p_m^e(v))_{m=0}^{n-1}, (q_m^u(v), p_m^u(v))_{m=1}^{n}\} \) given by \( \{ r, v_- , v_+ ; t \} \) is

\[
\begin{cases}
q_m^e(v) = 0 & \text{if } v_1 < r, \text{ or } v_1 \in [r, v_-) \text{ and } m \geq 1 \\
q_m^e(v) = 1/(m + k), \ p_m^e(v) = v_- & \text{if } v_1 \in [v_- , v_+] \text{ and } m + k \geq 2 \\
q_m^e(v) = 1, \ p_m^e(v) = (v_- + (m + k)v_+)/(m + k + 1) & \text{if } v_1 > v_+, \text{ and } v_2 \in [v_- , v_+] \text{ or } m \geq 1 \\
q_m^e(v) = 1, \ p_m^e(v) = \max\{v_2, r\} & \text{if otherwise,}
\end{cases}
\]

and

\[
\begin{cases}
q_m^u(v) = 0 & \text{if } v_1 > v_+ \\
q_m^u(v) = 1/(m + k), \ p_m^u(v) = t & \text{if otherwise.}
\end{cases}
\]

Suppose that observant buyers truthfully report their valuations in an equal priority mechanism. Then using the allocation rule, we can calculate the probability \( Q^e(w) \) with which an observant buyer with valuation \( w \) trades:

\[
\begin{cases}
0 & \text{if } w < r \\
(1 - \alpha)^{n-1}F^{n-1}(w) & \text{if } w \in [r, v_-) \\
\sum_{m=0}^{n-1} B(m; n - 1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+)/(m + k + 1) & \text{if } w \in [v_- , v_+] \\
\sum_{m=0}^{n-1} B(m; n - 1, \alpha)F^{n-1-m}(w) & \text{if } w > v_+,
\end{cases}
\]

where

\[
B_k^{n-1-m}(v_-, v_+) = \binom{n-1-m}{k} (F(v_+) - F(v_-))^k F^{n-1-m-k}(v_-).
\]
We now provide more convenient formulas for $Q$. For $w > v_+$, we have
\[ Q^e(w) = ((1 - \alpha) F(w) + \alpha)^{n-1}. \]

For $w \in [v_-, v_+]$, the trading probability $Q^e(w)$ plays a critical role in the analysis below, and for convenience we denote it as $\chi(v_-, v_+)$. We re-do the double summations over $m$ and $k$ by first summing over $k$ for fixed $l = m + k$ then summing over $l$, and rewrite $\chi(v_-, v_+)$ as
\[
\sum_{l=0}^{n-1} \binom{n-1}{l} (1 - \alpha) F(v_-)^{n-1-l} \frac{1}{l+1} \sum_{k=0}^{l} \binom{l}{k} ((1 - \alpha)(F(v_+) - F(v_-)))^{k} \alpha^{l-k}
= \sum_{l=0}^{n-1} \binom{n-1}{l} (1 - \alpha) F(v_-)^{n-1-l} \frac{1}{l+1} ((1 - \alpha)(F(v_+) - F(v_-)) + \alpha)^{l}.
\]

It follows that
\[
\chi(v_-, v_+) = \frac{(1 - \alpha)F(v_+) + \alpha)^n - ((1 - \alpha)F(v_-))^n}{n((1 - \alpha)(F(v_+) - F(v_-)) + \alpha)}.
\] (8)

The function $\chi$ gives the probability that a buyer whose valuation is in the pooling interval $[v_-, v_+]$ receives an offer. The logic in $\chi(v_-, v_+)$ is that the buyer has the same chance of receiving an offer as any of the unobservant buyers and any other observant buyer whose reported valuation is in the interval $[v_-, v_+]$, as long as there are no observant buyers with valuation above $v_+$. This explains why in the formula (8) the denominator is the expected number of buyers in the equal priority pool, and the numerator is the total probability that there is at least one buyer, observant or unobservant, with that priority.

The following result gives the necessary and sufficient condition for the equal priority mechanism $\{r,v_-,v_+;t\}$ to be incentive compatible.

**Lemma 1** The equal priority mechanism $\{r,v_-,v_+;t\}$ is incentive compatible if and only if
\[
\int_{r}^{v_-} (1 - \alpha)^{n-1} F^{n-1}(w)dw \geq \chi(v_-, v_+)(v_- - t)
\] (9)

Two arguments are needed to establish Lemma 1. The first is to show that the rules of allocation and offers are the ones that make truthful reporting incentive compatible by
observant buyers. Note that when observant buyers report their valuations truthfully, they accept their offers with probability one. Since the allocation rule is monotone, we just need to show that the payoff of observant buyers $U^\epsilon(w)$ from truthful reporting satisfies (5) for each $w$.$^{11}$

The second is to show that when $t$ satisfies condition (9) no observant buyer can improve his payoff by pretending to be unobservant. Since $Q^\epsilon(w) = \chi(v_-, v_\epsilon)$ for all $w \in [v_-, v_\epsilon]$, it follows from (5) that $U^\epsilon(w)$ is linear with slope $\chi(v_-, v_\epsilon)$. By construction, this is the same slope as the increasing part of the payoff function $U^\mu(w)$ for unobservant buyers, which is

$$U^\mu(w) = \chi(v_-, v_\epsilon) \max\{w - t, 0\},$$

because unobservant buyers have the same allocation priority as observant buyers whose valuations are in $[v_-, v_\epsilon]$. Moreover, since by construction $Q^\epsilon(w)$ is strictly increasing for $w \in [r, v_-)$ and $w > v_\epsilon$, it follows from (5) that the payoff function $U^\epsilon(w)$ is strictly convex for $w \geq r$ outside $[v_-, v_\epsilon]$. The equal priority auction $\{r, v_-, v_\epsilon; t\}$ is therefore incentive compatible if and only if $U^\epsilon(v_-) \geq U^\mu(v_-)$. This is precisely (9).

![Figure 1. An equal priority mechanism with a binding incentive condition.](image-url)

$^{11}$Indeed, the offer rule is constructed from the allocation rule of the equal priority mechanism to ensure that it is incentive compatible with respect to valuations.
Figure 1 above shows the payoffs to observant and unobservant buyers in an equal priority mechanism with a binding incentive compatible constraint (9). The green line represents the payoff function $U^\mu(\cdot)$ of an unobservant buyer or an observant buyer that acts as one. (It is zero for valuations below $t$.) The slope of the green line is $\chi(v_-, v_+).$ The red curve represents the payoff function $U^\epsilon(w)$ to an observant buyer. It coincides with the green line for valuations in the pooling interval $[v_-, v_+]$ because the incentive condition (9) is binding, and is strictly convex for valuations between $r$ and $v_-$, and above $v_+$. (It is equal to zero for valuations below $r$.)

In any equal priority mechanism, observant buyers with low valuations, between $r$ and $v_-$, and those with high valuations, above $v_+$, are strictly worse off by pretending to be unobservant. If the incentive compatibility constraint (9) is binding, it is a matter of indifference for observant buyers with valuations in $[v_-, v_+]$ whether they truthfully report their valuations or wait for the take-it-or-leave-it offer $t$ just like an unobservant buyer. Indeed, the same truth telling equilibrium among observant buyers is implemented if we change the offer rule, so that the offer received by an observant buyer with valuations in the pooling interval $[v_-, v_+]$ is always $t$, instead of the maximum of the second highest bid and reserve price $r$ when there are no other buyers in the equal priority pool, and $v_-$ when there is at least one unobservant buyer in the pool. Furthermore, by revenue equivalence, the seller’s revenue from observant buyers is the same if all observant buyers with valuations in the pooling interval $[v_-, v_+]$ behave in the same way as unobservant buyers. Since the allocation probability $q^\mu(v)$ and the offer $p^\mu(v)$ for unobservant buyers depend only on the size of the equality priority pool, i.e., $m + k$, and not on its composition, the seller’s revenue from unobservant buyers is also unaffected by whether or not observant buyers with valuations in $[v_-, v_+]$ pretend to be unobservant.

Any equal priority mechanism $\{r, v_-, v_+; t\}$ with a binding incentive condition (9) is therefore payoff-equivalent for all buyers and the seller to the following indirect mechanism. All buyers, observant or unobservant, are asked to place their bids; the seller reveals a random password; unobservant buyers do not know the password and their bids are treated as meaningless babbles; observant buyers can match the password and have their bids accepted as valid, except that those in the pooling interval $[v_-, v_+]$ are treated as babbles; the allocation
and offer rules otherwise mimic those in the equal priority mechanism, with \( l \) representing the total number of buyers who babble:

- When the highest bid is less than \( r \): the seller keeps the object if \( l = 0 \); otherwise, with probability \( 1/l \) the seller makes an offer \( t \) to each babbling buyer.

- When the highest bid is between \( r \) and \( v_- \): if \( l = 0 \), the bidder wins and pays the maximum of the second highest reported valuation and \( r \); if \( l \geq 1 \), with probability \( 1/l \), the seller makes an offer \( t \) to each babbling buyer.

- When the highest bid is above \( v_+ \): the bidder wins; he pays the second highest bid if it is above \( v_+ \), the maximum of \( r \) and the second highest bid if it is below \( v_- \) and \( l = 0 \), and otherwise \((v_- + lv_+)/l + 1)\).

The above indirect mechanism is what we refer to as the equal priority “auction” in the introduction. As in a standard auction, the winner is the one with the highest valid bid, and the auction commits all bidders to paying the price charged by the seller when they win. To bidders who submit valid bids, it looks like a second price auction: the winner pays the maximum of the second highest bid and a reserve price. A non-standard part is that bids in the interval \([v_-, v_+]\) are treated as uninformative. The other non-standard part is that the seller’s reserve price depends on the number \( l \) of babbling buyers: it is \( r \) when \( l = 0 \), and otherwise it is \((v_- + lv_+)/l + 1)\). To bidders in the auction, the reserve price is therefore “secret.” This is because the seller has the outside option of offering the object to a babbling buyer.

### 4.1 Optimal equal priority mechanism

Under an equal priority mechanism \( \{r, v_-, v_+; t\} \), the seller’s expected revenue from observant buyers is given by (6), with \( Q'(w) \) specified in (7), and the revenue from unobservant buyers is given by

\[
\sum_{m=1}^{n} B(m; n, \alpha) \sum_{k=0}^{n-m} B_k^{n-m}(v_-, v_+) \frac{m}{m+k} \pi(t) = n\alpha \chi(v_-, v_+) \pi(t). \tag{10}
\]
The optimal equal priority mechanism \( \{r, v_-, v_+; t\} \) maximizes the sum of (6) and (10) subject to \( r \leq v_- \leq v_+ \) and (9).

The following lemma characterizes optimal equal priority mechanisms. We assume that \( \pi(\cdot) \) is strictly concave. This implies that \( \phi(w) \) crosses 0 only once. Let the crossing point be \( r^* \); this is also the unique maximizer of \( \pi(w) \). Furthermore, \( \phi(w) \) is strictly increasing in \( v \) for \( w \geq r^* \). The valuation \( r^* \) represents the optimal reserve price in a standard auction, regardless of the number of buyers.\(^{12}\)

**Lemma 2** Suppose that \( \pi(\cdot) \) is strictly concave. If \( \{r, v_-, v_+; t\} \) is an optimal equal priority mechanism, then

\[
0 < r < r^* < t < v_- < v_+ < 1.
\]

Further, (9) holds with equality, and

\[
\alpha(\pi(t) - \phi(v_+)) = (1 - \alpha) \left( (v_- - t)(\phi(v_+) - \phi(v_-))f(v_-) + \int_{v_-}^{v_+} (\phi(v_+) - \phi(w))f(w)dw \right); \tag{11}
\]

\[
-\alpha \pi'(t) = (1 - \alpha)(\phi(v_+) - \phi(v_-))f(v_-); \tag{12}
\]

\[
-\phi(r)f(r) = (\phi(v_+) - \phi(v_-))f(v_-). \tag{13}
\]

Our proof (in the appendix) first uses variational arguments to establish that the optimal mechanism is interior, satisfying \( r < t < v_- < v_+ \). In particular, \( v_- < v_+ \), so that the pooling interval \([v_-, v_+]\) is non-degenerate as long as unobservant buyers are present in the model, i.e., \( \alpha > 0 \). If the interval were degenerate, then the seller could cut the offer \( t \) to unobservant buyers and pool observant buyers with them by decreasing \( v_- \) and increasing \( v_+ \). We show that cutting the price offer \( t \) has a first order revenue gain from unobservant buyers, and the corresponding pooling has only a second order revenue loss from observant buyers.

In any optimal equal priority mechanism, the incentive condition (9) for observant buyers

\(^{12}\)At any \( w \in (0, 1) \), if \( f(w) \) is non-decreasing, then by definition \( \phi(w) \) is strictly increasing; if \( f(w) \) is strictly decreasing at \( w \) and if \( \phi(w) \geq 0 \), then \( \phi(w) \) is strictly increasing in \( w \), because concavity of \( \pi(w) \) implies that \( \phi(w)f(w) \) is strictly increasing in \( w \).

\(^{13}\)In much of the auction literature, the seller has the fixed outside option of keeping the object. The virtual valuation function \( \phi(w) \) is assumed to be strictly increasing to simplify the analysis (the “regular case” in Myerson [1981]). In our model, the seller’s outside option in an auction with observant buyers is to give it to an unobservant buyer with a take-it-or-leave-it offer, and is endogenous because it is chosen by the seller. We do not need to assume that \( \phi(w) \) is strictly increasing for valuations below \( r^* \).
with valuations in the pooling interval \([v_-, v_+]\) is binding. Otherwise, in Figure 1 we would have a line segment in the payoff function \(U^*(\cdot)\) for observant buyers parallel to, and above, the linear part of the payoff function \(U^\mu(\cdot)\) for unobservant buyers. The seller would then want to either shrink the pooling interval, by increasing \(v_-\) and decreasing \(v_+\), or raise the take-it-or-leave-it offer \(t\) to unobservant buyers.

The three conditions (11), (12) and (13) are the first order conditions for an interior optimum.\(^{14}\) In an optimal equal priority mechanism, the reserve price \(r\) for selling to observant buyers when there are no unobservant buyers is set below the standard optimal reserve price \(r^*\) in the absence of unobservant buyers, as can be seen from (13). This sacrifices revenue when all observant buyers have low valuations and there are no unobservant buyers, but provides incentives for observant buyers to truthfully report their valuations instead of pretending to be unobservant. Correspondingly, (12) implies that the take-it-or-leave-it price \(t\) to unobservant buyers is raised above the optimal monopoly price \(r^*\) in the absence of observant buyers. This reduces the revenue when all buyers are unobservant, but provides disincentive for observant buyers to pretend to be unobservant.

If the seller does not give the object to an observant buyer, she can always make a take-it-or-leave-it offer to an unobservant buyer if there is one. Absent incentives, the seller would set the reserve price \(r(t)\) for observant buyers so that the virtual valuation is equal to the expected profit \(\pi(t)\) of making the offer \(t\) to an unobservant buyer:

\[
\phi(r(t)) = \pi(t).
\]

By condition (11), the optimal equal priority mechanism has \(\phi(v_+) < \pi(t)\). This means that the seller gives the object to observant buyers even though their virtual valuations are lower than the value of the seller’s “outside option” \(\pi(t)\). This reason for doing this is to provide incentives for truthful reporting by observant buyers with valuations just above \(v_+\) rather than wait for the take-it-or-leave-it offer by pretending to be unobservant.

\(^{14}\)They are all derived with variational arguments without explicitly using a multiplier for (9). For example, condition (11) is obtained by marginally changing \(v_-\) and \(v_+\) such that (9) is satisfied and then considering the effects on the seller’s revenue. From the proof in the appendix, it can be seen that the value of the multiplier associated with (9) is the right hand side of (12) multiplied by \(n\). This turns out to be the integral of the multiplier function \(\lambda(\cdot)\) in the proof of Theorem 2 over the valuation support \([0, 1]\).
When all buyers are surely observant the revenue from the optimal equal priority mechanism converges to the revenue from the standard auction with reserve price $r^*$, as it becomes optimal for the seller not to distort the reserve price $r$ at all to provide incentives (equation (13)). The pooling interval shrinks to a single valuation $v_0$ as $\alpha$ goes to 0,\footnote{The limit of $\chi(v_-, v_+) = \chi(v_-, v_+)$ as $\alpha$ goes to 0 \ and \ $v_-$ and \ $v_+$ shrink to the same point of $v_0$ is $F_{n-1}(v_0)$. That is, when all other buyers are almost surely observant, a deviating observant buyer will be the only buyer in the equal priority pool and will win the object with probability one if all other buyers (who are observant) have valuation below $v_0$.} satisfying the binding constraint (9) that an observant buyer with valuation $v_0$ is indifferent between truthfully reporting it and receiving a take-it-or-leave-it offer $t_0$ when all other buyers have valuations below $v_0$, 

$$\int_{r^*}^{v_0} F^{-1}(w) dw = F_{n-1}(v_0)(v_0 - t).$$

The limit values of $v_0$ and $t_0$ satisfy the above indifference condition and the limit version of first order conditions (11) and (12), given by

$$\pi'(t_0)(v_0 - t) + \pi(t_0) - \phi(v_0) = 0.$$

We have $t_0 > r^*$ and $\pi(t_0) > \phi(v_0)$. When $\alpha$ is arbitrarily close to 0, the incentives for observant buyers not to pretend to be unobservant are provided by raising the take-it-leave-it offer to an unlikely unobservant buyer above $r^*$, and not selling to unobservant buyers even when the profit from doing so exceeds virtual valuations of observant buyers.

In the opposite limit of $\alpha = 1$, buyers are surely unobservant, and the revenue from the optimal equal priority mechanism converges to the revenue from a take-it-or-leave-it offer $r^*$. By (12), the seller no longer distorts $t$ to provide incentives for observant buyers. From (11), the upper-bound of the pooling interval converges to $\pi(r^*)$, satisfying

$$\phi(\pi(r^*)) = \pi(r^*),$$

as the need for the seller to provide incentives for observant buyers with valuations just above the upper-bound becomes second order. From the binding constraint (9), the lower-bound of the pooling interval becomes $r^*$.\footnote{The limit of $\chi(v_-, v_+)$ as $\alpha$ goes to 1 is $1/n$, as an unlikely observant buyer will surely face $n - 1$} This is to prevent an unlikely observant buyer with
a valuation equal to the lower bound from pretending to be unobservant, as the buyer has
close to zero chance of making the winning bid with the limit reserve price \( r_1 \) satisfying (13)

\[-\phi(r_1)f(r_1) = \pi(r^*)f(r^*).\]

As long as \( \alpha \) is strictly less than 1, however, the mechanism is what provides incentives for
observant buyers with valuations just below the lower bound of the interval not to pretend
to be unobservant.

### 4.2 Optimal direct mechanisms

We want to show that an optimal equal priority mechanism provides the seller the highest
expected revenue among all direct mechanisms. Optimizing over all incentive compatible
direct mechanisms is difficult, due to the continuum of incentive constraints for observant
buyers with any valuation \( w \) not to pretend to be unobservant. Instead we adopt an indirect
approach, by incorporating the continuum of constraints through a multiplier function. This
is known as the Lagrangian relaxation method.

Recall that a direct mechanism \( \delta \) consists of a series of functions \( (q^\epsilon_m(v), p^\epsilon_m(v))_{m=0}^{n-1} \) and
\( (q^\mu_m(v), p^\mu_m(v))_{m=1}^n \). We first use the assumption that \( \pi(\cdot) \) is strictly concave to simplify
the optimal design problem. Replacing all these offers with the expected offer reduces the
deviation payoff to observant buyers from pretending to be un observant. Concavity then
implies a greater revenue from unobservant buyers.

**Lemma 3** If \( \pi(\cdot) \) is strictly concave, then in any optimal direct mechanism, \( p^\mu_m(v) \) is inde-
pendent of \( m \) and \( v \).

Using Lemma 3, we denote the constant price offered to the unobservant as \( p^\mu \). Define

\[ Q^\mu = \sum_{m=0}^{n-1} B(m; n - 1, \alpha) E_v[q^\mu_{m+1}(v)] \]

to be the total probability of an offer expected by an unobservant buyer (or a deviating
unobservant buyers in the equal priority pool after pretending to be unobservant.
observant buyer).

Next, we drop the transfers \((p'_m(v))^{n-1}_{m=0}\) to observant buyers, and construct the relaxed Lagrangian using only allocations \((q'_m(v))^{n-1}_{m=0}\). Once we show that an optimal equal priority \(\{r, v_-, v_+; t\}\) solves the relaxed Lagrangian, we can then use the offer rule in section 4 to construct the transfers \((p'_m(v))^{n-1}_{m=0}\) and the resulting payoff function \(U^\epsilon(\cdot)\), and apply Lemma 1 to conclude that the solution is incentive compatible.

We are thus led to the following maximization problem: Choose \((q'_m(v))^{n-1}_{m=0}\), \((q^\mu_m(v))^{n}_{m=1}\), and \(p^\mu\) to maximize

\[
n(1 - \alpha) \int_0^1 Q^\epsilon (w) \phi(w)f(w)dw + n\alpha Q^\mu \pi (p^\mu),
\]

subject to the feasibility constraint (4), \(Q^\epsilon (\cdot)\) is non-decreasing, and for every \(w\),

\[
\int_0^w Q^\epsilon (x) dx \geq Q^\mu \max \{w - p^\mu, 0\}. \tag{14}
\]

Let \(\lambda(\cdot)\) be an arbitrary non-negative valued Lagrangian function from \([0, 1]\) into \(\mathbb{R}\). The relaxed problem is to maximize

\[
n(1 - \alpha) \int_0^1 Q^\epsilon (w) \phi(w)f(w)dw + n\alpha Q^\mu \pi (p^\mu) \\
+ \int_0^1 \lambda (w) \left( \int_0^w Q^\epsilon (x) dx - Q^\mu \max \{w - p^\mu, 0\} \right) dw,
\]

with the same choice variables and constraints except (14). That is, by introducing the Lagrangian function, we incorporate a continuum of constraints (14) into the objective function of the relaxed problem as an extra term.

The above relaxed problem has different solutions depending on the choice of \(\lambda(\cdot)\). Regardless of the choice of \(\lambda(\cdot)\), however, the value of the relaxed problem is an upper bound on the value of the full problem, because the solution to the full problem is feasible for the relaxed problem and because the extra term in the objective function of the relaxed problem is non-negative by construction. We will try to construct a function \(\lambda (\cdot)\) such that the solution to the relaxed problem is an optimal equal priority mechanism. Since the equal priority
mechanism yields an upper bound on the seller’s revenue in the full problem, and since it satisfies all the constraints in the full problem, it solves the full problem.

The multiplier function $\lambda(\cdot)$ is the shadow cost (benefit) of violating (relaxing) the constraints (14). The second term in the relaxed Lagrangian is the total shadow value. The relaxed problem is then choosing feasible allocations $(q^v_m(v))_{m=0}^{n-1}$ and $(q^u_m(v))_{m=1}^n$, together with $p^\mu$, to maximize the sum of the resulting revenues from observant and unobservant buyers and the shadow values. The key to our construction of the desired $\lambda(\cdot)$ is that, first, it satisfies complementary slackness so that the extra term in the relaxed Lagrangian is zero; and second, the allocations of an optimal equal priority auction characterized by Lemma 2 maximize the sum of the revenues and the shadow values. More precisely, we use integration by parts and rewrite the Lagrangian as

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha) \int_0^1 \left( (n-1-\alpha) \phi(w) f(w) + \int_w^1 \lambda(x) dx \right) Q^v_m(w) dw$$

$$+ \sum_{m=0}^{n-1} B(m; n-1, \alpha) \left( n\alpha \pi(p^\mu) - \int_0^1 \lambda(w) max\{w-p^\mu, 0\} dw \right) Q^\mu_{m+1}.$$ 

We want to choose $\lambda(\cdot)$ to have the following properties: (i) It is equal to 0 outside of $[v_-, v_+]$ where the constraint (14) is slack. (ii) It is non-negative on $[v_-, v_+]$ and makes the expression in the first bracket in the above Lagrangian constant, so that it is point wise maximizing to have constant $Q^v_m(w)$ for all $w \in [v_-, v_+]$. (iv) The constant value of the expression in the first bracket in the above Lagrangian matches the constant value of the expression in the second bracket, so that it is point wise maximizing to give the same allocation priority to observant buyers with valuations in the pooling interval and unobservant buyers. (iv) The value of the expression in the first bracket is strictly increasing and greater than that in the second bracket for $w > v_+$, and increasing and smaller for $w < v_-$, so that observant buyers have increasing and higher priorities than unobservant buyers if their valuations are higher than $v_+$, and increasing and lower priorities if their valuations are lower than $v_-$. 

**Theorem 2** Suppose that $\pi(\cdot)$ is strictly concave. Then, there is no incentive compatible direct mechanism that yields a strictly greater revenue than an optimal equal priority mechanism.
Putting together Theorems 2 and 1, we have shown that when $\pi(\cdot)$ is concave, the outcome of a symmetric equilibrium of the game $G(\alpha)$ where unobservant buyers babble corresponds to an optimal equal priority mechanism. Conversely, once we solve for the optimal equal priority mechanism, we can construct an equal priority auction with password to support a symmetric equilibrium of the game as discussed at the end of section 4. Since equal priority mechanisms are relatively straightforward to describe and optimize over, our result provides a simple characterization of equilibrium outcomes of the unobserved mechanism design game in the important class where unobservant buyers babble.\(^\text{17}\)

The relative simplicity of optimal equal priority mechanisms also allows us to understand welfare implications of unobserved mechanism design. The seller is of course worse off compared to when all buyers are observant, as unobservability reduces the power of commitment necessary for standard optimal auctions. This means that the seller has incentives to “educate” buyers about her mechanism. But such attempt would be thwarted so long as the commitments in the mechanism remain unverifiable.

When all $n$ buyers are surely observant, they face the standard optimal reserve price of $r^*$. In a symmetric equilibrium of the unobserved mechanism design game $G(\alpha)$ with $\alpha > 0$, the seller sets $r < r^*$, so an observant buyer with a valuation between $r$ and $r^*$ is better off than when there are no unobservant buyers around. Observant buyers with higher valuations are affected by the presence of unobservant buyers in two opposing ways: they can win even though some unobservant buyer has a higher valuation, but they may also lose to an unobservant with a lower valuation. The net effect is generally ambiguous, but we can show that observant buyers with sufficiently high valuations benefit from having unobservant buyers around if the number of buyers is sufficiently large.\(^\text{18}\)

\(^{17}\)Indeed, the first order conditions (11), (13) and (12), together with the binding constraint (9), are sufficient as well as necessary for an optimal equal priority mechanism. The sufficiency comes from the fact that the proof of Theorem 2 uses only the first order conditions. That is, Theorem 2 actually shows that an equal priority mechanism that satisfies the first order conditions are optimal among all direct mechanisms, and a fortiori, optimal among all equal priority mechanisms.

\(^{18}\)To see this, note that

$$U'(1) = \int_r^1 Q'(w)dw > \int_{v_+}^1 ((1 - \alpha)F(w) + \alpha)^{n-1}dw.$$  

The above is greater than $\int_r^1 F^{n-1}(w)dw$ when $n$ is sufficiently large, because by integration by parts, it is
For unobservant buyers, the relevant welfare comparison question is how they are affected by the presence of observant buyers. If there are no observant buyers, unobservant buyers have an equal chance of receiving a take-it-or-leave-it offer equal to \( r^* \). Since in a symmetric babbling equilibrium of \( \mathcal{G}(\alpha) \) the seller sets the take-it-or-leave-it offer \( t \) strictly above \( r^* \), an unobservant buyer with a valuation \( w \) just above \( r^* \) is worse off in equilibrium than when there are no observant buyers around. For unobservant buyers with higher valuations, they have a higher priority than observant buyers with valuations below \( v_- \), which makes them better off in equilibrium, but lose out to observant buyers with valuations above \( v_+ \). The net effect is again ambiguous, but we can show that unobservant buyers are all worse off in equilibrium than when there are no observant buyers if the number of buyers is large.\(^\text{19}\)

5 Discussion

We have assumed that the output of the seller’s mechanism is a single take-it-or-leave-it offer in the unobserved mechanism design game. If this offer is rejected, which it will sometimes be if it is made to an unobservant bidder, the game ends without trade. We view this game form as a “canonical” one, because under the standard mechanism design problem with observant agents, there is no benefit for the mechanism to generate more offers. As we have mentioned in the introduction, a separate motivation for this particular game form is that we can obtain empirically relevant results through equal priority auctions.

The assumption of a single take-it-or-leave-it offer is without loss for observant buyers, since, as in a standard auction, they will always want to accept the offer when they are made one. For the unobservant this assumption is perhaps unrealistic. Once the seller learns who the unobservant buyers are, the seller is likely to approach them in sequence with offers. One question is how this might change if the seller could follow up a rejection by making a

implied by

\[
(1 - \alpha) \int_{v_-}^{1} ((1 - \alpha)F(w) + \alpha)^{n-2} f(w)wdw < \int_{r^*}^{1} F^{n-2}(w)f(w)wdw,
\]

which is true for large enough \( n \) by using another integration by parts.

\(^\text{19}\)We have

\[
U'^{\text{h}}(1) = \chi(v_-, v_+)(1 - t) < ((1 - \alpha)F(v_+) + \alpha)^{n-1}(1 - r^*),
\]

The above is less than \((1 - r^*)/n\) when \( n \) is sufficiently large. Since the payoff functions are piece wise linear, an unobservant buyer with any valuation is worse off in equilibrium.
possibly lower offer to one of the other unobservant bidders.

A general approach to unobserved mechanisms is to model the output of a mechanism as an “algorithm,” which is a sequence of take-it-or-leave-it offers and the identities of the buyers to whom the offers are made. As in the present model, the seller first makes a commitment in terms of how a particular sequence of offers is chosen in response to the messages sent by the buyers, who however may not observe it. It is straightforward to generalize the analysis in the present paper to the case in which algorithms are restricted to at most one take-it-or-leave-it offer for each buyer, and unobservant buyers babble. The main insights are intact - an unobservant buyer receives an expected offer independent of the buyer’s valuation, while observant buyers face an outside option of waiting for their turn to receive an offer if they decide to pretend to be unobservant. We conjecture that the equilibrium outcome with babbling by unobservant buyers can be characterized by a similar equal priority mechanism as in the present model, with the single offer to unobservant buyers replaced with a decreasing sequence of offers. The seller’s equilibrium revenue should be higher than the present single-offer model, because being able to make a sequence of offers improves the seller’s revenue from unobservant buyers, without necessarily increasing the value of outside option to observant buyers who pretend to be unobservant.

A more challenging question with multiple offers arises if the seller’s algorithm is not restricted to at most one take-it-or-leave-it offer for each buyer. Since an unobservant buyer does not observe the seller’s deviations to other algorithms, rejecting an offer from the seller could reveal information about his valuation that could be exploited later by the seller. Yet we can make one observation. The equilibrium when the seller’s algorithm is restricted to at most one take-it-or-leave-it offer for each buyer can be supported as an equilibrium when the algorithms are unrestricted. Imagine that an unobservant buyer disappears after rejecting an offer, believing the seller’s algorithm makes at most one offer to each buyer. Given this belief by unobservant buyers, committing to an algorithm that potentially makes multiple offers to a given buyer would only affect the behavior of observant buyers. This then becomes unprofitable because observant buyers observe the seller’s commitment.

There may be other equilibria when the seller’s algorithm is not restricted to at most one take-it-or-leave-it offer for each buyer. It would be interesting to find out if any of these
equilibria makes the seller better off compared to the equilibrium when the seller can make at most one offer to a buyer. We defer these questions to future research since it not clear at this point what is the best way to generalize to multiple offers to each buyer.

We have assumed that buyers are either fully observant or fully unobservant. A more reasonable assumption might be that buyers have partial information about commitments. For example, we could assume that some buyers may only be able to understand commitments to actions based on their own messages, but not commitments that depend on the messages of others. If all buyers have this type of partial information, then there is an equilibrium in which the seller implements the optimal auction of Myerson [1981] through a first-price sealed bid auction. This corresponds to the main result of Akbarpour and Li [2020]), who frame the issue of partial observability in terms of limited commitment by the seller. When buyers have differential information about the seller’s commitments - for example, if buyers either fully observe the seller’s commitment or only observe the part based on their own message - we nonetheless believe that our basic insight could be extended to this kind of assumption. Yet we are reluctant to pursue without a better model of what buyers can and cannot understand.

6 Appendix: Omitted Proofs

Proof of Theorem 1

Recall that a perfect Bayesian equilibrium in $G(\alpha)$ is given by some mixture $\psi$ for the seller, and strategy rules $(\sigma_i(\cdot, \epsilon, \gamma), \sigma_i(\cdot, \mu))_{i=1}^n$ for observant and unobservant buyers respectively. By the assumption of symmetry, we can drop the subscript $i$ from $\sigma_i(\cdot, \epsilon, \gamma)$ for observant buyers. Since unobservant buyers babble, their messages won’t affect any outcomes can be implemented without these message. We ignore $\sigma_i(\cdot, \mu)$ entirely.

For each $\gamma$ in the support of $\psi$, we reorder the $n$ buyers such that the first $n-m$ of them are observant. By symmetry of the equilibrium, we can use the same permutation device $\rho$ in direct mechanisms on messages from observant buyers to rewrite $\gamma$. Denote as $\tilde{q}_m^\mu(b, \gamma)$ the probability with which an offer $\tilde{p}_m^\mu(b, \gamma)$ is made to an unobservant buyer given that
there are \( m \) unobservant buyers and the profile of \( n - m \) messages from observant buyers is \( b = \{ b_1, \ldots, b_{n-m} \} \). The function \( \tilde{q}_m^\mu(b, \gamma) \) gives the probability with which an offer \( \tilde{\nu}_m^\epsilon(b, \gamma) \) is made to the buyer with the first value in \( b \), given that there are \( m \) unobservant buyers and the other \( n - m - 1 \) observant buyers send messages \( b_{-1} = \{ b_2, \ldots, b_{n-m} \} \). Feasibility of \( \gamma \) requires \( \sum_{i=1}^{n-m} \tilde{q}_m^\mu(\rho_m^\epsilon(b), \gamma) + m \tilde{q}_m^\mu(b, \gamma) \leq 1 \) for all \( m \).

Fix any \( \gamma \) in the support of \( \psi \). Define a direct mechanism \( \delta^* = \{(q_m^\mu, p_m^\epsilon)^{n-1}_{m=0}, (q_m^\mu, p_m^\epsilon)^n_{m=1}\} \). For each \( m = 1, \ldots, n \) and \( v = (v_1, \ldots, v_n) \), let \( q_m^\mu(v) = \tilde{q}_m^\mu(\sigma(v_1, \epsilon, \gamma), \ldots, \sigma(v_{n-m}, \epsilon, \gamma), \gamma) \), and \( p_m^\epsilon(v) = \tilde{p}_m^\epsilon(\sigma(v_1, \epsilon, \gamma), \ldots, \sigma(v_{n-m}, \epsilon, \gamma), \gamma) \); for each \( m = 0, \ldots, n-1 \) and \( v = (v_1, \ldots, v_n) \), if \( \tilde{\nu}_m^\epsilon(\sigma(v_1, \epsilon, \gamma), \ldots, \sigma(v_{n-m}, \epsilon, \gamma), \gamma) \geq v_1 \), let \( q_m^\mu(v) = \tilde{q}_m^\mu(\sigma(v_1, \epsilon, \gamma), \ldots, \sigma(v_{n-m}, \epsilon, \gamma), \gamma) \) and \( p_m^\epsilon(v) = \tilde{q}_m^\mu(\sigma(v_1, \epsilon, \gamma), \ldots, \sigma(v_{n-m}, \epsilon, \gamma), \gamma) \), and otherwise let \( q_m^\mu(v) = 0 \).

It is easy to see that \( \delta^* \) is incentive compatible in that truthful reporting of valuations is a Bayesian Nash equilibrium among observant buyers. The feasibility constraint (4) is satisfied. The direct mechanism achieves the same revenue as \( \gamma \).

In any equilibrium of \( \mathcal{G}(\alpha) \), the seller’s revenue is the same for each realization \( \gamma \) in the support of \( \psi \). Thus the expected revenue \( R(\delta^*) \) from \( \delta^* \) achieves the same equilibrium revenue for the seller. Further, there is no incentive compatible direct mechanism \( \delta \) that achieves a strictly revenue \( R(\delta) \) than \( R(\delta^*) \). If there were, then given that unobservant buyers babble and observant buyers can condition their strategies on the mechanism, the seller would deviate to the more profitable direct mechanism \( \delta \). This is possible because by assumption the message space \( \mathcal{M} \) in \( \mathcal{G}(\alpha) \) is assumed to be sufficiently rich to embed the support of valuations. This contradicts the equilibrium condition (3) for the seller.

The reverse direction of Theorem 1 follows by constructing a symmetric equilibrium of \( \mathcal{G}(\alpha) \) using password mechanisms that are derived from an optimal direct mechanism \( \delta^* = \{(q_m^\mu, p_m^\epsilon)^{n-1}_{m=0}, (q_m^\mu, p_m^\epsilon)^n_{m=1}\} \). The seller’s equilibrium strategy \( \psi \) is a mixture \( \psi \) over password mechanisms \( \gamma(\zeta) \), where each \( \zeta \) is a realization of a uniform random variable on \([0, 1]\). Each password mechanism \( \gamma(\zeta) \) has message space \([0, 1]^2\), with the first component representing a report of the password and the second component representing a report of the valuation. Given a realized password mechanism \( \gamma(\zeta) \), the equilibrium strategy of an observant buyer \( i \) with valuation \( v_i \) is to send message \((\zeta, v_i)\). The equilibrium strategy of an unobservant buyer with any valuation is a pair of independent and random draws.
from the uniform distribution over $[0, 1]$. For each password mechanism $\gamma(\zeta)$, the trading probabilities and offers $(\hat{q}_i, \hat{p}_i)_{i=1}^n$ are derived from $\delta^*$ as follows. For each profile of messages $(b_1, \ldots, b_n) = ((z_1, v_1), \ldots, (z_n, v_n))$, let $m = \#\{i : z_i \neq \zeta\}$, and reorder the buyers so that $z_i = \zeta$ for each $i = 1, \ldots, n - m$. Define $\hat{q}_i(b, \zeta) = q^\epsilon_m(v), \hat{p}_i(b, \zeta) = p^\epsilon_m(v)$ for each $i = 1, \ldots, n - m$, and $\hat{q}_i(b, \zeta) = q^\mu_m(v), \hat{p}_i(b, \zeta) = p^\mu_m(v)$ for each $i = n - m + 1, \ldots, n$.

Now we verify that the strategies of the seller, observant and unobservant buyers form an equilibrium. Since $\delta^*$ is incentive compatible, the equilibrium condition (1) for observant buyers is satisfied for any $\gamma(\zeta)$. Given that the seller ignores any valuation report by buyer $i$ when $i$ does not match the realized password $\zeta$, it is an optimal response for unobservant buyers to report a random number from $[0, 1]$ as his valuation since he does not observe the password. The equilibrium condition (2) for unobservant buyers is satisfied. Finally, by construction the seller gets the same revenue from each $\gamma(\zeta)$. There is no other mechanism $\gamma$ that gives the seller a strictly higher revenue, given that unobservant buyers babble. Any revenue achieved by $\gamma$ can be replicated by a direct mechanism, and so the optimality of $\delta^*$ among direct mechanisms implies that the equilibrium condition (3) for the seller is satisfied.

**Proof of Lemma 1**

We verify that the expected payoff of an observant buyer with valuation $w$ matches $U^\epsilon(w)$ given by (5) and (7). There are four cases.

(i) By truthfully reporting his valuation, an observant buyer with $w < r$ never wins the object, and thus the expected payoff is 0, matching $U^\epsilon(w)$ in (7) and (5) for $w < r$.

(ii) By truthful reporting, an observant buyer with $w \in [r, v_-)$ wins the object only when $m = 0$ and all $n - 1$ other observant buyers have valuation at most $w$, pays the maximum of $r$ and the second highest valuation. Thus, the expected payoff is

$$w(1 - \alpha)^{n-1}F^{n-1}(w) - \left(r(1 - \alpha)^{n-1}F^{n-1}(r) + \int_r^w x \, d\left((1 - \alpha)^{n-1}F^{n-1}(x)\right)\right).$$

By integration by parts, the above matches $U^\epsilon(v)$ in (5) and (7) for $v \in [r, v_-)$.

(iii) By truthful reporting, an observant buyer with $w \in [v_-, v_+]$ wins the object with probability one when $m = 0$ and all $n - 1$ other observant buyers have valuation at most
event to the buyer’s expected payoff is

\[ w(1 - \alpha)^{n-1} F^{n-1}(v_-) - \left( r(1 - \alpha)^{n-1} F^{n-1}(r) + \int_r^{v_-} x d((1 - \alpha)^{n-1} F^{n-1}(x)) \right), \]

which equals \( U^\epsilon(v_-) + (w - v_-)(1 - \alpha)^{n-1} F^{n-1}(v_-) \). The buyer also wins the object with probability \( 1/(m+k+1) \) when there are \( m \) unobservant buyers, all \( n-m-1 \) other observant buyers have valuation at most \( v_+ \), and \( m+k \) is at least 1 (where \( k \) is the number of observant buyers with valuation on \([v_-, v_+])\), and pays \( v_- \). The contribution of this event to the buyer’s expected payoff is \((w - v_-) (\chi(v_-, v_+) - (1 - \alpha)^{n-1} F^{n-1}(v_-)) \). The sum of the above two expressions matches \( U^\epsilon(w) \) in (5) and (7) for \( w \in [v_-, v_+] \).

(iv) By truthful reporting, an observant buyer with \( w > v_+ \) wins the object with probability one when \( m = 0 \) and the second highest bid is below \( v_- \), and he pays the maximum of the second highest bid and the reserve price \( r \). The contribution to the expected payoff is \( U^\epsilon(v_-) + (w - v_-)(1 - \alpha)^{n-1} F^{n-1}(v_-) \). He also wins with probability one when the second highest bid is below \( v_+ \) and \( m+k \geq 1 \), and pays \((v_- + v_+(m+k))/(m+k+1)\). The contribution to the expected payoff is

\[
\sum_{m=0}^{n-1} \sum_{k=0}^{n-1-m} B(m; n-1, \alpha) B_k^{n-1-m}(v_-, v_+) \left( w - \frac{v_- + v_+(m+k)}{m+k+1} \right) - (w - v_-)(1 - \alpha)^{n-1} F^{n-1}(v_-) \]

\[= (w - v_+)((1 - \alpha) F(v_+) + \alpha)^{n-1} + (v_+ - v_-) \chi(v_-, v_+) - (w - v_-)(1 - \alpha)^{n-1} F^{n-1}(v_-). \]

Finally, the observant buyer with \( w > v_+ \) wins with probability one and pays the second highest bid \( x \) when it is above \( v_+ \), which occurs with probability \( \sum_{m=0}^{n-1} B(m; n-1, \alpha)(F^{n-1-m}(x) - F^{n-1-m}(v_+)) \). By integration by parts, the contribution is

\[
\int_{v_+}^{w} \sum_{m=0}^{n-1} B(m; n-1, \alpha)(F^{n-1-m}(x) - F^{n-1-m}(v_+))dx \]

\[= \int_{v_+}^{w} \sum_{m=0}^{n-1} B(m; n-1, \alpha)F^{n-1-m}(x)dx - (w - v_+)((1 - \alpha) F(v_+) + \alpha)^{n-1}. \]

The sum of the three expressions above matches \( U^\epsilon(w) \) in (5) and (7) for \( w > v_+ \).
Proof of Lemma 2

Fix an incentive compatible, optimal equal priority mechanism \( \{r, v_-; t\} \) with \( r \leq v_- \leq v_+ \). When \( r \leq t \leq v_- \), define

\[
D = U^c(v_-) - U^u(v_-) = \int_r^{v_-} (1 - \alpha)^{-1} F^{n-1}(w) dw - \chi(v_-, v_+)(v_- - t),
\]

and let \( R \) be the revenue, which is the sum of (6) and (10). If \( 0 < r < v_- \), or if \( 0 = r < v_- \) and \( dr > 0 \), or if \( 0 < r = v_- \) and \( dr < 0 \), we have

\[
\frac{\partial D}{\partial r} = -(1 - \alpha)^{-1} F^{n-1}(r); \quad \frac{\partial R}{\partial r} = -n(1 - \alpha)^{n-1}(r)\phi(r)f(r).
\]

If \( 0 < t < v_- \), or \( 0 = t < v_- \) and \( dt > 0 \), or \( 0 < t = v_- \) and \( dt < 0 \), we have

\[
\frac{\partial D}{\partial t} = \chi(v_-, v_+); \quad \frac{\partial R}{\partial t} = n\alpha\chi(v_-, v_+)\pi'(t).
\]

If \( t < v_- < v_+ \), or if \( t = v_- < v_+ \) and \( dv_- > 0 \), or \( t < v_- = v_+ \) and \( dv_- < 0 \), we have

\[
\frac{\partial \chi(v_-, v_+)}{\partial v_-} = \frac{(1 - \alpha)f(v_-)}{(1 - \alpha)(F(v_+ - F(v_-)) + \alpha(\chi(v_-, v_+) - ((1 - \alpha)F(v_-))^{n-1})};
\]

\[
\frac{\partial D}{\partial v_-} = (1 - \alpha)^{-1} F^{n-1}(v_-) - \chi(v_-, v_+) - \frac{\partial \chi(v_-, v_+)}{\partial v_-}(v_- - t);
\]

\[
\frac{\partial R}{\partial v_-} = n(1 - \alpha)((1 - \alpha)^{-1} F^{n-1}(v_-) - \chi(v_-, v_+))\phi(v_-)f(v_-)
\]

\[
+ n((1 - \alpha)(\pi(v_-) - \pi(v_+)) + \alpha\pi(t))\frac{\partial \chi(v_-, v_+)}{\partial v_-}.
\]

If \( v_- < v_+ < 1 \), or if \( v_- = v_+ < 1 \) and \( dv_+ > 0 \), or if \( v_- < v_+ = 1 \) and \( dv_+ < 0 \), we have

\[
\frac{\partial \chi(v_-; v_+)}{\partial v_+} = \frac{(1 - \alpha)f(v_+)}{(1 - \alpha)(F(v_+ - F(v_-)) + \alpha)((1 - \alpha)F(v_+ + \alpha)^{n-1} - \chi(v_-, v_+));
\]

\[
\frac{\partial D}{\partial v_+} = -\frac{\partial \chi(v_-; v_+)}{\partial v_+}(v_- - t);
\]

\[
\frac{\partial R}{\partial v_+} = n(1 - \alpha)(\chi(v_-, v_+) - ((1 - \alpha)F(v_+ + \alpha)^{n-1})\phi(v_+)))f(v_+)
\]

\[
+ n((1 - \alpha)(\pi(v_-) - \pi(v_+)) + \alpha\pi(t))\frac{\partial \chi(v_-, v_+)}{\partial v_+}.
\]
The proof of the lemma is divided into seven steps.

(i) We claim that \( r \leq t \leq v_- \). We can rule out \( t < r \) right away, because it violates (9). To rule out \( t > v_- \), note that in this case (9) is slack. From the expression of \( \partial R/\partial t \), concavity of \( \pi(\cdot) \) and the optimality of \( \{r,v_-;v_+;t\} \) together imply that \( t = r^* \). If \( r < v_- \), then since \( v_- < t = r^* \), we have \( r < r^* \). From the expression of \( \partial R/\partial r \), a marginal increase in \( r \) would increase \( (6) \), contradicting the optimality of \( \{r,v_-;v_+;t\} \). Thus, \( r = v_- \). If \( v_- < v_+ \), then from the expression of \( \partial R/\partial v_- \), a marginal increase in \( v_- \) would increase the revenue, contradicting the assumption of optimality. Thus, \( r = v_- = v_+ < t = r^* \). From the expressions of \( \partial R/\partial v_- \) and \( \partial R/\partial v_+ \), a increase in \( v_- \) and \( v_+ \) by the same marginal amount would increase the revenue, a contradiction. Thus, \( t \leq v_- \).

(ii) We claim that \( r < t < v_- \). We can rule out \( r = t < v_- \) right away, because it violates (9). To rule out \( r < t = v_- \), note that in this case (9) is slack. Since \( r < t \), either \( r < r^* \) or \( t > r^* \), or both. If \( r < r^* \), then by raising \( r \) marginally, the seller could increase the revenue because \( \partial R/\partial r > 0 \). If \( t > r^* \), then by lowering \( t \) marginally, the seller could increase the revenue because \( \partial R/\partial t < 0 \). Either way, we have a contradiction to the assumption of optimality. Finally, we rule out \( r = t = v_- \). If \( r = t = v_- < r^* \), then by raising \( t \) marginally, the seller relaxes (9), and increases the revenue because \( \partial R/\partial t > 0 \). If \( r = t = v_- > r^* \), then by lowering \( r \) marginally, the seller relaxes (9), and increases the revenue because \( \partial R/\partial r < 0 \). If \( r = t = v_- = r^* \), then by lowering \( r \) marginally, the seller relaxes (9) because \( \partial D/\partial r < 0 \), without changing the revenue because \( \partial R/\partial r = 0 \). With (9) slack, the seller could then increase the revenue by either further raising \( v_- \) marginally if \( v_- = r^* < v_+ \), because \( \phi(v_-) = 0 \) implies \( \partial R/\partial v_- > 0 \), or by raising both \( v_- \) and \( v_+ \) by the same infinitesimal amount if \( v_- = v_+ = r^* \), because \( \partial R/\partial v_- + \partial R/\partial v_+ > 0 \). In each case, we have a contradiction to the assumption of optimality.

(iii) We claim that \( r < t < v_- < v_+ \). Suppose instead \( v_- = v_+ = \hat{w} \), and consider decreasing both \( v_- \) and \( v_+ \) by the same marginal amount. We have \( \partial D/\partial v_- + \partial D/\partial v_+ < 0 \), and \( \partial R/\partial v_- + \partial R/\partial v_+ \) has the same sign as \( \pi(t) - \phi(\hat{w}) \). Thus, we must have \( \pi(t) > \phi(\hat{w}) \); otherwise, the seller relaxes (9) without decreasing the revenue, which would then allow the seller to increase the revenue by either raising \( r \) or lowering \( t \), as \( r < t \) implies \( r < r^* \) or \( t > r^* \), or both. Since \( \phi(1) = 1 \), it follows from \( \pi(t) > \phi(\hat{w}) \) that \( \hat{w} < 1 \). Consider perturbing
the equal priority mechanism by reducing $v_-$ from $\hat{w}$ and raising $v_+$ from $\hat{w}$ such that

$$-(\chi(\hat{w}, \hat{w}) - (1 - \alpha)^{n-1}F^{n-1}(\hat{w}))dv_- = \left(\left((1 - \alpha)F(\hat{w}) + \alpha\right)^{n-1} - \chi(\hat{w}, \hat{w})\right)dv_+.$$  

By construction, (9) is relaxed, because

$$\frac{\partial D}{\partial v_-}dv_- + \frac{\partial D}{\partial v_+}dv_+ = ((1 - \alpha)^{n-1}F^{n-1}(\hat{w}) - \chi(\hat{w}, \hat{w}))dv_-,$$

which is strictly positive. The seller’s revenue is unchanged, because

$$\frac{\partial R}{\partial v_-}dv_- + \frac{\partial R}{\partial v_+}dv_+ = n(1 - \alpha)f(\hat{w}) \left(\chi(\hat{w}, \hat{w}) - (1 - \alpha)^{n-1}F^{n-1}(\hat{w})\right) (\pi(t) - \phi(\hat{w}))dv_-.$$

which is equal to 0 by construction. The seller could now increase the revenue by either raising $r$ or lowering $t$, as $r < t$ implies $r < r^*$ or $t > r^*$, or both. This contradicts the assumption of optimality.

(iv) We claim that (9) binds, $r < r^* < t$, and $\pi(t) > \phi(v_+)$. If (9) is slack, then since $r < t$ implies that $r < r^*$ or $t > r^*$, or both, the seller could increase the revenue by either raising $r$ or lowering $t$, a contradiction to the assumed optimality. If $r^* \leq r < t$, the seller could relax (9) by lowering $r$ marginally without decreasing the revenue, which then would allow the seller to increase the revenue by lowering $t$. Similarly, if $r < t \leq r^*$, the seller could relax (9) by raising $t$ marginally without decreasing the revenue, which then would allow the seller to increase the revenue by raising $r$. Finally, we show that $\pi(t) > \phi(v_+)$. Otherwise, by lowering $v_+$ marginally, the seller relaxes (9) because $\partial D/\partial v_+ < 0$, and increases the revenue, as $\partial R/\partial v_+$ has the same sign as

$$\alpha(\pi(t) - \phi(v_+)) + (1 - \alpha)(\pi(v_-) - \pi(v_+)) - \phi(v_+)(F(v_+) - F(v_-))$$

$$= \alpha(\pi(t) - \phi(v_+)) - \int_{v_-}^{v_+} (\phi(v_+) - \phi(w))f(w)dw,$$

which is strictly less than $\alpha(\pi(t) - \phi(v_+))$, contradicting the assumed optimality. Note that $\pi(t) > \phi(v_+)$ implies $v_+ < 1$.  

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(v) To obtain (11), consider perturbations $dv_-$ and $dv_+$, while keeping $r$ and $t$ unchanged. An optimality condition is that

$$\frac{\partial R}{\partial v_-} dv_- + \frac{\partial R}{\partial v_+} dv_+ = 0,$$

for all perturbations $dv_-$ and $dv_+$ satisfying

$$\frac{\partial D}{\partial v_-} dv_- + \frac{\partial D}{\partial v_+} dv_+ = 0.$$

Thus we have

$$\frac{\partial R/\partial v_-}{\partial D/\partial v_-} = \frac{\partial R/\partial v_+}{\partial D/\partial v_+}.$$

Using the expressions for $\chi(v_-, v_+)$, $\partial \chi(v_-, v_+)/\partial v_-$ and $\partial \chi(v_-, v_+)/\partial v_+$, straightforward algebra lead us to the first-order condition (11) for an optimal equal priority mechanism with respect to $v_-$ and $v_+$. Also, (11) implies that

$$\frac{\partial R/\partial v_+}{\partial D/\partial v_+} = -n(1 - \alpha)(\phi(v_+) - \phi(v_-))f(v_-).$$

(vi) To obtain (12), consider perturbations $dt$ and $dv_+$. The optimality condition is

$$\frac{\partial R/\partial t}{\partial D/\partial t} = \frac{\partial R/\partial v_+}{\partial D/\partial v_+}.$$

This gives the first order condition (12) with respect to $t$ and $v_+$.

(vii) Lastly, to obtain (13), consider perturbations $dr$ and $dv_+$, while keeping $t$ and $v_-$ unchanged. The resulting optimality condition is

$$\frac{\partial R/\partial r}{\partial D/\partial r} \geq \frac{\partial R/\partial v_+}{\partial D/\partial v_+},$$

and $r \geq 0$, with complementary slackness. This gives the first-order condition

$$-\phi(r)f(r) \leq (\phi(v_+) - \phi(v_-))f(v_-).$$
Note that \( -\phi(0) f(0) = 1 \). Since \( \phi(v_+) < \pi(t) < \pi(r^*) < r^* \), and \( v_- > t > r^* \),

\[
(\phi(v_+) - \phi(v_-)) f(v_-) = (\phi(v_+) - v_-) f(v_-) + 1 - F(v_-) < 1.
\]

It follows that the optimal \( r \) is interior and so (13) holds.

**Proof of Lemma 3**

Fix a direct mechanism \((q^*_{m_n}, p^*_{m_n})_{m=0}^{n-1} \) and \((q^\mu_{m_n}, p^\mu_{m_n})_{m=1}^n \). Define \( p^\mu \in [0,1] \) to be the expected offer to unobservant buyers, given by

\[
\sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_v[q^\mu_{m+1}(v)(p^\mu - p^\mu_{m+1}(v))] = 0.
\]

Since \( \max\{w - p, 0\} \) is convex in \( p \) for any \( w \),

\[
U^\mu(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_v[q^\mu_{m+1}(v) \max\{w - p^\mu_{m+1}(v), 0\}]
\geq \sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_v[q^\mu_{m+1}(v)] \max\{w - p^\mu, 0\}.
\]

Thus, replacing all functions \( \{p^\mu_{m_n}(\cdot)\}_{m=1}^n \) with a single offer \( p^\mu \) reduces the deviation payoff of an observant buyer. The seller’s revenue from unobservant buyers is

\[
\sum_{m=1}^n B(m; n, \alpha) \mathbb{E}_v[mq^\mu_{m}(v) \pi (p^\mu_{m}(v))] = n\alpha \sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_v[q^\mu_{m+1}(v) \pi (p^\mu_{m+1}(v))].
\]

The lemma then follows from the strict concavity of \( \pi(\cdot) \).

**Proof of Theorem 2**

Suppose that \( \{r, v_-, v_+; t\} \) is an optimal equal priority mechanism. By Lemma 2, the first order conditions (11)-(13) are satisfied. We construct a non-negatively valued multiplier function \( \lambda(w) \) for all \( w \in [0,1] \) such that the allocative rule \((q^*_{m_n}(v))_{m=0}^{n-1} \) and \((q^\mu_{m_n}(v))_{m=1}^n \) defined by \( \{r, v_-, v_+; t\} \), together with \( p^\mu = t \), solves the Lagrangian relaxation. By Lemma 1,
the offer rule \( (p_m^*(v))^n_{m=0} \) we have specified for an equal priority mechanism supports a truth-
ful reporting equilibrium among observant buyers. The conclusion then follows immediately.
The proof is divided into four steps.

(i) Construction of the multiplier function. Let \( \lambda(w) = 0 \) for all \( w \not\in [v_-, v_+] \), and let

\[
\lambda(w) = n(1 - \alpha) \frac{d}{dw} (f(w)(\phi(w) - \phi(v_+))) = n(1 - \alpha)(2f(w) + f'(w)(w - \phi(v_+)))
\]

for all \( w \in (v_-, v_+) \), with \( \lambda(v_-) \) and \( \lambda(v_+) \) given by the corresponding limit from above and from below. Since by assumption \( \pi(\cdot) \) is strictly concave, \( f(w)\phi(w) \) is strictly increasing in \( w \), and thus \( \lambda(w) > 0 \) at any \( w \in [v_-, v_+] \) such that \( f'(w) \leq 0 \). By (11) we have \( \phi(v_+) < \pi(t) < \pi(r^*) < r^* \). Since \( w \geq v_- > t > r^* \), we have \( \lambda(w) > 0 \) at any \( w \in [v_-, v_+] \) such that \( f'(w) > 0 \). Thus, \( \lambda(w) \) as constructed is non-negative for any \( w \).

For each \( w \in [0, 1] \), denote

\[
K^\epsilon(w) = n(1 - \alpha)\phi(w) + \int_w^1 \lambda(x) dx / f(w); \\
K^\mu = n\alpha\pi(p^\mu) - \int_0^1 \lambda(x) \max\{x - p^\mu, 0\} dx.
\]

We can then rewrite the Lagrangian as

\[
(1 - \alpha)^{n-1} \int_0^1 K^\epsilon(w)Q_0^\epsilon(w)f(w)dw + \alpha^{n-1}K^\mu q^\mu_n \\
+ \sum_{m=1}^{n-1} \left( \int_0^1 B(m; n - 1, \alpha)K^\epsilon(w)Q_m^\epsilon(w)f(w)dw + B(m; n - 1, \alpha)K^\mu Q_m^\mu \right),
\]

where \( Q_0^\epsilon(w) \) is the probability that an observant buyer with valuation \( w \) wins the object when all buyers are observant, and \( q^\mu_n \) is the probability that each unobservant buyer wins the object when all buyers are unobservant.

(ii) We claim that \( p^\mu = t \) maximizes the Lagrangian. For any \( w \in [v_-, v_+] \), by construction we have

\[
\int_w^1 \lambda(x) dx = n(1 - \alpha)f(w)(\phi(v_+) - \phi(w)).
\]
Using integration by parts, we have

\[
\int_0^1 \lambda(w) \max\{w - p^\mu, 0\} dw
= - \int_{v_-}^{v_+} (w - p^\mu) \left( \int_w^{v_+} \lambda(x) dx \right) \nonumber
= n(1 - \alpha) \left( (v_- - p^\mu)f(v_-)(\phi(v_+) - \phi(v_-)) + \int_{v_-}^{v_+} f(w)(\phi(v_+) - \phi(w)) dw \right)
= n(1 - \alpha) ((v_- - p^\mu)f(v_-)(\phi(v_+) - \phi(v_-)) + \phi(v_+)(F(v_+) - F(v_-)) - (\pi(v_-) - \pi(v_+))).
\]

By (11), we have

\[K^\mu = n\alpha\phi(v_+) + n\alpha(\pi(p^\mu) - \pi(t)) + (p^\mu - t)n(1 - \alpha)f(v_-)(\phi(v_+) - \phi(v_-)).\]

The above is strictly concave in \(p^\mu\). By (12), it is maximized at \(p^\mu = t\), with the maximum

\[K^\mu_t = n\alpha\phi(v_+).\]

(iii) Comparison of \(K^\epsilon(\cdot)\) and \(K^\mu_t\). For \(w \in [v_-, v_+]\), we have

\[
\frac{B(m; n - 1, \alpha)}{n - m} K^\epsilon(w) = \frac{B(m - 1; n - 1, \alpha)}{m} K^\mu_t.
\]

For all \(w > v_+\), since \(\pi(\cdot)\) is strictly concave,

\[K^\epsilon(w) = n(1 - \alpha)\phi(w) > n(1 - \alpha)\phi(v_+) = K^\epsilon(v_+),\]

and so

\[
\frac{B(m; n - 1, \alpha)}{n - m} K^\epsilon(w) > \frac{B(m - 1; n - 1, \alpha)}{m} K^\mu_t.
\]

For all \(w < v_-\),

\[K^\epsilon(w) = n(1 - \alpha)\phi(w) + \int_{v_-}^{v_+} \lambda(x) dx / f(w) = n(1 - \alpha)(\phi(w) + f(v_-)(\phi(v_+) - \phi(v_-)) / f(w)).\]
We claim that
\[ \phi(w) + \frac{f(v_-)(\phi(v_+) - \phi(v_-))}{f(w)} < \phi(v_+) \]
for all \( w < v_- \), and thus \( K^\epsilon(w) < K^\epsilon(v_+) \) and
\[ \frac{B(m; n-1, \alpha)}{n-m} K^\epsilon(w) \leq \frac{B(m-1; n-1, \alpha)}{m} K^\mu_0. \]

To establish the claim, recall that in showing that the constructed multiplier function \( \lambda(w) \) is positive for \( w \in [v_-, v_+] \), we have proved that \( f(w)(\phi(w) - \phi(v_+)) \) is strictly increasing in \( w \) for all \( w \geq \phi(v_+) \). This immediately implies that the claim holds for any \( w \in [\phi(v_+), v_-] \).

For \( w < \phi(v_+) \), we have
\[ f(w)(\phi(w) - \phi(v_+)) = f(w)(w - \phi(v_+)) - (1 - F(w)) < -(1 - F(w)) < -(1 - F(r^*)), \]
where the last inequality follows because \( \phi(v_+) < \pi(t) < \pi(r^*) < r^* \), while
\[ f(v_-)(\phi(v_+) - \phi(v_-)) < f(r^*) \phi(v_+) < f(r^*) r^*, \]
where the first equality comes from \( f(w)(\phi(w) - \phi(v_+)) \) being strictly increasing in \( w \) for all \( w \geq \phi(v_+) \). The claim then follows from the definition of \( r^* \).

(iv) We claim that the allocations \((q^\epsilon_m(v))_{m=0}^{n-1}\) and \((q^\mu_m(v))_{m=1}^n\) specified by \( \{r, v_-, v_+; t\} \) are a point-wise maximizer of the Lagrangian relaxation. We disaggregate \( Q^\epsilon_m(w) \) and write the Lagrangian as
\[
(1 - \alpha)^{n-1} \int_0^1 K^\epsilon(w)Q^\epsilon_0(w)f(w)dw + \alpha^{n-1}K^\mu_0 q^\mu_0
+ \sum_{m=1}^{n-1} \mathbb{E}_v \left[ \frac{B(m; n-1, \alpha)}{n-m} \sum_{i=1}^{n-m} K^\epsilon(v_i)q^\epsilon_m(v_i) + B(m-1; n-1, \alpha)K^\mu_1 q^\mu_m(v) \right].
\]

Fix any realized number \( m \) of unobservant buyers such that \( 1 \leq m \leq n-1 \), and consider the last term in the above objective function. Suppose that for some realized valuation profile \( v \) we have \( v_i > v_+ \) for some \( i = 1, \ldots, n-m \), but \( q^\mu_m(v) > 0 \). By (4), we can decrease
\( q_m^\mu(v) \) marginally by \( dq_m^\mu(v) > 0 \) and increase \( q_m^\epsilon(\rho_m^i(v)) \) by \( mdq_m^\epsilon(v) \). Since

\[
\frac{m}{n - m} B(m; n - 1, \alpha) K^\epsilon(v_i) > B(m - 1; n - 1, \alpha) K^\mu_i,
\]

the effect on the seller’s revenue is strictly positive. Therefore, \( q_m^\mu(v) = 0 \) for any \( v \) such that \( v_i > v_+ \) for some \( i = 1, \ldots, n - m \). Further, since \( K^\epsilon(w) \) is strictly increasing for \( w > v_+ \), we have \( q_m^\epsilon(\rho_m^i(v)) = 1 \) for \( v_i = \max\{v_1, \ldots, v_{n-m}\} \). Finally, since

\[
\frac{B(m; n - 1, \alpha)}{n - m} K^\epsilon(w) \leq \frac{B(m - 1; n - 1, \alpha)}{m} K^\mu_i,
\]

for all \( w \leq v_+ \), with equality if \( w \in [v_-, v_+] \), if \( v \) is such that \( \max\{v_1, \ldots, v_{n-m}\} \leq v_+ \), there is a maximizer of the Lagrangian such that \( q_m^\epsilon(\rho_m^i(v)) = 0 \) whenever \( v_i < v_- \), and \( q_m^\epsilon(\rho_m^i(v)) = q_m^\mu(v) \) if \( v_i \in [v_-, v_+] \).

For \( m = 0 \) and the first term in the Lagrangian, the strict concavity of \( \pi(\cdot) \) implies \( K^\epsilon(w) \) for \( w < v_- \) crosses 0 at most once and only from below. Thus, for \( r \) that satisfies (13), it is point-wise maximizing to set \( q_0^\epsilon(\rho_0^i(v)) = 1 \) if \( v_i = \max\{v_1, \ldots, v_n\} \) and \( v_i > v_+ \), or if \( v_i = \max\{v_1, \ldots, v_n\} \) and \( v_i \in [r, v_-] \); set \( q_0^\epsilon(\rho_0^i(v)) = 1/k \) if \( v_i \in [v_-, v_+] \), \( \max\{v_1, \ldots, v_n\} \in [v_-, v_+] \) and \#\{\( j : v_j \in [v_-, v_+] \)\} = \( k \); and set \( q_0^\epsilon(\rho_0^i(v)) = 0 \) otherwise.

For \( m = n \) and the second term in the Lagrangian, it is optimal to set \( q_n^\mu = 1/n \) because \( K^\mu_i > 0 \).

**References**


