

Price Discrimination and Efficient Matching

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ABSTRACT: This paper considers the problem of a monopoly matchmaker that uses a schedule of entrance fees to sort different types of agents on the two sides of a matching market into exclusive meeting places, where agents randomly form pairwise matches. We make the standard assumption that the match value function exhibits complementarities, so that matching types at equal percentiles maximizes total match value and is efficient. We provide necessary and sufficient conditions for the revenue-maximizing sorting to be efficient. These conditions require complementarities in the match value function to be sufficiently strong along the efficient matching path.

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1. Introduction

Many users of Internet dating agencies such as Match.com complain about the problem of misrepresentations and exaggerations by some users in the information they provide to the agencies.² This problem, and the perception of it among the public, is responsible for reducing the quality of Internet search and matching and for preventing many lonely people from fully utilizing the online dating services, in spite of the advantages in cost, safety, anonymity and breadth of the reach offered by the new technology compared to more traditional means of finding dates. Although Internet dating agencies rely on individual users to report information about themselves truthfully and have little resource or capability of validating the information, economic theory suggests self-selection as an alternative way of screening information and improving match quality.³ After all, self-selection is evident in more traditional meeting places. Night clubs that cater people with more expensive tastes have higher cover charges. More exclusive singles clubs charge more for membership fees.

In this paper we look at the theoretical problem of a monopoly matchmaker that uses a schedule of entrance fees to sort different types of agents on the two sides of a matching market into different “meeting places,” in which agents are randomly pairwise matched. This problem is presented in section 2. The monopoly matchmaker faces two constraints in revenue maximization. First, the matchmaker does not observe the one-dimensional characteristic (“type”) of each agent. This information constraint means that the matchmaker must provide incentives in terms of match quality and fees for agents to self-select into the meeting places. We refer to the menu of meeting places created by the matchmaker as the “sorting structure.” Second, the monopoly matchmaker faces a technology constraint that restricts match formation in each meeting place to random pairwise matching. This primitive matching technology allows us to focus on the impact

² See for example *The New York Times*, Jan. 18, 2001, “On the Internet, Love Really is Blind.”

³ Most of the Internet dating agencies in North America, including the dominant companies such as Match.com, at present charge a uniform fee for all participants. Lavalife.com, an industry pioneer founded in Canada in 1987, now offers a fee schedule based on the number of initial contact messages that a participant wishes to purchase. This is not the same kind of price discrimination discussed in the present paper, and is unlikely to solve the problem of misrepresentation or improve matching efficiency.

of revenue-maximization on the sorting structure and matching efficiency. We make the standard assumption that the match value function exhibits complementarities between types. Under this assumption, the “perfect sorting structure,” or matching types at equal percentiles with a continuum of meeting places, maximizes the total match value and is efficient. The goal of this paper is to understand when the perfect sorting is revenue-maximizing.

Our framework fits various two-sided market environments characterized by sorting or self-selection based on prices.⁴ For example, online job searching has become a major way to explore potential employer-employee relationships. However, existing job searching services such as Monster.com are plagued by job intermediaries (head hunters) that post entries only to collect information from job applicants and positions and then profit from the information. Our framework is also potentially useful in understanding how private schools can use tuition policies to exploit the concern of students about the peer group effect (students care about what kind of other students are also attending same school), and to compete with public schools. The results in the present paper show that a monopoly matchmaker can have the same incentive as a social planner to implement the efficient matching. In this case, the matchmaker makes directed search possible by creating one meeting place for each type and achieves the first best matching outcomes, in spite of the technological constraint of random pairwise matching.

In section 3 we show how the matchmaker’s problem of designing fee schedules and the corresponding sorting structure can be transformed into a problem of monopoly price discrimination. The assumption of complementarity in the match value function implies that the standard single-crossing condition in the price discrimination literature is satisfied for both sides of the market, and results in the incentive compatibility constraint that a higher type receives a higher match quality. The transformation is then achieved by combining this incentive compatibility constraint with the feasibility constraint that match

⁴ A limitation of this paper is the assumption of a monopoly matchmaker, as competition exists in most two-sided markets. We believe that understanding the monopoly revenue-maximization problem is necessary for an analysis of price competition in two-sided markets where participants have heterogeneous qualities and sorting is important. See our companion paper Damiano and Li (2004) for an application of the present framework to issues of price competition.

qualities are generated in a two-sided matching environment. One important aspect of the feasibility constraint is the assumption that each type participates in at most one meeting place. The outcome of the transformation is a sorting structure in which the efficient matching path in the type space (pairwise matching of types at equal percentiles) is partitioned into pooling intervals and sorting intervals: each pooling interval on the efficient matching path represents a meeting place with the corresponding intervals of types on the two sides being pooled together, while each sorting interval represents a continuum of meeting places with the types efficiently matched. We refer to this sorting structure as “weak sorting.” Since meeting places are mutually exclusive in type, in weak sorting if two types on the same side the market participate in two different meeting places, then the higher type not only has a higher average match type, but never gets a lower match. Weak sorting allows us to rewrite the objective function of the monopolist by using a “virtual match value function,” which is the match value function adjusted for the incentive costs of eliciting private type information.

Unlike a standard price discrimination problem, the optimal sorting structure of the monopolist cannot be solved by pointwise maximization due to the feasibility constraint on how the monopolist provides match quality to the two sides of the market. In section 4 we first provide a necessary condition for the optimal sorting structure to be the perfect sorting. This condition requires that the virtual match value function be locally supermodular along the efficient matching path, that is, have positive cross partial derivatives at equal percentiles. If it is not satisfied at any percentile, the monopoly matchmaker can increase revenue by pooling adjacent types into a single meeting place. Local supermodularity of the virtual match value function along the efficient matching path turns out to be insufficient for the optimal sorting structure to be the perfect sorting, because it does not guarantee that a greater revenue cannot be generated by pooling a large set of types on the two sides. A sufficient condition for the perfect sorting to be optimal is that the virtual match value function is “globally” supermodular along the efficient matching path. Intuitively, the inability to observe the type of agents creates an incentive cost for the matchmaker to extract surplus because the matchmaker has to rely on self-selection by the agents. The perfect sorting structure maximizes revenues for the monopolist matchmaker

if this incentive cost does not dominate the complementarities in the match value function. In this case, there is no distortion in match quality provision for any type, in contrast to the standard result in the price discrimination literature that quality is under-provided for all types except the highest. Finally, if the virtual match value function satisfies an even stronger condition that it is supermodular over the entire type space (as opposed to global supermodularity along the efficient matching path), then the matchmaker's revenue is increasing in the number of meeting places created. In this case, even when there are technological constraints on creation of meeting places, revenue-maximization always leads to improvement in matching efficiency.

2. The Model

Consider a two-sided matching market. Without loss of generality, we assume that the two sides have the same size. For convenience, agents of the two sides are called men and women, respectively. Men and women have heterogeneous one-dimensional characteristics, called types. The type distribution is $F(\cdot)$ for men and $G(\cdot)$ for women. Both type distributions are assumed to have differentiable densities, denoted as f and g , respectively. The support is $[a_m, b_m]$ for men and $[a_w, b_w]$ for women, with both subsets of \mathbb{R}_+ , and b_m and b_w possibly infinite. A match between a type x man and a type y woman produces value xy to both the man and the woman. Men and women are risk neutral and have quasi-linear preferences. They care only about the difference between the expected match value and the entrance fee they pay. An unmatched agent gets a payoff of 0, regardless of type.

An important assumption about the matching preferences that we have made above is that matching characteristics of each agent can be summarized in one-dimensional type. This is a simplification relative to the reality of any matching market. However, since the assumption of one-dimensional type is standard, it facilitates comparison with the existing literature. Implicit in our specification of the matching preferences is that all agents on each side of the market have homogeneous preferences. For the same price, they all prefer the highest type agents on the other side. Clearly there are matching

characteristics that are ranked differently by agents in real matching markets. For example, in online dating, it is sometimes argued that not everyone wishes to date the smartest person. Rather, matching preferences may be single peaked. However, since the most desirable match differs across agents, there is no competition among the agents and hence little incentive to misrepresent this kind of matching characteristics. Since this paper is about how the monopoly matchmaker uses price discrimination to mitigate the problem of misrepresentation in a matching market, we will focus on matching characteristics that all agents rank identically and compete for.⁵

Another important assumption about the matching preferences we have made is that types are complementary in generating match values. This assumption is embedded in the match value function xy : each agent’s willingness to pay for an improvement in match type increases with the type of the agent.⁶ Complementarity is a standard assumption in the literature on matching. Under this assumption, matching types at equal percentiles maximizes the total value of pairwise two-sided matches and is efficient (Koopmans and Beckmann, 1953; Becker, 1981.) Formally, for each $x \in [a_m, b_m]$, let

$$s_m(x) = G^{-1}(F(x))$$

be the female type at the same percentile of the male type x . We refer to the pairs of types at equal percentiles $\{(x, s_m(x)) | x \in [a_m, b_m]\}$ as the “efficient matching path.” We adopt the specific match value function xy for analytical convenience. Since we allow the type distributions to be different for the two sides of the market, this specification is without loss of generality in so far as the match value function is multiplicatively separable and monotone in male and female types. To be precise, any match value function of the form $u(x)v(y)$, with u and v being positive-valued and monotone, can be transformed into the match value function xy by redefining types and changing the distribution functions

⁵ Users of online dating tend to segregate into services that cater groups that share the same preferences for non-competing characteristics. One such example is religious affiliation. Jdate.com attracts only Jewish users while Eharmony.com targets the Christian population.

⁶ In online dating, a more attractive individual is more likely to have a successful first date than a less attractive individual, so even if both derive the same utility from a given potential match, the more attractive individual is willing to pay more for an improvement in the quality of the potential match.

appropriately.⁷ The separability assumption implies that each agent in a pairwise random matching market cares only about the average agent type on the other side, as opposed to the entire distribution. As a result, the monopolist problem of designing the sorting structure can be reduced to be a one-dimensional problem of match quality provision. The importance of this assumption will become clear in section 3.

A monopoly matchmaker, unable to observe types of men and women, can create a menu of meeting places with a pair of schedules of entrance fees p_m and p_w . Each man or woman participates in only one meeting place. We will restrict each meeting place to have equal measure of men and women. We assume that men and women in each meeting place form pairwise matches randomly, with the probability of finding a match equal to 1 for all agents, and that the probability a type x man meets a type y woman is given by the density of type y in that meeting place. In other words, the meeting technology in our model is random matching. For simplicity, we assume that meeting places cost nothing to organize. The objective function of the matchmaker is to maximize the sum of entrance fees collected from men and women.

The technology side of our framework is modeled on the motivating example of online dating. Imagine that each meeting place consists of two data bases, of men and women who have paid the corresponding subscription fees. Any man in the meeting place has access to the data base of women and can “search” it for a match. We have assumed that the probability of finding a match is 1 for all agents. This assumption rules out any size effect, which postulates a different probability of finding a match depending on the size of the market, and allows us to focus on the issue of price discrimination. The search technology in each meeting place, which is pairwise random matching, is admittedly primitive, compared to the actual matching technology used by online dating services where agents can search according to the information available on the data base and exchange further information through anonymous email correspondence.⁸ We have adopted the

⁷ When u is monotonically increasing, the new type \tilde{x} is $u(x)$; if it is monotonically decreasing, the new type \tilde{x} is $1/u(x)$.

⁸ In a benchmark model of Damiano, Li and Suen (2004), we consider the question of how repeated, costless, random pairwise meeting with no new entry or replacements can improve efficiency over random matching. In what we call “dynamic sorting” there, a higher type gets more chances to match with

pairwise random matching technology in order to focus on the misrepresentation problem, by implicitly assuming that any information volunteered by participants beyond what is signaled by their choices of meeting place is not credible and therefore cannot be used to improve matching efficiency. For the same reason, we have ignored the possibility of verifying certain information by providers of online dating services.⁹ We concentrate on unverifiable information and the consequent problem of misrepresentation.

We refer to a menu of meeting places as a sorting structure. Let ϕ_m be a set-valued function that maps any male type x in $[a_m, b_m]$ to a subset $\phi_m(x)$ of $[a_w, b_w]$. The set $\phi_m(x)$ represents the set of female types that the male type x men can hope to meet. We sometimes refer to $\phi_m(x)$ as type x men's "match set." We allow the possibility that male type x is excluded by the monopolist matchmaker, with $\phi(x) = \emptyset$. Define ϕ_w similarly, and denote $\phi = \langle \phi_m, \phi_w \rangle$. For any $X \subseteq [a_m, b_m]$, define

$$\Phi_m(X) = \{y | y \in \phi_m(x) \text{ for some } x \in X\}.$$

That is, the female type set $\Phi_m(X)$ represents the union of match sets of male types in X . Define Φ_w similarly.

DEFINITION 2.1. *A sorting structure ϕ is feasible if for any $x, \tilde{x} \in [a_m, b_m]$, $y, \tilde{y} \in [a_w, b_w]$, $X \subseteq [a_m, b_m]$ and $Y \subseteq [a_w, b_w]$, i) $y \in \phi_m(x)$ implies $x \in \phi_w(y)$, and $x \in \phi_w(y)$ implies $y \in \phi_m(x)$; ii) $\phi_m(x) \neq \phi_m(\tilde{x})$ implies $\phi_m(x) \cap \phi_m(\tilde{x}) = \emptyset$, and $\phi_w(y) \neq \phi_w(\tilde{y})$ implies $\phi_w(y) \cap \phi_w(\tilde{y}) = \emptyset$; and iii) $\Phi_m(X)$ has the same measure as $\{x | \phi_m(x) \subseteq \Phi_m(X)\}$, and $\Phi_w(Y)$ has the same measure as $\{y | \phi_w(y) \subseteq \Phi_w(Y)\}$.*

Condition i) is analogous to the standard symmetry condition for matching correspondences. It states that if type x men are participating in a meeting place where there are type y women, then type y women are participating in a meeting place where there are type x men, and vice versa. This condition is needed for a meeting place to have

other higher types on the other side than a lower type. It is possible to allow dynamic sorting in each meeting place in the present paper. We conjecture that this would not change the qualitative nature of the conclusions.

⁹ For example, claims of college education are in principle verifiable at some cost.

the interpretation of a matching market. Condition ii) requires that each type participate in at most one meeting place. This simplifies the analysis. Condition iii) requires that each meeting place consist of men and women of equal measures. This ensures that match probability is one for each agent in any meeting place. This condition helps us minimize the role of search technologies and focus on the impact of revenue-maximization on the sorting structure and matching efficiency.

3. Weak Sorting

The monopolist's problem is to choose a sorting structure and the corresponding two fee schedules, one for males and one for females. A sorting structure assigns to each male type a set of potential female matches and to female types a set of potential male partners. The design problem appears multi-dimensional because what a type buys from the matchmaker is a type distribution on the other side of the meeting place. However, the assumption of a multiplicatively separable match value function allows us to reduce the problem to one dimension. Our first step of analysis is to substitute a pair of expected match types for each meeting place in the design problem, and transform the market design problem to a more familiar price discrimination problem.

A feasible sorting structure ϕ generates two schedules of expected match types, q_m and q_w . The function $q_m : [a_m, b_m] \rightarrow [a_w, b_w] \cup \{0\}$ assigns to each male type the expected value of his match; the function $q_w : [a_w, b_w] \rightarrow [a_m, b_m] \cup \{0\}$ is the corresponding function for female types. We refer to $q = \langle q_m, q_w \rangle$ as a pair of “match quality schedules.” Given ϕ , we obtain q_m and q_w as¹⁰

$$\begin{aligned} q_m(x) &= \mathbb{E}[y | y \in \phi_m(x)]; \\ q_w(y) &= \mathbb{E}[x | x \in \phi_w(y)] \end{aligned} \tag{3.1}$$

for all $x \in [a_m, b_m]$ such that $\phi_m(x) \neq \emptyset$ and $y \in [a_w, b_w]$ such that $\phi_w(y) \neq \emptyset$. We adopt the convention that if any type is excluded by the matchmaker, the match quality

¹⁰ Without the restrictions of types participating in at most one meeting place, ϕ would not be sufficient to define q_m and q_w and we would need additional notation to specify the fraction of agents of a given type who participate in any given meeting place.

assignment is 0, which is the reservation utility. For notational simplicity all the lemmas in the remainder of this section refer to types that are not excluded by the monopoly matchmaker. With the convention we have adopted, the lemmas can be easily restated to cover the excluded types.

As in a price discrimination problem, the monopolist does not observe agent types and must rely on self-selection of agents into their assigned expected match quality.¹¹ The incentive compatibility constraints for men are

$$xq_m(x) - p_m(x) \geq xq_m(\tilde{x}) - p_m(\tilde{x})$$

for all $x, \tilde{x} \in [a, b]$, where $p_m(x)$ is the participation fee for type x . A similar set of incentive compatibility constraints must hold for women. Under the complementarity assumed in the match value function, standard arguments imply that q_m and q_w being nondecreasing is both necessary and sufficient conditions for the incentive compatibility constraints to be satisfied (see, e.g., Maskin and Riley, 1984). Further, the associated indirect utility $U_m(x)$ of male type x , defined as

$$U_m(x) = xq_m(x) - p_m(x),$$

satisfies the envelope condition

$$U'_m(x) = q_m(x). \tag{3.2}$$

at every x such that $q_m(x)$ is continuous. Similar condition holds for the indirect utility $U_w(y)$ of female type y .

Unlike in a typical price discrimination problem, the monopolist can only choose schedules q_m and q_w consistent with some feasible sorting structure. Through a series of lemmas, we show how the feasibility constraints on the sorting structure translate into direct restrictions on quality schedules. Take a pair of nondecreasing match quality schedules q . Monotonicity of the schedules leads to the following definition.¹²

¹¹ Since the matching market is two-sided, self-selection involves a coordination problem that is absent in a standard price discrimination problem. We ignore such problem in this paper by assuming that the monopoly matchmaker can decide how agents self-select so long as the sorting structure is feasible and incentive compatible. See our companion paper (Damiano and Li, 2004) for a discussion of how to resolve the coordination problem.

¹² There is no need to specify whether a maximal pooling interval contains the two end points. The assignment of values of q_m and q_w to the end points does not affect the revenue function stated later in Proposition 3.6.

DEFINITION 3.1. An interval $T_m \subseteq [a_m, b_m]$ is a maximal pooling interval under q_m if $q_m(\cdot)$ is constant on T_m , and there is no interval $T'_m \supset T_m$ such that $q_m(\cdot)$ is constant on T'_m .

Maximal pooling intervals T_w under q_w can be similarly defined. We say that $q = \langle q_m, q_w \rangle$ is “feasible” if there is a feasible $\phi = \langle \phi_m, \phi_w \rangle$ such that equations (3.1) are satisfied for almost all x and y . We call ϕ the “associated” sorting structure.

LEMMA 3.2. If q is feasible, then for any maximal pooling interval T_m under q_m and any associated sorting structure ϕ , $\Phi_m(T_m)$ is a maximal pooling interval under q_w .

PROOF. Suppose $\Phi_m(T_m)$ is not a maximal pooling interval under q_w . There are two cases.

Case 1. Suppose that q_w is not constant on $\Phi_m(T_m)$. Then, we can find $y, \tilde{y} \in \Phi_m(T_m)$ such that $q_w(y) < q_w(\tilde{y})$. It follows from condition ii) in Definition 2.1 that $\phi_w(y) \cap \phi_w(\tilde{y}) = \emptyset$ and $\Phi_m(\phi_w(y)) \cap \Phi_m(\phi_w(\tilde{y})) = \emptyset$. Since T_m is a maximal pooling interval and $y, \tilde{y} \in \Phi_m(T_m)$, we have

$$\mathbb{E}[t | t \in \Phi_m(\phi_w(y))] = \mathbb{E}[t | t \in \Phi_m(\phi_w(\tilde{y}))],$$

which is possible only if

$$\inf \Phi_m(\phi_w(\tilde{y})) < \sup \Phi_m(\phi_w(y)).$$

Then, there exist $y_1 \in \Phi_m(\phi_w(y))$ and $\tilde{y}_1 \in \Phi_m(\phi_w(\tilde{y}))$ such that $y_1 > \tilde{y}_1$. It follows that $q_w(y_1) = q_w(y) < q_w(\tilde{y}) = q_w(\tilde{y}_1)$, which contradicts the assumption that q_w is nondecreasing.

Case 2. Suppose that there is a $W \supset \Phi_m(T_m)$ such that q_w is constant on W . By a symmetric argument as in Case 1, we can show that q_m is constant on $\Phi_w(W)$. Since $W \supset \Phi_m(T_m)$, we can write

$$\Phi_w(W) = \Phi_w(\Phi_m(T_m)) \cup \Phi_w(W \setminus \Phi_m(T_m)).$$

We claim that $\Phi_w(\Phi_m(T_m)) \supseteq T_m$: if $x \in T_m$, then there exists $y \in \Phi_m(T_m)$ such that $y \in \phi_m(x)$, which by condition i) of Definition 2.1 implies that $x \in \phi_w(y)$, and therefore $x \in \Phi_w(\Phi_m(T_m))$. Further, $\Phi_w(W \setminus \Phi_m(T_m)) \neq \emptyset$ because $W \supset \Phi_m(T_m)$, and q_w is

constant and different from 0 on W . Finally, $\Phi_w(W \setminus \Phi_m(T_m)) \cap T_m = \emptyset$, because $y \notin \Phi_m(T_m)$ implies that $\phi_w(y) \cap T_m \neq \emptyset$ by condition i) of Definition 2.1. It follows that $\Phi_w(W) \supset T_m$. Since q_m is constant over $\Phi_w(W)$, we have reached a contradiction to the assumption that T_m is a maximal pooling interval under q_m . *Q.E.D.*

By symmetry, if a pair of nondecreasing schedules q is feasible, then $\Phi_w(T_w)$ is a maximal pooling interval under q_m for any maximal pooling interval T_w under q_w and any associated sorting structure ϕ . A corollary of Lemma 3.2 is thus $\Phi_w(\Phi_m(T_m)) = T_m$, and symmetrically $\Phi_m(\Phi_w(T_w)) = T_w$. Another implication is that for any associated sorting structure ϕ , and for any maximal pooling interval T_m under q_m , we have $q_m(x) = \mathbb{E}[y|y \in \Phi_m(x)]$ for all $x \in T_m$.¹³ Symmetrically, for any maximal pooling interval T_w under q_w and for any $y \in T_w$, we have $q_w(y) = \mathbb{E}[x|x \in \Phi_w(T_w)]$.

Lemma 3.2 is the first step in showing that a pair of nondecreasing, feasible schedules q defines two sequences $\{T_m^l\}_{l=1}^L$ and $\{T_w^l\}_{l=1}^L$ of maximal pooling intervals in $[a_m, b_m]$ and $[a_w, b_w]$ respectively, with $T_w^l = \Phi_m(T_m^l)$ and $T_m^l = \Phi_w(T_w^l)$ for each l . The next step is to identify the end points of each maximal pooling interval.

LEMMA 3.3. *If q is feasible, then for any maximal pooling interval T_m under q_m and any associated sorting structure ϕ , $s_m(\inf T_m) = \inf \Phi_m(T_m)$ and $s_m(\sup T_m) = \sup \Phi_m(T_m)$.*

PROOF. We first establish that if $x < \inf T_m$ then $\sup \phi_m(x) \leq \inf \Phi_m(T_m)$. To prove this claim by contradiction, suppose that there exists $y > \inf \Phi_m(T_m)$ such that $y \in \phi_m(x)$. Since T_m is a maximal pooling interval and $x \notin T_m$, we have $\phi_m(x) \cap \Phi_m(T_m) = \emptyset$. By Lemma 3.2, $\Phi_m(T_m)$ is an interval and hence $y \geq \sup \Phi_m(T_m)$. If $\inf \phi_m(x) \geq \sup \Phi_m(T_m)$, then since $q_m(x) = \mathbb{E}[y|y \in \phi_m(x)]$ and $x < \inf T_m$, we have a contradiction to the assumption that q_m is nondecreasing. If instead $\inf \phi_m(x) \leq \inf \Phi_m(T_m)$, then there exists $\tilde{y} \in \phi_m(x)$ such that $\tilde{y} \leq \inf \Phi_m(T_m)$. By condition ii) of Definition 2.1

¹³ In general, it may not be true that $\phi_m(x) = \Phi_m(T_m)$ for all $x \in T_m$, as there can be more than one way of assigning match sets for x in T_m such that $q_m(x)$ is constant. However, by condition iii) of Definition 2.1, we have $\int_{T_m} \mathbb{E}[y|y \in \phi_m(x)] dF(x) = (F(\sup(T_m)) - F(\inf(T_m)))\mathbb{E}[y|y \in \Phi_m(T_m)]$. Since $\mathbb{E}[y|y \in \phi_m(x)]$ equals $q_m(x)$ and is constant on T_m , it follows that $q_m(x) = \mathbb{E}[y|y \in \Phi_m(T_m)]$ for all $x \in T_m$.

and the definition of q_w we have $q_w(y) = q_w(\tilde{y})$. Monotonicity of q_w implies that q_w is constant on $[\tilde{y}, y] \supset \Phi_m(T_m)$ therefore $\Phi_m(T_m)$ is not a maximal pooling interval under q_w , contradicting Lemma 3.2.

It follows from the above claim that $\phi_m(x) \subseteq \Phi_m([a_m, \inf T_m])$ for any $x < \inf T_m$, and hence $[a_w, \inf \Phi_m(T_m)] \supseteq \Phi_m([a_m, \inf T_m])$. Thus,

$$G(\inf \Phi_m(T_m)) \geq \int_{\Phi_m([a_m, \inf T_m])} dG.$$

By condition iii) of Definition 2.1,

$$\int_{\Phi_m([a_m, \inf T_m])} dG = \int_{\{x | \phi_m(x) \subseteq \Phi_m([a_m, \inf T_m])\}} dF,$$

which implies that

$$G(\inf \Phi_m(T_m)) \geq F(\inf T_m).$$

By a symmetric argument, we have $\sup \phi_w(y) \leq \inf T_m$ for any $y < \inf \Phi_m(T_m)$. Hence, $[a_m, \inf T_m] \supseteq \Phi_w([a_w, \inf \Phi_m(T_m)])$ and

$$F(\inf T_m) \geq G(\inf \Phi_m(T_m)).$$

It follows that $G(\inf \Phi_m(T_m)) = F(\inf T_m)$, which by the definition of s_m implies that $s_m(\inf T_m) = \inf \Phi_m(T_m)$.

The argument for $s_m(\sup T_m) = \sup \Phi_m(T_m)$ is symmetric. *Q.E.D.*

Lemma 3.2 and Lemma 3.3 completely characterize a nondecreasing, feasible q for x and y in maximal pooling intervals. It remains to characterize $q_m(x)$ and $q_w(y)$ for any x and y not in a maximal pooling interval respectively.

LEMMA 3.4. *If q is feasible, then $q_m(x) = s_m(x)$ for any $x \in [a_m, b_m]$ such that x does not belong to any maximal pooling interval under q_m .*

PROOF. Fix any sorting structure ϕ associated with q . We first show that, if x does not belong to any maximal pooling interval, $\phi_m(x)$ is a singleton. Suppose $y, \tilde{y} \in \phi_m(x)$ for some $y < \tilde{y}$. By condition ii) of Definition 2.1, $\phi_w(y) = \phi_w(\tilde{y})$, and $q_w(y) = q_w(\tilde{y})$.

Since q_w is monotone, it must be constant on the interval $[y, \tilde{y}]$. Therefore, there exists a maximal pooling interval $W \supseteq [y, \tilde{y}]$. By construction, x belongs to $\Phi_w(W)$ which is a maximal pooling interval by Lemma 3.2; a contradiction.

Let $\phi_m(x) = \{y_x\}$. Since $q_m(x) = \mathbb{E}[y|y \in \phi_m(x)]$, we can write $q_m(x) = y_x$. By monotonicity of q_m , if $\tilde{x} < x$ and \tilde{x} does not belong to a maximal pooling interval then $y_{\tilde{x}} \leq y_x$ where $\phi_m(\tilde{x}) = \{y_{\tilde{x}}\}$. Together with Lemma 3.3, this implies that $\Phi_m[a_m, x] \subseteq [a_w, \phi_m(x)]$. Clearly, y_x does not belong to a maximal pooling interval under ϕ_w and $\phi_w(y_x) = \{x\}$, so an identical argument yields $\Phi_w[a_w, y_x] \subseteq [a_m, x]$. Then, by condition iii) in Definition 2.1, we have $F(x) = G(y_x)$, or $q_m(x) = y_x = s_m(x)$. Q.E.D.

The following proposition summarizes the feasibility restrictions on incentive compatible quality schedules that we have derived from the restrictions imposed on feasible sorting structure (Definition 2.1). For the ease of notation, for any $a_m \leq x < \tilde{x} \leq b_m$, let

$$\mu_m(x, \tilde{x}) = \mathbb{E}[t|x \leq t \leq \tilde{x}]$$

be the mean male type on the interval $[x, \tilde{x}]$. Define $\mu_w(y, \tilde{y})$ similarly.

PROPOSITION 3.5. *A pair of nondecreasing match quality schedules $\langle q_m, q_w \rangle$ is feasible if and only if i) for any maximal pooling interval T_m under q_m and each $x \in T_m$, $q_m(x) = \mu_w(s_m(\inf T_m), s_m(\sup T_m))$ and $q_w(s_m(x)) = \mu_m(\inf T_m, \sup T_m)$; and ii) for any x not in any maximal pooling interval under q_m , $q_m(x) = s_m(x)$ and $q_w(s_m(x)) = x$.*

PROOF. Necessity of i) and ii) follow immediately from Lemmas 3.2–3.4. For sufficiency, fix any q that is nondecreasing and feasible. For each maximal pooling interval T_m under q_m , construct the set-valued function ϕ_m such that $\phi_m(x) = [s_m(\inf T_m), s_m(\sup T_m)]$ for any x in the closure of T_m , and correspondingly ϕ_w such that $\phi_w(y) = [\inf T_m, \sup T_m]$ for any $y \in [s_m(\inf T_m), s_m(\sup T_m)]$. For all other x , let $\phi_m(x) = \{s_m(x)\}$ and $\phi_w(s_m(x)) = \{x\}$. By conditions i) and ii) stated in the proposition, $\phi_m(x)$ and $\phi_w(y)$ are well-defined for all $x \in [a_m, b_m]$ and $y \in [a_w, b_w]$ respectively, and further, ϕ_m and ϕ_w satisfy equations (3.1) for almost all x and y . Thus, $\langle q_m, q_w \rangle$ is feasible. Q.E.D.

The above result can be viewed as a characterization of any feasible sorting structure associated with an incentive compatible, feasible pair of quality schedules. We refer to the characterization as “weak sorting.” Since meeting places are mutually exclusive in type, in weak sorting if two types on the same side the market participate in two different meeting places, then the higher type not only has a higher average match type, but never gets a lower match.

We have completed transforming the design problem from choosing a feasible and incentive compatible sorting structure ϕ_m and ϕ_w to a problem of choosing a pair of nondecreasing match quality schedules that satisfies Proposition 3.5. The advantage of this transformation is that from the mechanism design literature we know how to manipulate the incentive compatibility and individual rationality constraints associated with one-dimensional schedules q to rewrite the matchmaker’s revenue. Define

$$J_m(x) = x - \frac{1 - F(x)}{f(x)};$$

$$J_w(y) = y - \frac{1 - G(y)}{g(y)}$$

to be the “virtual type” for male type x and female type y respectively. Virtual type of x takes into account the incentive cost of eliciting private type information from type x . These are familiar definitions from the standard price discrimination literature (e.g., Maskin and Riley, 1984). Next, we combine the virtual types and define

$$K(x, y) = xJ_w(y) + yJ_m(x) \tag{3.3}$$

as the “virtual match value” for male type x and female type y . Virtual match value of types x and y is based on the match value between x and y with proper adjustment of the incentive costs of eliciting truthful information from the two types.

For the following proposition, note that for any q that is nondecreasing and feasible, there are at most countable many maximal pooling intervals. This is because for any maximal pooling interval T_m , the quality schedule q_m is discontinuous at $\inf T_m$ (unless $\inf T_m = a_m$) and $\sup T_m$ (unless $\sup T_m = b_m$). Since q_m is monotone, it can only have a countable number of discontinuities. Let L be the total number of maximal pooling intervals under q_m ; note that we allow L to be infinite.

PROPOSITION 3.6. Let $\langle q_m, q_w \rangle$ be a pair of nondecreasing and feasible match quality schedules, and let $\{T_m^l\}_{l=1}^L$ be the collection of all maximal pooling interval under q_m . The maximum revenue generated by $\langle q_m, q_w \rangle$ is given by:

$$\int_{[a_m, b_m] \setminus (\cup_{l=1}^L T_m^l)} K(x, s_m(x)) dF(x) + \sum_{l=1}^L \int_{\inf T_m^l}^{\sup T_m^l} \int_{s_m(\inf T_m^l)}^{s_m(\sup T_m^l)} \frac{K(x, y) dG(y) dF(x)}{F(\sup T_m^l) - F(\inf T_m^l)}.$$

As already mentioned above, given our characterization of incentive compatible, feasible quality schedules in Proposition 3.5, the proof of the above proposition follows the standard steps of integration by parts and application of the envelope conditions (3.2) (see, e.g., Stole, 1996). Clearly, incentive compatibility and feasibility of the quality schedules imply that the monopoly matchmaker's exclusion policy takes the form of a cutoff male type $c_m \in [a_m, b_m]$ such that male types $x < c_m$ and female types $y < s_m(c_m)$ are excluded. The cutoff types c_m and $s_m(c_m)$ receive their reservation utility of 0 in any optimal price discrimination mechanism for the monopolist. Under our convention that the match quality assignment is 0 for any excluded type, the revenue formula in Proposition 3.6 applies with c_m replacing a_m , regardless of the value of c_m .

The restatement of the monopolist's revenue in Proposition 3.6 contains two terms, corresponding respectively to the revenue from the types that are perfectly sorted and the revenue from a sequence of pooling regions. The revenue from perfectly sorting the types in a region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ is the integral of the virtual match value function K along the efficient matching path $\{(x, s_m(x)) | x \in (x_1, x_2)\}$, while the revenue from pooling the types in a region $T_m^l \times \Phi_m(T_m^l)$ is the integral of the virtual match value function over the entire region. Note that the quality schedule q does not appear explicitly in the revenue formula; by Proposition 3.5, the feasibility constraint on q , together with the incentive compatibility constraint, has already pinned down q once the sequence of maximal pooling intervals $\{T_m^l\}_{l=1}^L$ is given. Thus, the monopolist's sorting structure design problem is reduced to choosing the break points of the maximal pooling intervals.¹⁴ We can think of the monopolist partitioning the male type space $[a_m, b_m]$ into a sequence of pooling

¹⁴ By definition two sorting intervals cannot be adjacent to each other. However, it is possible, and it can be optimal, to have two pooling intervals adjacent to each other.

intervals and sorting intervals, with the female type space correspondingly partitioned. Since the revenue is written as sum of revenues from these intervals in the formula of Proposition 3.6, whether it is optimal to pool or to sort the types in one particular interval can be determined in isolation. This feature will be repeatedly used in the analysis of the next section.

4. Perfect Sorting

Proposition 3.5 establishes weak sorting as the outcome of satisfying both the incentive compatibility constraint in self-selection and the feasibility constraint on the sorting structure. Weak sorting can take different forms, from pooling the entire population of agents into a single meeting place to segregating each type into separate meeting places. Due to the assumption of complementarity in the match value function xy , the finer the agents are partitioned into separate meeting places, the higher the matching efficiency in terms of the total match value.¹⁵ The question is whether the monopoly's revenue is also increased.

In this section, we use the revenue formula of Proposition 3.6 to study the implications of revenue maximization to the sorting structure and matching efficiency. We are particularly interested in the perfect sorting structure. If the perfect sorting maximizes the monopolist's revenue, then the monopolist has the same incentive to create meeting places as a social planner who maximizes the total match value. In this case, the incentive cost of eliciting private type information generates no distortion in terms of match quality provision. This is in contrast with the standard price discrimination result that quality is under-provided for all types but the very highest (Mussa and Rosen, 1978; Maskin and Riley, 1984).

Before proceeding with the main analysis, we note that there are two aspects of the efficient matching. In order to maximize the total match value, the monopolist not only needs to create a continuum of meeting places to implement the perfect sorting, but also needs to have the same exclusion policy as the planner. By assumption, the match value of

¹⁵ Although this statement is intuitively obvious, we are not aware of a proof in the existing literature. Proposition 4.5 below provides such a proof.

any pair of types is positive and the reservation utility of each type is zero, implying that the planner will have full market coverage—every type will be matched by the planner. In contrast, the virtual match value of a pair of types need not be positive, and so the monopolist may find it optimal to exclude some types. The issue of optimal exclusion policy is orthogonal to the focus of the present paper of when the revenue-maximizing sorting structure is the perfect sorting. The analysis below may be viewed as characterizing necessary and sufficient conditions for the optimality of the perfect sorting for any given exclusion policy.¹⁶

4.1. Necessary condition

We proceed now to find necessary conditions for the optimality of the perfect sorting structure. Unlike the price discrimination problems in mechanism design, due to the feasibility constraint characterized in Proposition 3.5, we cannot apply pointwise maximization and then check whether the solution satisfies the monotonicity constraint implied by incentive compatibility. Instead, we use a local approach to identify a necessary condition for the perfect sorting to be optimal. The idea is simple.¹⁷ We start with the revenue from the perfect sorting and study how it changes when we pool types in a small neighborhood around a point along the efficient matching path while keeping all other types separated. We then let the neighborhood become arbitrarily small.

¹⁶ The optimal sorting structure design problem is necessary for a characterization of the monopoly matchmaker’s optimal exclusion policy. In an earlier version of the paper, we focused on two aspects of optimal exclusion policy: cross-subsidy in fee schedules, where the two sides of the matching market have different reservation values, and endogenous outside options, where the reservation value is type-dependent and determined by participating in a costless meeting place.

¹⁷ Bergemann and Pesendorfer (2001) use the same techniques to answer the question of how much private information an auctioneer should allow the bidder to learn about his valuation. The analogy between our sorting structure design problem and theirs can be seen if one thinks of a partition element in an information structure in their paper as a pooling of types in our problem. There are at least two important technical differences. First, in Bergemann and Pesendorfer types do not know who they are in a partitional element of an information structure and form expectations about their types, with all types in the element sharing the same expectation. This implies that the important incentive compatibility constraint is the one for the conditional average type. In our problem agents know their types, so the relevant incentive compatibility constraint for all types in a pooling meeting place is that for the lowest type. Second, we have an additional constraint in the revenue maximization problem that the match quality schedules must be generated from a feasible sorting structure.

Start with the match quality schedules under the perfect sorting $s = \langle s_m, s_w \rangle$, where s_w denotes the inverse of s_m . For some $t \in (a_m, b_m)$ and a small positive ϵ , construct a new pair of match quality schedules $q(\epsilon)$ by pooling the male types on the interval $[t, t + \epsilon]$ with the female types on the corresponding interval $[s_m(t), s_m(t + \epsilon)]$, while retaining the perfect sorting structure outside the region $[t, t + \epsilon] \times [s_m(t), s_m(t + \epsilon)]$. Let $\Delta_t(\epsilon)$ be the revenue difference between s and $q(\epsilon)$. We note that $q(\epsilon)$ is nondecreasing by construction, and feasible because it satisfies Proposition 3.5. Thus, we can apply the revenue formula of Proposition 3.6 and write $\Delta_t(\epsilon)$ as:

$$\int_t^{t+\epsilon} K(x, s_m(x)) dF(x) - \int_t^{t+\epsilon} \int_{s_m(t)}^{s_m(t+\epsilon)} K(x, y) \frac{dG(y) dF(x)}{F(t+\epsilon) - F(t)}.$$

We will study the behavior of $\Delta_t(\epsilon)$ for $\epsilon \rightarrow 0$. In the following lemma, K_{12} is the cross partial derivative of K . The proof is in the appendix.

LEMMA 4.1. $\Delta_t(0) = \Delta'_t(0) = \Delta''_t(0) = 0$, and $\Delta'''_t(0) = \frac{1}{2}K_{12}(t, s_m(t))f(t)s'_m(t)$.

The next proposition follows immediately from the above lemma and gives a necessary condition for the perfect sorting to be optimal. This condition requires that the virtual match value function have nonnegative cross partial derivative along the efficient matching path $\{(x, s_m(x)) | x \in [a_m, b_m]\}$. For any $x \in (a_m, b_m)$, we say that K satisfies “local path supermodularity” (local PS) at x if $K_{12}(x, s_m(x)) \geq 0$.¹⁸ If local PS is not satisfied at some point on the efficient matching path, then the continuity of $\Delta_t(\epsilon)$ implies that there exists $\epsilon > 0$ such that the monopoly matchmaker can increase revenue by pooling male types in $[t, t + \epsilon]$ and corresponding female types in $[s_m(t), s_m(t + \epsilon)]$ into a single meeting place, instead of perfectly sorting these types.

PROPOSITION 4.2. *Suppose that $K_{12}(x, s_m(x)) < 0$ for some $x \in (a_m, b_m)$. Then, any nondecreasing, feasible pair of match quality schedules such that x does not belong to the closure of a maximal pooling interval is non-optimal.*

¹⁸ For general expositions of the concept of supermodularity and its economic applications, see Milgrom and Roberts (1990), Topkis (1998), and Vives (1990).

From the definition of virtual match value function K (equation 3.3), we have

$$K_{12}(x, s_m(x)) = J'_m(x) + J'_w(s_m(x)). \quad (4.1)$$

In the special case where the two sides of the market have the same type distributions, the condition in Proposition 4.2 boils down to the virtual type function being nondecreasing. With asymmetric distributions, monotonicity of the virtual types on both male and female sides is not required, though for the perfect sorting to be optimal, at least one side must have nondecreasing virtual type along the efficient matching path.

In a standard price discrimination problem (e.g., Maskin and Riley, 1984), monotonicity of the virtual type is also necessary for a strictly increasing quality schedule to be optimal. Although in the special case where the two sides have the same type distribution our conclusion coincides with the standard monotonicity condition, the logic is different between the two models. The local PS condition cannot be reduced to a monotonicity condition when the type distributions are different. Further, monotonicity of the sum of the virtual type functions, $J_m(x) + J_w(y)$, along the efficient matching path is equivalent to

$$J'_m(x) + J'_w(s_m(x))s'_m(x) \geq 0,$$

so the sum of the virtual type functions may be decreasing at some x while the virtual match value function satisfies local PS, and conversely the sum of the virtual type functions may be increasing while the virtual match value function fails local PS.

4.2. Sufficient conditions

By Proposition 4.2, local PS of the virtual match value function is necessary for the optimal sorting structure to be the perfect sorting. Failure of this condition at any point on the efficient matching path implies that the monopoly matchmaker can increase the revenue by pooling adjacent types into a single meeting place. However, the local PS does not impose any constraint on the behavior of the virtual match value function away from the neighborhood of the efficient matching path. As a result, it fails to ensure that a greater revenue cannot be generated by pooling a large set of types. For the perfect sorting to be

optimal, the virtual match value function needs to satisfy a global version of the necessary condition in Proposition 4.2. We say that the virtual match value function K satisfies “global” PS in the region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, if for any $x, \tilde{x} \in (x_1, x_2)$,

$$K(x, s_m(x)) + K(\tilde{x}, s_m(\tilde{x})) \geq K(x, s_m(\tilde{x})) + K(\tilde{x}, s_m(x)). \quad (4.2)$$

When the above inequality holds with the strict sign we say that K satisfies strict global PS. Clearly, global PS implies local PS of Proposition 4.2: letting \tilde{x} converge to x and $s_m(\tilde{x})$ converge to $s_m(x)$ in (4.2) and using the definition of cross partial derivatives imply that $K_{12}(x, s_m(x)) \geq 0$. However, global PS is weaker than requiring that K be supermodular in the region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$. Supermodularity requires that for any $x, \tilde{x} \in (x_1, x_2)$ and $y, \tilde{y} \in (s_m(x_1), s_m(x_2))$ such that $x \leq \tilde{x}$ and $y \leq \tilde{y}$,

$$K(x, y) + K(\tilde{x}, \tilde{y}) \geq K(x, \tilde{y}) + K(\tilde{x}, y).$$

Instead, global PS only requires that the above inequality hold when (x, y) and (\tilde{x}, \tilde{y}) are on the efficient matching path.

PROPOSITION 4.3. *If K satisfies strict global PS in $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, then any nondecreasing, feasible pair of match quality schedules with $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ as the interior of a maximal pooling region is non-optimal.*

PROOF. Let $\hat{q} = \langle \hat{q}_m, \hat{q}_w \rangle$ be a pair of nondecreasing and feasible match quality schedules, with $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ as the interior of a maximal pooling region. Construct a pair of match quality schedule $q^* = \langle q_m^*, q_w^* \rangle$ such that: i) q_m^* and q_w^* are identical to \hat{q}_m and \hat{q}_w outside the maximal pooling intervals that contain (x_1, x_2) and $(s_m(x_1), s_m(x_2))$, respectively; and ii) $q_m^*(x) = s_m(x)$ and $q_w^*(s_m(x)) = x$ for any x in the maximal pooling intervals that contain (x_1, x_2) and $(s_m(x_1), s_m(x_2))$, respectively. By construction, q^* is nondecreasing and feasible. Let Δ denote the revenue difference between q^* and \hat{q} . Applying the revenue formula of Proposition 3.6 to \hat{q} and q^* , we have

$$\Delta = \int_{x_1}^{x_2} K(x, s_m(x)) dF(x) - \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(x, y) \frac{dG(y) dF(x)}{F(x_2) - F(x_1)}. \quad (4.3)$$

The first term on the right-hand-side of the above can be also written as

$$\int_{x_1}^{x_2} K(x, s_m(x)) dF(x) = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(x, s_m(x)) \frac{dG(y) dF(x)}{F(x_2) - F(x_1)}.$$

With a change of variable $y = s(x)$, we can also write

$$\int_{x_1}^{x_2} K(x, s_m(x)) dF(x) = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(s_w(y), y) \frac{dG(y) dF(x)}{F(x_2) - F(x_1)}.$$

In a similar way, after two changes of variable $x = s_w(\tilde{y})$ and $y = s_m(\tilde{x})$, the second term on the right-hand-side of (4.3) can be written as

$$\int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(x, y) \frac{dG(y) dF(x)}{F(x_2) - F(x_1)} = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(s_w(y), s_m(x)) \frac{dG(y) dF(x)}{F(x_2) - F(x_1)}.$$

Hence, Δ is equal to one-half of

$$\int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} (K(x, s_m(x)) + K(s_w(y), y) - K(x, y) - K(s_w(y), s_m(x))) \frac{dG(y) dF(x)}{F(x_2) - F(x_1)},$$

which is strictly positive because K satisfies strict global PS in the range of the integration.

Q.E.D.

The idea of the above proposition comes from the revenue formula in Proposition 3.6. The revenue from perfectly sorting the types in the region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ is the integral of the virtual match value function K along the segment of the efficient matching path $\{(x, s_m(x)) | x \in (x_1, x_2)\}$, while the revenue from pooling the types is the integral of K over the entire region. By changes of variables we can write the revenue difference Δ as one-half of the integral of the inequality (4.2) over the region.¹⁹

Proposition 4.3 establishes that if the virtual match value function K satisfies global PS in $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, perfectly sorting types in that region does better than pooling the same types all together. This does not imply that the optimal sorting structure

¹⁹ The proof of Proposition 4.3 also provides a direct argument of Proposition 4.2. If $K_{12}(x, s_m(x)) < 0$ for all $x \in (x_1, x_2)$, then any non-decreasing, feasible pair of match quality schedules that perfectly sorts types in any subset of the region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ is non-optimal. We choose to present the original proof of Proposition 4.3 because it is constructive and illustrates the local nature of the necessary condition for the perfect sorting to be optimal.

calls for the perfect sorting in the region, because it may be optimal to have a larger pool than $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$. On the hand, since by definition global PS in a region implies global PS in any subset of the region, if K satisfies global PS everywhere, then the perfect sorting is optimal. Thus global PS is a sufficient condition for the optimality of the perfect sorting.

The next proposition, proved in the appendix, shows that the revenue from perfectly sorting types in any region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, can be approximated by breaking up the intervals (x_1, x_2) and $(s_m(x_1), s_m(x_2))$ in a finite number of sufficiently small pooling intervals. This result is useful in practice because creating a continuum of markets may be prohibitively costly.

PROPOSITION 4.4. *Let X be an interval in $[a_m, b_m]$ and q^* a pair of nondecreasing, feasible match quality schedules that perfectly sorts types in X . For any $\epsilon > 0$, there exists a pair of nondecreasing, feasible match quality schedules \hat{q} that pools the types in X into finitely many subintervals of X and generates a revenue that differs from the revenue from q^* by at most ϵ .*

The proof of Proposition 4.4 is intuitively straightforward. The revenue $R(q^*)$ from the perfect sorting quality schedules q^* in the region $X \times \Phi_m(X)$ is the integral of the virtual match value function K along the efficient matching path $\{(x, s_m(x)) | x \in X\}$. In the appendix we show that the revenue is equal to the expected value of K with respect to an appropriately constructed “line measure” on the product space, which puts weight only along the efficient matching path. We construct a sequence of nondecreasing, feasible quality schedules that partition types in X and $\Phi_m(X)$ in increasingly smaller pools. Each pair of quality schedules in the sequence generates a revenue that is equal to the expected value of K with respect to a probability measure on the product space that puts weight only on the pooling regions. The sequence of probability measures so constructed converges weakly to the line measure, and the proposition follows from the continuity of K . Note that this convergence result does not require any assumption on K regarding path supermodularity.

Proposition 4.3 and Proposition 4.4 together imply that when the virtual match value function K satisfies global PS in some region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, then breaking up

the region into sufficiently many small pooling regions generates more revenue than pooling all types in $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ together. However, global PS does not guarantee that dividing the market into meeting places always increases the revenue. The next proposition establishes that supermodularity of K is sufficient for this stronger result.²⁰ This monotone convergence result is useful in practice because it implies that setting up a new meeting place always strictly increases revenue. It also illustrates a difference between supermodularity and global PS.

PROPOSITION 4.5. *Let q^* be a pair of nondecreasing, feasible match quality schedules with $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ as the interior of a maximal pooling region. If K is (strictly) supermodular on $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, then for any $t \in (x_1, x_2)$, any pair of nondecreasing, feasible match quality schedules \hat{q} such that \hat{q} is identical to q^* outside $[x_1, x_2] \times [s_m(x_1), s_m(x_2)]$ and \hat{q} has $(x_1, t) \times (s_m(x_1), s_m(t))$ and $(t, x_2) \times (s_m(t), s_m(x_2))$ as the interiors of two maximal pooling regions generates (strictly) more revenue than q^* .*

PROOF. Let the revenue difference between \hat{q} and q^* be Δ . Using the revenue formula from Proposition 3.6 we can write

$$\begin{aligned} \Delta = & \int_{x_1}^t \int_{s_m(x_1)}^{s_m(t)} K(x, y) dF_l(x) dG_l(y) + \int_t^{x_2} \int_{s_m(t)}^{s_m(x_2)} K(x, y) dF_h(x) dG_h(y) \\ & - \int_{x_1}^t \int_{s_m(t)}^{s_m(x_2)} K(x, y) dF_l(x) dG_h(y) - \int_t^{x_2} \int_{s_m(x_1)}^{s_m(t)} K(x, y) dF_h(x) dG_l(y), \end{aligned}$$

where for any $x \in (x_1, t)$ and $\tilde{x} \in (t, x_2)$

$$F_l(x) = \frac{F(x)}{F(t) - F(x_1)}; \quad F_h(\tilde{x}) = \frac{F(\tilde{x})}{F(x_2) - F(t)},$$

and for $y \in (s_m(x_1), s_m(t))$ and any $\tilde{y} \in (s_m(t), s_m(x_2))$

$$G_l(y) = \frac{G(y)}{F(t) - F(x_1)}; \quad G_h(\tilde{y}) = \frac{G(\tilde{y})}{F(x_2) - F(t)}.$$

²⁰ An implication is that increasing the number of meeting places always strictly increases match efficiency in terms of total match value. This can be seen by replacing K with the match value function xy in the proof of Proposition 4.5 below and noting that by assumption the match value function satisfies supermodularity.

Next, apply the change of variables $F_h(x) = F_l(\tilde{x})$ to x in the second integral and in the fourth integral, and $G_h(y) = G_l(\tilde{y})$ to y in the second integral and in the third integral.

We have

$$\begin{aligned} \Delta = \int_{x_1}^t \int_{s_m(x_1)}^{s_m(t)} & (K(x, y) + K(F_h^{-1}(F_l(x)), G_h^{-1}(G_l(y))) \\ & - K(x, G_h^{-1}(G_l(y))) - K(F_h^{-1}(F_l(x)), y)) \, dF_l(x) \, dG_l(y). \end{aligned}$$

The above is (strictly) positive because $F_h^{-1}(F_l(x)) > x$, $G_h^{-1}(G_l(y)) > y$ and K is (strictly) supermodular on $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$. *Q.E.D.*

The idea behind Proposition 4.5 is to write the revenue difference Δ between sorting the types in $(x_1, x_2) \times (s_m(t_1), s_m(t_2))$ into two meeting places and pooling all types in the region, as the integral of the inequality (4.2) where x varies between x_1 and t and correspondingly y between $s_m(x_1)$ and $s_m(t)$. This is achieved by change of variables. Note that other changes of variables would also work. For example, one can define new integration variables by setting $F_h(\tilde{x}) = 1 - F_l(x)$ and $G_h(\tilde{y}) = 1 - G_l(y)$. The proof of the proposition proceeds in a similar fashion; the only difference is that for each $x \in [x_1, t]$, inequality (4.2) applies to a different set of four points.

Proposition 4.3 implies that if K satisfies global PS then the perfect sorting maximizes the revenue. We have already noted in the beginning of this subsection that local PS of K is not sufficient to imply the optimality of the perfect sorting. Indeed, Proposition 4.3 does not imply that if $K_{12}(x, s_m(x)) > 0$ for some type x then this type must be separated from the adjacent types. One special case where the gap between the necessary condition of Proposition 4.2 and the sufficient condition of Proposition 4.3 is closed is when the type distributions are identical on the two sides. In this case, local PS implies that the common virtual type function is increasing. As a result, the virtual match value function is supermodular everywhere (both along and off the efficient matching path), and therefore global PS is satisfied.

More generally we can show that whenever the efficient matching path is linear, local PS is equivalent to supermodularity, and hence the necessary and sufficient conditions for the optimality of the perfect sorting coincide. To be precise, let F and G be such that

$s_m(x) = \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{R}$ and $\alpha > 0$. Then, the following are equivalent: i) K satisfies local PS at each $x \in [a_m, b_m]$; ii) K satisfies global PS on $[a_m, b_m]$; and iii) K is supermodular on $[a_m, b_m] \times [a_w, b_w]$. To see this, note that since supermodularity implies both local PS and global PS, and local PS implies global PS, it suffices to show that local PS implies supermodularity. By assumption $F(x) = G(\alpha x + \beta)$ for all $x \in [a_m, b_m]$. Differentiating, we obtain $f(x) = \alpha g(\alpha x + \beta)$ and $f'(x) = \alpha^2 g'(\alpha x + \beta)$. Substituting in $J'_w(s_m(x))$ we have $J'_m(x) = J'_w(s_m(x))$. Hence when K satisfies local PS on $(a_m, b_m) \times (a_w, b_w)$, both $J_m(x)$ and $J_w(y)$ are increasing. Thus, K is supermodular on the entire type space.

5. Related Literature

A classical reference in the price discrimination literature is Maskin and Riley (1984) (see also Mussa and Rosen, 1978). In both the standard price discrimination problem and our sorting structure design problems, the monopolist faces consumers with one-dimensional private information about their willingness to pay, and must provide incentives for self-selection. In a price discrimination problem, the monopolist controls the quality (or quantity) of the good provided. Consumers of different types self-select by choosing a price-quality combination from the schedule offered by the monopolist. In contrast, in our sorting structure design problem, the monopolist offers a pair of fee schedules for an array of meeting places with different match quality, and the match quality schedules must be consistent with the type of agents that self-select into that meeting place.

The most closely related paper in the price discrimination literature is Rayo (2002).²¹ He considers the price-discrimination problem of a monopolist that sells a status good. In his benchmark model, there is no intrinsic quality dimension to different varieties of the good, and buyers of one variety care only about who else are buying the same variety. Our result of weak sorting implies that this is essentially the same price discrimination problem considered here if one restricts to a symmetric, matching environment. His results on

²¹ We thank Jonathan Levin for alerting us to the paper.

when providing different varieties to different types is optimal can therefore be obtained as a special case of our necessary and sufficient conditions for the perfect sorting to be optimal.

Inderst (2001) challenges the classical result in optimal contract under incomplete information that it is optimal for the uninformed side (the monopolist) to offer low types distorted contracts in order to extract more rents from higher types. His paper looks at contract design in a matching market environment with frictions and shows that the distortion result does not hold anymore. In particular, for low enough search frictions all contracts are free of distortion. The driving force of the result is that in a search and matching environment reservation values are type dependent as higher types will generally have more match opportunity and therefore higher reservation values. In contrast, our no-distortion result under conditions of path supermodularity does not rely on type-dependent reservation values, and is generated by feasibility restrictions on match quality provision in a two-sided matching market.

There is a growing literature on competing search markets in which sorting of heterogeneous types plays a role (Moen, 1997; Mortensen and Wright, 1997; Shimer, 1995). The main idea of this literature is to allow for more than one search markets. A new search markets can be opened at any point, and workers and firms, at any point in time, choose in which of the currently active markets to search. Participation is either free, or there is a fixed participation cost, equal across the markets. In equilibrium, markets differ in terms of average quality of the participating workers as well as tightness (the ratio of participating workers to job openings). The tradeoff across markets is between quality of the expected match and duration of the unemployment spell. This tradeoff is driven by the matching technology: the speed at which workers (firms) draw matching opportunities is decreasing (increasing) in the market tightness. These models assume limited type heterogeneity, with typically two types of workers (high quality and low quality) and one type of firm. In contrast, in our model both sides of the matching market have a continuum of types and sorting (or self-selection) is based on the tradeoff between price and match quality.

The present paper grew out of our previous work on dynamic sorting (Damiano, Li and Suen, 2004). The two papers share the same interest in efficiency of matching markets

in the presence of search frictions.²² In both papers, search frictions are modeled by the primitive search technology of random meeting. In Damiano, Li and Suen, dynamic sorting provides higher types more search opportunities and improves matching efficiency. In the present paper, price discrimination by the monopolist creates directed search markets and can achieve the efficient matching. In a companion paper (Damiano and Li, 2004), we use a simplified framework of the present paper to study how competition among matchmakers can affect the sorting structure and matching efficiency.

²² McAfee (2002) also considers the issue of matching efficiency in random matching markets. He shows a relatively large efficiency gain can be made by splitting one market into two, but he does not consider the incentives of market participants to reveal their type information.

Appendix

A.1. Proof of Lemma 4.1

PROOF. For notational convenience, denote $t + \epsilon$ as \tilde{t} . Define

$$R_t(\epsilon) = \int_t^{\tilde{t}} s_m(x) J_m(x) dF(x) + \int_{s_m(t)}^{s_m(\tilde{t})} s_w(y) J_w(y) dG(y);$$

$$\hat{R}_t(\epsilon) = \int_t^{\tilde{t}} \mu_w(s_m(t), s_m(\tilde{t})) J_m(x) dF(x) + \int_{s_m(t)}^{s_m(\tilde{t})} \mu_m(t, \tilde{t}) J_w(y) dG(y).$$

Then, $\Delta_t(\epsilon) = R_t(\epsilon) - \hat{R}_t(\epsilon)$. Clearly, we have $\Delta_t(0) = 0$. We are going to take derivative with respect to ϵ of $R_t(\epsilon)$ and $\hat{R}_t(\epsilon)$, and study the difference as ϵ goes to zero.

For the first derivatives, let

$$A_t(\epsilon) = s_m(\tilde{t}) J_m(\tilde{t}) + (\tilde{t}) J_w(s_m(\tilde{t})).$$

We have

$$R'_t(\epsilon) = f(\tilde{t}) A_t(\epsilon).$$

Next, we have

$$\begin{aligned} \hat{R}'_t(\epsilon) = & \mu_w(s_m(t), s_m(\tilde{t})) J_m(\tilde{t}) f(\tilde{t}) + \mu_m(t, \tilde{t}) J_w(s_m(\tilde{t})) g(s_m(\tilde{t})) s'_m(\tilde{t}) \\ & + \mu'_w(s_m(t), s_m(\tilde{t})) s'_m(\tilde{t}) \int_t^{\tilde{t}} J_m(x) dF(x) + \mu'_m(t, \tilde{t}) \int_{s_m(t)}^{s_m(\tilde{t})} J_w(y) dG(y), \end{aligned}$$

where $\mu'_w(s_m(t), s_m(\tilde{t}))$ denotes the derivative of $\mu_w(s_m(t), s_m(\tilde{t}))$ with respect to the second argument, and similarly $\mu'_m(t, \tilde{t})$ denotes the derivative of $\mu_m(t, \tilde{t})$ with respect to the second argument. Let

$$\begin{aligned} \hat{A}_t(\epsilon) = & \mu_w(s_m(t), s_m(\tilde{t})) J_m(\tilde{t}) + \mu_m(t, \tilde{t}) J_w(s_m(\tilde{t})) \\ & + (s_m(\tilde{t}) - \mu_w(s_m(t), s_m(\tilde{t}))) J_m(t, \tilde{t}) + (\tilde{t} - \mu_m(t, \tilde{t})) J_w(s_m(t), s_m(\tilde{t})), \end{aligned}$$

where $J_m(t, \tilde{t}) = \int_t^{\tilde{t}} J_m(x) dF(x)$ and $J_w(s_m(t), s_m(\tilde{t})) = \int_{s_m(t)}^{s_m(\tilde{t})} J_w(y) dG(y)$. We can write

$$\hat{R}'_t(\epsilon) = f(\tilde{t}) \hat{A}_t(\epsilon).$$

It is easy to see that $A_t(0) = \hat{A}_t(0)$; therefore, $\Delta'_t(0) = 0$.

Now consider the second derivative:

$$\begin{aligned} R''_t(\epsilon) &= f'(\tilde{t})A_t(\epsilon) + f(\tilde{t})A'_t(\epsilon); \\ \hat{R}''_t(\epsilon) &= f'(\tilde{t})\hat{A}_t(\epsilon) + f(\tilde{t})\hat{A}'_t(\epsilon). \end{aligned}$$

Taking derivatives, we have

$$A'_t(\epsilon) = s'_m(\tilde{t})J_m(\tilde{t}) + s_m(\tilde{t})J'_m(\tilde{t}) + J_w(s_m(\tilde{t})) + (\tilde{t})J'_w(s_m(\tilde{t}))s'_m(\tilde{t}),$$

and therefore

$$A'_t(0) = s'_m(t)J_m(t) + s_m(t)J'_m(t) + J_w(s_m(t)) + tJ'_w(s_m(t))s'_m(t).$$

On the other hand, we have

$$\begin{aligned} \hat{A}'_t(\epsilon) &= \mu'_w(s_m(t), s_m(\tilde{t}))s'_m(\tilde{t})J_m(\tilde{t}) + J'_m(\tilde{t})\mu_w(s_m(t), s_m(\tilde{t})) + \mu'_m(t, \tilde{t})J_w(s_m(\tilde{t})) \\ &\quad + \mu_m(t, \tilde{t})J'_w(s_m(\tilde{t}))s'_m(\tilde{t}) + (s_m(\tilde{t}) - \mu_w(s_m(t), s_m(\tilde{t})))J'_m(t, \tilde{t}) \\ &\quad + (1 - \mu'_m(t, \tilde{t}))J_w(s_m(t), s_m(\tilde{t})) + (s'_m(\tilde{t}) - \mu'_w(s_m(t), s_m(\tilde{t})))s'_m(\tilde{t})J_m(t, \tilde{t}) \\ &\quad + (\tilde{t} - \mu_m(t, \tilde{t}))J'_w(s_m(t), s_m(\tilde{t}))s'_m(\tilde{t}), \end{aligned}$$

where $J'_m(t, \tilde{t})$ denotes the derivative of $J_m(t, \tilde{t})$ with respect to the second argument, and $J'_w(s_m(t), s_m(\tilde{t}))$ similarly denotes the derivative of $J_w(s_m(t), s_m(\tilde{t}))$ with respect to the second argument. Noting that

$$\lim_{\epsilon \rightarrow 0} \mu'_m(t, \tilde{t}) = \lim_{\epsilon \rightarrow 0} \mu'_w(s_m(t), s_m(\tilde{t})) = \frac{1}{2},$$

we obtain:

$$\hat{A}'_t(0) = s'_m(t)J_m(t) + s_m(t)J'_m(t) + J_w(s_m(t)) + tJ'_w(s_m(t))s'_m(t).$$

Therefore, $\Delta''_t(0) = 0$.

We are led to compute the third derivatives of $\Delta_t(\epsilon)$:

$$\begin{aligned} R'''_t(\epsilon) &= f''(\tilde{t})A_t(\epsilon) + 2f'(t + \epsilon)A'_t(\epsilon) + f(\tilde{t})A''_t(\tilde{t}); \\ \hat{R}'''_t(\epsilon) &= f''(\tilde{t})\hat{A}_t(\epsilon) + 2f'(\tilde{t})\hat{A}'_t(\epsilon) + f(\tilde{t})\hat{A}''_t(\tilde{t}). \end{aligned}$$

Taking derivatives and evaluating at $\epsilon = 0$, we have

$$\begin{aligned} A_t''(0) &= s_m''(t)J_m(t) + 2s_m'(t)J_m'(t) + s_m(t)J_m''(t) + 2J_w'(s_m(t))s_m'(t) \\ &\quad + t(J_w''(s_m(t))(s_m'(t))^2 + J_w'(s_m(t))s_m''(t)), \end{aligned}$$

and

$$\begin{aligned} \hat{A}_t''(0) &= s_m''(t)J_m(t) + s_m'(t)J_m'(t) + s_m'(t)J_w'(s_m(t)) + s_m(t)J_m''(t) \\ &\quad + t(J_w''(s_m(t))(s_m'(t))^2 + J_w'(s_m(t))s_m''(t)) + \frac{1}{2}s_m'(t)(J_m'(t) + J_w'(s_m(t))). \end{aligned}$$

Therefore,

$$\Delta_t'''(0) = \frac{1}{2}(J_m'(t) + J_w'(s_m(t)))f(t)s_m'(t).$$

This completes the proof of the lemma. Q.E.D.

PROOF OF PROPOSITION 4.4. Let $X = (x_1, x_2]$ and define $Y = (s_m(x_1), s_m(x_2)]$. For any positive integer n , let $\{t_0^n, t_1^n, \dots, t_{2^n}^n\}$ be a set of points in $[x_1, x_2]$ with $t_0^n = x_1$ and for any positive integer $i \leq 2^n$:

$$F(t_i^n) - F(t_{i-1}^n) = \frac{F(x_2) - F(x_1)}{2^n}.$$

For each $1 \leq i \leq 2^n$, let $T_i^n = (t_{i-1}^n, t_i^n]$, and $Q_i^n = (s_m(t_{i-1}^n), s_m(t_i^n)]$. We will use T^n and Q^n to denote the collection of T_i^n 's and Q_i^n 's respectively. For any $x \in X$, $T^n(x)$ is the element of T^n that contains x . Similarly, for any $y \in Y$, $Q^n(y)$ is the element of Q^n that contains y . For given n , q^n is a pair of match quality schedules $\langle q_m^n, q_w^n \rangle$ defined by:

$$\begin{aligned} q_m^n(x) &= \begin{cases} q_m^*(x), & \text{if } x \notin X; \\ \mu_w(Q^n(s_m(x))), & \text{otherwise.} \end{cases} \\ q_w^n(y) &= \begin{cases} q_w^*(y), & \text{if } y \notin Y; \\ \mu_m(T^n(s_w(y))), & \text{otherwise.} \end{cases} \end{aligned}$$

Note that q^n is feasible and incentive compatible and differs from q^* only in the region $X \times Y$. From the revenue formula of Proposition (3.6) we can write the difference Δ^n in the revenue generated by q^n and q^* as:

$$\Delta^n = \sum_{i=1}^{2^n} \int_{t_{i-1}^n}^{t_i^n} \int_{s_m(t_{i-1}^n)}^{s_m(t_i^n)} K(x, y) \frac{dG(y) dF(x)}{F(t_i^n) - F(t_{i-1}^n)} - \int_{x_1}^{x_2} K(x, s_m(x)) dF(x).$$

For any $Z \subset [a_m, b_m] \times [a_w, b_w]$, let $\mu_K(Z)$ denote the conditional expectation of K in Z .

We can rewrite the above as:

$$\Delta^n = \sum_{i=1}^{2^n} (F(t_i^n) - F(t_{i-1}^n)) \mu_K(T_i^n \times Q_i^n) - \int_{x_1}^{x_2} K(x, s_m(x)) dF(x). \quad (\text{A.1})$$

Given (A.1), to complete the proof it suffices to show that Δ^n converges to 0 as n tends to infinity. We proceed by first showing that Δ^n is proportional to the difference in the expected value of K with respect to two different probability measures, λ^n and λ , on the space $(X \times Y, \mathcal{B}(X) \times \mathcal{B}(Y))$, where $\mathcal{B}(X), \mathcal{B}(Y)$ denote the Borel sigma-fields on X and Y respectively. In a second step, we show that λ^n converges weakly to λ as n becomes large. The claim, then, follows immediately from the continuity of K .

Let π_X and π_Y denote the probability measures, on $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ respectively, defined according to the density functions $f_X(x) = f(x)/(F(x_2) - F(x_1))$ and $g_Y(y) = g(y)/(G(s_m(x_1)) - G(s_m(x_2)))$. Let π denote the product measure. First, we construct a line measure λ on the product space $(X \times Y, \mathcal{B}(X) \times \mathcal{B}(Y))$ in the following way. For any $x \in X$ let γ_x be a measure on $(Y, \mathcal{B}(Y))$ such that for any $V \in \mathcal{B}(Y)$:

$$\gamma_x V = \begin{cases} 1 & \text{if } s_m(x) \in V; \\ 0 & \text{if otherwise.} \end{cases}$$

By continuity of s_m , for any $x \in X$ and $V \in \mathcal{B}(Y)$,

$$\{x | \gamma_x V = 1\} = \{x | s_m(x) \in V\} \in \mathcal{B}(X);$$

$$\{x | \gamma_x V = 0\} = \{x | s_m(x) \notin V\} \in \mathcal{B}(X);$$

$$\{x | \gamma_x V = k\} = \emptyset \in \mathcal{B}(X) \quad \text{for all } k \neq 0, 1.$$

Thus, for each $V \in \mathcal{B}(Y)$, the mapping from X to $[0, 1]$ defined by $\gamma_x V$ is $\mathcal{B}(X)$ -measurable, and the family of measures $\Gamma = \{\gamma_x\}_{x \in X}$ is a kernel from $(X, \mathcal{B}(X))$ to $(Y, \mathcal{B}(Y))$. Moreover, Γ is a probability kernel as $\gamma_x Y = 1$ for all x . Since π_X is a probability measure on $\mathcal{B}(X)$, the iterated integral $\pi_X \otimes \Gamma$ defines a probability measure λ on $\mathcal{B}(X) \times \mathcal{B}(Y)$. (A more detailed account of this procedure for constructing probability measures on product spaces can be found in Pollard (2001), Chapter 4.) For each $Z \in \mathcal{B}(X) \times \mathcal{B}(Y)$, λZ is given by $\pi_X(\Gamma Z)$, where ΓZ is a function from X into $[0, 1]$ defined by $\Gamma Z(x) = \gamma_x \{y \in Y | (x, y) \in Z\}$. That is, λZ is given by

$$\pi_X(\Gamma Z) = \pi_X \{x | (x, s_m(x)) \in Z\}.$$

The integral of K with respect to λ is $\lambda K_+ - \lambda K_-$, where $K_+(x, y) = \max\{K(x, y), 0\}$ and $K_- = \max\{-K(x, y), 0\}$. By the monotone convergence theorem, we can write λK_+ as the limit of

$$\sum_{j=1}^{4^n} \left(\frac{j-1}{2^n} \right) \lambda \left\{ (x, y) \mid \frac{j-1}{2^n} \leq K_+(x, y) < \frac{j-1}{2^n} \right\} + 2^n \lambda \{(x, y) \mid K_+(x, y) \geq 2^n\},$$

which by definition of λ is equal to

$$\sum_{j=1}^{4^n} \left(\frac{j-1}{2^n} \right) \pi_X \left\{ x \mid \frac{j-1}{2^n} \leq K_+(x, s_m(x)) < \frac{j-1}{2^n} \right\} + 2^n \lambda \{(x, y) \mid K_+(x, s_m(x)) \geq 2^n\}.$$

The limit of the above as $n \rightarrow \infty$ is $\pi_X K_+(x, s_m(x))$. An analogous result holds for K_- . Thus, we can write the expectation of K with respect to λ as

$$\lambda K = \int_{x_1}^{x_2} K(x, s_m(x)) \frac{dF(x)}{F(x_2) - F(x_1)}. \quad (\text{A.2})$$

We now construct a sequence of probability measures on $(X \times Y, \mathcal{B}(X \times Y))$. For any $x \in X$ let γ_x^n be the measure on $(Y, \mathcal{B}(Y))$ such that for any $V \in \mathcal{B}(Y)$:

$$\gamma_x^n V = \frac{\pi_Y(V \cap Q^n(s_m(x)))}{\pi_Y Q^n(s_m(x))}.$$

Note that for any $V \in \mathcal{B}(Y)$, if $\gamma_x^n V = k$ then $\gamma_{\tilde{x}}^n V = k$ for all $\tilde{x} \in T^n(x)$. It follows that the set $\{x \in X \mid \gamma_x^n V = k\} \in \mathcal{B}(X)$ because it is either empty or is the union of some elements of T^n . Therefore the mapping from X to $[0, 1]$ defined by $\gamma_x V$, is $\mathcal{B}(X)$ -measurable for all $V \in \mathcal{B}(Y)$ and the family of measures $\Gamma^n = \{\gamma_x^n\}_{x \in X}$ is a kernel from $(X, \mathcal{B}(X))$ to $(Y, \mathcal{B}(Y))$. Moreover, since $\gamma_x^n Y = 1$ for all x , Γ^n is a probability kernel. As in the construction of λ , this implies that the iterated integral $\pi_X \otimes \Gamma^n$ defines a probability measure λ^n on $\mathcal{B}(X) \times \mathcal{B}(Y)$. For any $Z \in \mathcal{B}(X) \times \mathcal{B}(Y)$, we have

$$\lambda^n Z = \pi_X (\Gamma^n Z),$$

where $\Gamma^n Z$ is a function from X into $[0, 1]$ defined by $\Gamma^n Z(x) = \gamma_x^n \{y \in Y \mid (x, y) \in Z\}$.

For any $Z \in \mathcal{B}(X) \times \mathcal{B}(Y)$ we can always write $\lambda^n Z$ as

$$\begin{aligned} \lambda^n Z &= \sum_{i=1}^{2^n} \lambda^n (T_i^n \times Y) \frac{\lambda^n (Z \cap (T_i^n \times Y))}{\lambda^n (T_i^n \times Y)} \\ &= \sum_{i=1}^{2^n} \int_{T_i^n} dF(x) \frac{\int_{T_i^n} \int_{\{y \in Y | (x,y) \in Z\} \cap Q_i^n} dG(y) dF(x)}{\int_{T_i^n} dF(x) \int_{Q_i^n} dG(y)} \\ &= \sum_{i=1}^{2^n} \pi(T_i^n \times Y) \frac{\pi((T_i^n \times Q_i^n) \cap Z)}{\pi(T_i^n \times Q_i^n)}. \end{aligned}$$

From the above it can be seen that λ^n is obtained by restricting π to each set of types $T_i^n \times Y$ and conditioning it onto $T_i^n \times Q_i^n$.

Using the notation $\mu_K(T_i^n \times Q_i^n)$ for the conditional expectation of K on the set $T_i^n \times Q_i^n$, we can write the expectation of K with respect to λ^n as:

$$\lambda^n K = \sum_{i=1}^{2^n} \pi(T_i^n \times Y) \mu_K(T_i^n \times Q_i^n) = \sum_{i=1}^{2^n} \frac{(F(t_i^n) - F(t_{i-1}^n))}{F(x_2) - F(x_1)} \mu_K(T_i^n \times Q_i^n). \quad (\text{A.3})$$

From (A.1), (A.2) and (A.3) we obtain

$$\Delta^n = (F(x_2) - F(x_1))(\lambda^n K - \lambda K).$$

We are left to show that λ^n converges weakly to λ . We prove the equivalent claim that the distribution function Λ^n of λ^n , converges to the distribution function Λ of λ for every continuity point of Λ . First notice that, for any $z = (z_m, z_w) \in X \times Y$:

$$\Lambda(z) = \lambda((x_1, z_m] \times (s_m(x_1), z_w]) = \min \left\{ \frac{F(z_m) - F(x_1)}{F(x_2) - F(x_1)}, \frac{G(z_w) - F(x_1)}{F(x_2) - F(x_1)} \right\}.$$

Thus, Λ is continuous on $X \times Y$ and we need to show that Λ^n converges pointwise to Λ . For all $z \in X \times Y$, define

$$I^n(z) = \min\{\min\{i | t_i^n \geq z_m\}, \min\{i | s_m(t_i^n) \geq z_w\}\}.$$

By construction,

$$\begin{aligned} \Lambda^n(z) &= \lambda^n([x_1, z_m] \times [s_m(x_1), z_w]) \\ &= \sum_{i=1}^{2^n} \pi(T_i^n \times Y) \frac{\pi((T_i^n \times Q_i^n) \cap ([x_1, z_m] \times [s_m(x_1), z_w]))}{\pi(T_i^n \times Q_i^n)}. \end{aligned}$$

Note that $T_i^n \times Q_i^n \subseteq [x_1, z_m] \times [s_m(x_1), z_w]$ for all $i \leq I^n(z) - 1$, while $(T_i^n \times Q_i^n) \cap ([x_1, z_m] \times [s_m(x_1), z_w]) = \emptyset$ for $i \geq I^n(z) + 1$. We can rewrite $\Lambda^n(z)$ as:

$$\sum_{n=1}^{I^n(z)-1} \pi(T_i^n \times Y) + \frac{\pi(A_{I^n(z)} \times Y)}{\pi(A_{I^n(z)} \times Q_{I^n(z)})} \pi\left((t_{I^n(z)-1}^n, \min\{t_{I^n(z)}^n, z_m\}] \times (s_m(t_{I^n(z)-1}^n), \min\{s_m(t_{I^n(z)}^n), z_w\}]\right).$$

The second term in the expression above goes to zero when n is large. The first term is equal to $(F(t_{I^n(z)}^n) - F(x_1))/(F(x_2) - F(x_1))$. By continuity of F we have:

$$\lim_{n \rightarrow \infty} \Lambda^n(z) = \frac{F\left(\lim_{n \rightarrow \infty} t_{I^n(z)-1}^n\right) - F(x_1)}{F(x_2) - F(x_1)}.$$

Moreover, $\lim_{n \rightarrow \infty} t_{I^n(z)-1}^n = \min\{z_m, s_w(z_w)\}$. The claim follows immediately.

Q.E.D.

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