

# Strategizing Pressure - Was Cristiano Ronaldo Right?

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*Abstract.* After Portugal lost a penalty shootout with Ronaldo unused as the last penalty-taker before the sudden-death phase, he claimed that there was no mistake with his position. We show that Ronaldo would be right that the order of the penalty-takers before the sudden-death phase is irrelevant to the outcome of the shootout, if the probability of scoring of a penalty-taker is unaffected by whether his team is ahead or behind. However, Ronaldo would be wrong if a penalty-taker is less likely to score when his team is behind (under pressure) than when it is ahead (under no pressure). Under the assumption that pressure has proportionally decreasing impact on all penalty-takers of a team, which is satisfied if pressure reduces the probabilities of scoring by all penalty-takers by the same amount, we show that it is strictly dominant for the team to assign its penalty-takers in the decreasing order of their probabilities of scoring under no pressure.

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# 1 Introduction

After their 120-minute scoreless draw, Portugal and Spain needed a penalty shootout to decide which team would go through to the Euro 2012 final. Cristiano Ronaldo, the Portuguese captain and superstar, was supposed to be the last penalty-taker in the fixed-length phase of five kicks, against the Spanish captain and his Real Madrid teammate, Iker Casillas. But the football gods intervened. With Spain going first and missing the first attempt through Xabi Alonso but converting the next three including the fourth one through Sergio Ramos a la Panenka, Portugal managed to miss both the first one through Joao Moutinho and the fourth through Bruno Alves. When Cesc Fabregas converted for Spain, Portugal lost the shootout with their arguably best penalty-taker unused. When asked later by reporters, Ronaldo explained that he was told by his coach to take the last penalty, and volunteered that it was not reason why Portugal went out to Spain in the shootout. Was Ronaldo right? Is the order of penalty-taking irrelevant for winning a penalty shootout?

We show that Ronaldo was right if the probability of scoring of a penalty-taker is independent of whether his team is ahead or behind. When given the opportunity, penalty-takers assigned towards the end of the fixed-length phase can be decisive by scoring or by missing, or make their next rival penalty-taker decisive, as those assigned to the sudden-death phase. While it is strictly dominant for a team to assign penalty-takers to the sudden-death phase in decreasing order probability of scoring and those with the highest probabilities of scoring to the fixed-length phase, but the order in the fixed-length phase is irrelevant. If we think of a penalty-taker who has a greater probability of scoring under no pressure as the “better” penalty-taker, then a team should put its top 5 penalty-takers in the fixed-length phase of five kicks, with no particular order, the 6th best in the first position of the sudden-death phase, and so on. So, Ronaldo, being the best penalty-taker for Portugal, should be assigned to the fixed-length of five kicks, but whether he goes first, and thus always gets to take his penalty but is never decisive, or goes fifth, and thus gets to take a decisive penalty more often at the risk of not being used, is irrelevant to Portugal’s chance of winning the shootout.

The assumption that the probability of scoring of a penalty-taker is independent of whether his team is ahead or behind, however, has a counter-factual implication. We show

that for fixed probabilities of scoring of the penalty-takers on each team, and for any fixed order among them, the probability of winning the shootout of a team does not depend on whether it is the first mover or the second mover. This is contradicted by revealed preference of teams after they win the coin toss, as they overwhelmingly choose to go first.

The overwhelming answer given by teams of why they prefer to go first in a penalty shootout is that going first and scoring give them a chance to put their rivals under pressure. We modify the game of penalty shootout by assuming that penalty-takers have smaller probabilities of scoring when their team is behind – under pressure – than when the scores are tied or when their team is ahead.<sup>1</sup> This assumption has an immediate implication to the first mover in assigning penalty-takers to the sudden-death phase. Each penalty-taker in the sudden-death phase from the first mover is never under pressure himself, and when given the opportunity, one with a greater probability of scoring under no pressure is more likely to put the rival penalty-taker in the same round under pressure and reduce the latter’s probability of scoring. Thus, the incentive to order its penalty-takers in the sudden-death phase in decreasing probabilities of scoring under no pressure is strengthened for the first mover. This captures the “pressuring the rival” motivation for going first in a penalty shootout.

The motivation of pressuring the rival however becomes irrelevant to a penalty-taker when his team is already behind. For example, a penalty-taker from the second mover assigned to the sudden-death phase is already under pressure when the rival penalty-taker from the first mover in the same round has scored, and whether the former scores under pressure or misses under pressure, the next rival penalty-taker is never under pressure when given the opportunity. When given the opportunity, the present penalty-taker from the second mover can help his team win, but only if he scores under pressure to pull even and in the next round his rival misses so that his teammate can score, under no pressure, to win the shootout. Both the present and the next penalty-takers have their probability of scoring reduced by pressure, but if pressure has “proportionally decreasing impact on scoring” by the two penalty-takers, assigning the one with a greater probability of scoring under no pressure – the better penalty-taker – to the present round increases the chance of “winning under pressure.” The assumption that a better penalty-taker is less negatively impacted by

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<sup>1</sup>See Apesteguia and Palacios-Huerta (2010) for empirical support of this assumption.

pressure proportionally implies that pressure is order-preserving: a better penalty-taker also scores with a greater probability under pressure. As a result, assigning the better penalty-taker to the present round also reduces the second mover’s chance of losing under pressure.

The motivation of pressuring the rival is also irrelevant to a penalty-taker when his team is already ahead, or in the case of the second mover, when the scores are tied. An example of the second case is when a penalty-taker from the second mover assigned to the sudden-death phase steps forward under no pressure after the rival penalty-taker in the present round has just missed. The penalty-taker from the second mover in the present round cannot put the rival penalty-taker in the next round under pressure: the former wins the shootout for the second mover already by scoring under no pressure, and when he misses the scores remain tied so the latter is not under pressure. However, there is a risk of losing under no pressure for the second mover: its penalty-taker in the present round may miss under no pressure and its penalty-taker in the next round may miss again, but under pressure, after the rival penalty-taker scores. Both the present and the next penalty-takers have their probability of missing increased by pressure, but if pressure has “proportionally decreasing impact on missing” by the two penalty-takers, assigning the one with a greater probability of scoring under no pressure – the better penalty-taker – to the present round will reduce the chance of “losing under no pressure.” At the same time, assigning the better penalty-taker to the present round also increases the second mover’s chance of winning under no pressure.

Both the assumption of proportionally decreasing impact on scoring and the assumption of proportionally decreasing impact on missing by the same two players are satisfied if pressure reduces their probabilities scoring by the same amount. We show that under these assumptions, regardless of whether the team is the first mover or the second mover, conditional on reaching any round with any score difference, whether it is in the fixed-length phase or in the sudden-death phase, it is weakly dominant for the team to put a better penalty-taker – one with a greater probability of scoring under no pressure – in the present round rather than the next round. The reason behind this result combines the same three ingredients: the argument of pressuring the rival when the rival is less likely to score under pressure than under no pressure; the argument of increasing the chance of winning under pressure when it has proportionally decreasing impact on scoring, and the argument of de-

creasing the chance of losing under no pressure when it has proportionally decreasing impact on missing.

So, Ronaldo may be wrong after all: assuming he was a better penalty-taker than Alves, Portugal’s chance of winning the shootout would be increased by having Ronaldo go before Alves rather than the other way around. Indeed, under the assumptions of proportionally decreasing impact on scoring and on missing by all pairs of penalty-takers of a team, it is strictly dominant for the team to assign its best penalty-taker first, the second best the second, and so on. Cristiano Ronaldo should have gone first as the best penalty-taker on the Portugal team.

## 2 Model

There are two teams,  $T = A, B$ . Each team consists of  $N \geq 2$  penalty-takers. Before the shootout, each team submits an order of its  $N$  penalty-takers. Denote as  $l_j^T$  the penalty-taker assigned by team  $T$  to the  $j$ -th position on the list  $l^T = (l_1^T, \dots, l_N^T)$ .

### 2.1 Penalty shootout

The rules of penalty shootout are as follows. There are  $N$  rounds of penalty kicks.

In each round  $j = 1, \dots, N$ , penalty-taker  $l_j^A$  takes the penalty first and  $l_j^B$  second. We refer to team  $A$  as the first mover,  $B$  as the second mover.

There are two phases of the shootout. The first phase (“fixed-length”) consists of  $F$  rounds, with  $1 \leq F \leq N - 1$ .<sup>2</sup> If penalty-takers  $(l_1^A, \dots, l_F^A)$  score 1 or more kicks than  $(l_1^B, \dots, l_F^B)$  at the end of fixed-length phase, round  $F$ , team  $A$  wins the shootout; if penalty-takers  $(l_1^B, \dots, l_F^B)$  score 1 or more kicks than  $(l_1^A, \dots, l_F^A)$ , team  $B$  wins; if the two sets of penalty-takers score the same number of kicks after round  $F$ , the shootout continues to the second phase (“sudden-death”).

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<sup>2</sup>In association football (soccer),  $F = 5$  while  $N = 11$ . If some penalty-takers are removed from their team due to injury or red cards, the team with more penalty-takers must reduce the number of eligible penalty-takers for the shootout to match the other team.

In each round  $j \geq F + 1$  of the sudden-death phase: if  $l_j^A$  scores and  $l_j^B$  misses then team  $A$  wins; if  $l_j^A$  misses and  $l_j^B$  scores then  $B$  wins; if both  $l_j^A$  and  $l_j^B$  score or both miss, the shootout continues to round  $j + 1$ , unless  $j = N$ , in which case the shootout ends and the winner is determined by a fair coin toss.<sup>3</sup>

## 2.2 Score difference

We refer to the difference between the realized number of successful kicks by team  $A$  and that by team  $B$  as the “score difference.” For each  $T = A, B$ , we denote as  $s_j^T$  the score difference that faces penalty-taker  $l_j^T$  of team  $T$ ,  $j = 1, \dots, N$ . By the rules of the shootout, in the sudden-death phase  $s_j^A = 0$  for each penalty-taker  $l_j^A$  of team  $A$  with  $j \geq F + 1$ . For each penalty-taker  $l_j^B$  with  $j \geq F + 1$ , we have  $s_j^B = 0$  or  $1$ , corresponding to either  $l_j^A$  missed and scored for team  $A$ .

In the fixed-length phase, the score difference  $s_j^A$  for penalty-taker  $l_j^A$  from team  $A$ , with  $1 \leq j \leq F$ , satisfies

$$\max\{-(j-1), -(F-j+1)\} \leq s_j^A \leq \min\{j-1, F-j+1\} \quad (1)$$

The score difference is at least  $-(j-1)$ , which happens all proceeding  $j-1$  penalty-takers from team  $A$  missed and all from  $B$  scored, and is at most  $j-1$ , when the opposite has happened. Further, the shootout would have ended before penalty-taker  $l_j^A$ , either when  $s_j^A \leq -(F-j+2)$ , because team  $A$  would lose even if all penalty-takers from  $A$  from round  $j$  through to round  $F$  score and all from  $B$  from the same remaining round in the fixed-length phase miss, or when  $s_j^A \geq F-j+2$ , because team  $A$  would win even if all remaining penalty-takers from  $A$  in the fixed-length phase miss and all from  $B$  score.

We say that penalty-taker  $l_j^A$  is “decisive by scoring” if team  $A$  wins the shootout once  $l_j^A$  scores, and that  $l_j^A$  is “decisive by losing” if  $A$  loses once  $l_j^A$  misses. Penalty-taker  $l_j^A$ ,  $j = 1, \dots, F$ , assigned to the fixed-length phase by team  $A$ , is decisive by scoring if  $s_j^A =$

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<sup>3</sup>This is a simplification for the game to be well-defined. According to Law 10 (“Determining the Outcome of a Match”) of the International Football Association Board’s (IFAB) *Laws of the Game*, the shootout continues in the sudden-death phase when it is still tied after all eligible penalty-takers including the goalkeeper have taken their penalty kicks, but teams are not required to keep the same order.

$F - j + 1$ , and decisive by missing if  $s_j^A = -(F - j + 1)$ . From the bounds (1) that the score difference  $s_j^A$  has to satisfy, only penalty-takers assigned towards the end of the fixed-length phase can be decisive either way. More precisely,  $l_j^A$  can be decisive only if

$$2(j - 1) \geq F.$$

For example, for  $F = 5$ , only  $l_4^A$  and  $l_5^A$  can be decisive:  $l_4^A$  is decisive in scoring at  $s_4^A = 2$  and is decisive in missing at  $s_4^A = -2$ , and  $l_5^A$  is decisive in scoring at  $s_5^A = 1$  and is decisive in missing at  $s_5^A = -1$ . Also, regardless of the position, penalty-taker  $l_j^A$  cannot be decisive both by scoring and by missing for the same score difference  $s_j^A$ . Finally, since  $A$  is the first mover, any penalty-taker  $l_j^A$ ,  $j \geq F + 1$ , assigned to the sudden-death phase is never decisive by scoring or by losing.

For the second mover, team  $B$ , the score difference  $s_j^B$  for penalty-taker  $l_j^B$  with  $1 \leq j \leq F$  in the fixed-length satisfies

$$\max\{-(j - 1), -(F - j)\} \leq s_j^B \leq \min\{j, F - j + 1\} \quad (2)$$

We say that penalty-taker  $l_j^B$  is “decisive by scoring” if team  $B$  wins the shootout once  $l_j^B$  scores, and that  $l_j^B$  is “decisive by losing” if  $B$  loses once  $l_j^B$  misses. Penalty-taker  $l_j^B$ ,  $j = 1, \dots, F$ , assigned to the fixed-length phase is decisive by scoring if  $s_j^B = -(F - j)$ , and decisive by missing if  $s_j^B = F - j + 1$ . As is true for the first mover, the bounds (2) on  $s_j^B$  imply that  $l_j^B$  is decisive either way only if

$$2j - 1 \geq F.$$

For example, for  $F = 5$ ,  $l_1^B$  and  $l_2^B$  are never decisive either way, while  $l_j^B$ ,  $j = 3, 4, 5$ , is decisive in scoring at  $s_j^B = -(F - j)$  and is decisive in missing at  $s_j^B = F - j + 1$ . Note that  $l_F^B$  is always decisive, by scoring or by missing, because  $s_F^B$  is either 0 or 1 by (2). As is true for  $A$ , no penalty-taker  $l_j^B$  can be decisive both by scoring and by missing for the same score difference  $s_j^B$ . In contrast to team  $A$ , however, any penalty-taker  $l_j^B$ ,  $j \geq F + 1$ , assigned to the sudden-death phase is always decisive: he is decisive by scoring if  $s_j^B = 0$ , and is decisive by missing if  $s_j^B = 1$ .

### 3 Ronaldo Could be Right

We consider the following game of a penalty shootout. We assume that the probabilities of scoring are independent across all penalty-takers and both teams. Each of the  $N$  penalty-takers from either team is represented by an exogenous probability of scoring strictly between 0 and 1.<sup>4</sup> For convenience, we denote the probability  $l_j^T$  scores as  $q_j^T$ .<sup>5</sup> Each team is risk-neutral and cares only about winning.<sup>6</sup> We look for Nash equilibria of the zero-sum strategic game. A pure-strategy of each team  $T = A, B$  is  $l^T$ . A strategy profile of this game is denoted as  $(l^A, l^B)$ .

#### 3.1 Order irrelevance in fixed-length phase

We first establish an irrelevance result: the ordering of the first  $F$  penalty-takers with each team  $T = A, B$  for any strategy profile  $(l^A, l^B)$  determines when each team  $T$  wins the shootout in the fixed-length phase, but not the ex ante probability that each team  $T$  wins. The argument is based on pair-wise switching of two penalty-takers assigned to the fixed-length phase on a given team. Formally, for any strategy  $l^T$  of team  $T = A, B$ , and any  $j = 1, \dots, N-1$ , “switching”  $l_j^T$  and  $l_{j+1}^T$  yields another strategy  $\tilde{l}^T$  of team  $T$ , where  $\tilde{l}_j^T = l_{j+1}^T$ ,  $\tilde{l}_{j+1}^T = l_j^T$ , and  $\tilde{l}_k^T = l_k^T$  for all  $k \neq j, j+1$ .

**Lemma 1** *For any strategy profile  $(l^A, l^B)$ , the probability of each team winning the shootout is unchanged if any team  $T = A, B$  permutes the ordering of  $(l_1^T, \dots, l_F^T)$ .*

The argument for the irrelevance of the order of penalty-takers within the fixed-length phase, regardless of whether the team is the first mover or the second mover, is transparent when the shootout always continues from the first penalty-taker switched to the second penalty-taker. The score difference after the second penalty-taker depends only on the sum of the scores between the two penalty-takers, not on the order of their scores. Since

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<sup>4</sup>This is to ensure that all paths with score differences  $s_j^A$  and  $s_j^B$  satisfying (1) and (2) respectively occur with strictly positive probabilities under any strategy profile.

<sup>5</sup>Each team  $T$ ,  $T = A, B$ , chooses the order of its  $N$  penalty-takers, not their scoring probabilities.

<sup>6</sup>In particular, teams do not care about whether they win the shootout quickly in the first phase or they need to go to the sudden death phase.

probabilities of scoring are independent across penalty-takers and across teams, switching such penalty-takers has no impact on each team's ex ante probabilities of winning. Even when the first penalty-taker switched is decisive or his score makes the next opposing penalty-taker decisive, the shootout will continue to the second penalty-taker with a probability that makes the switching irrelevant to each team's ex ante probability of winning.

### 3.2 Order in transition and sudden death

We refer to the last round in the fixed-length phase, round  $F$ , as the “transition” round to the sudden-death phase. We consider switching  $l_j^T$  and  $l_{j+1}^T$  of team  $T = A, B$  for the transition round or any sudden-death round  $j \geq F$ . The following simple result provides a way to establish that switching  $l_j^T$  and  $l_{j+1}^T$  strictly increases  $T$ 's ex ante probability of winning, regardless of the strategy of the rival team.

**Claim 1** *If switching  $l_j^T$  and  $l_{j+1}^T$  for any  $j \geq F$  weakly increases  $T$ 's probability of winning and weakly decreases  $T$ 's probability of losing in rounds  $j$  and  $j + 1$ , with at least one strict inequality, then switching strictly increases  $T$ 's ex ante probability of winning.*

We first use Claim 1 to show that for each team  $T = A, B$ , switching of  $l_F^T$  and  $l_{F+1}^T$  when  $q_F^T < q_{F+1}^T$  increases team  $T$ 's ex ante probability of winning the shootout regardless of the opponent's strategy. We consider three cases, depending on whether  $l_F^T$  is decisive by scoring, decisive by missing, or not decisive either way.

**Lemma 2** *For each  $T = A, B$ , any strategy  $l^T$  with  $q_F^T < q_{F+1}^T$  is strictly dominated for  $T$  by a strategy that switches  $l_F^T$  and  $l_{F+1}^T$ .*

The contrast between Lemma 2 and Lemma 1 is interesting. For both results, switching of two adjacent penalty-takers can involve a first penalty-taker being switched who is decisive, and this can be the case for both the first mover and the second mover. Yet, while in Lemma 1 we have established an irrelevance result, in Lemma 2 we show switching the first penalty-taker  $l_{F+1}^T$  in the sudden-death phase for the last penalty-taker  $l_F^T$  on either team  $T = A, B$  strictly increases the ex ante probability of winning the shootout, if the former has a greater probability of scoring than the latter.

The reason for the contrasting conclusions is the following. In Lemma 1, if the first penalty-taker switched  $l_j^T$  is decisive, either by scoring or by missing, then the second penalty-taker  $l_{j+1}^T$  is also decisive by the same outcome with a probability that makes switching irrelevant to team  $T$ 's ex ante probability of winning. In comparison, in Lemma 2, the second penalty-taker switched  $l_{F+1}^A$  for the first mover  $A$  is never decisive, and the second penalty-taker  $l_{F+1}^B$  for the second mover is always decisive – by scoring if  $l_{F+1}^A$  misses and by missing if  $l_{F+1}^A$  scores. Switching in the transition round is no longer irrelevant to team  $T$ 's ex ante winning probability. Indeed, to increase its ex ante winning probability, the first mover  $A$  should assign a penalty-taker with a greater probability of scoring to the potentially decisive round  $F$  rather than the never-decisive round  $F + 1$ , and the second mover  $B$  should do the same because a penalty-taker is decisive more often in round  $F$  than in round  $F + 1$ .

The final lemma establishes that for each team  $T = A, B$ , switching of  $l_j^T$  and  $l_{j+1}^T$  when  $q_j^T < q_{j+1}^T$  for  $j \geq F$  increases team  $T$ 's ex ante probability of winning in the sudden-death phase of the shootout. The arguments are special cases of Lemma 2, and are omitted.<sup>7</sup>

**Lemma 3** *For each  $T = A, B$ , any strategy  $l^T$  with  $q_j^T < q_{j+1}^T$  for  $j$  such that  $F + 1 \leq j \leq N - 1$  is strictly dominated for  $T$  by a strategy that switches  $l_j^T$  and  $l_{j+1}^T$ .*

### 3.3 If Ronaldo was right

Combining Lemmas 1, 2, and 3, we immediately have the following result. We say that two strategies  $l^T$  and  $\tilde{l}^T$  of team  $T$  are “outcome equivalent” if they give  $T$  the same ex ante probability of winning against any strategy of the opposing team.

**Proposition 1** *If a strategy of a team is not strictly dominated, then penalty-takers assigned to the sudden-death phase are ordered in decreasing probability of scoring, and none of them has a greater probability of scoring than a penalty-taker assigned to the fixed-length phase. Further, all undominated strategies of the team are outcome equivalent.*

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<sup>7</sup> We have assumed that the shootout ends with a fair coin toss if the score difference is 0 after the last kick by  $l_N^B$ . If the coin toss is postponed to after an extended sudden-death phase where teams are allowed to go through all their  $N$  penalty-takers in any order they want, then the lemma below applies. The dominant strategy for each team is to go through in the decreasing order of their scoring probabilities.

All undominated strategies for each team  $T$  assign the top  $F$  penalty-takers in terms of scoring probabilities to the fixed-length phase, with any order among them, and assign the rest of  $N - F$  penalty-takers to the sudden-death phase in the decreasing order of their scoring probabilities. So, Ronaldo was right: so long as he, as the best penalty-taker for Portugal, is assigned to the fixed-length phase of five kicks, the chance of Portugal winning the shootout against Spain does not depend on whether he is the first to take penalty kick or he is the fifth. By going last in the fixed-length phase, Ronaldo has a chance to be decisive, but at the same time he may not get the chance at all, which was what happened in the penalty shootout of the Euro 2012 semi-final between Portugal and Spain. In expectation, Ronaldo going last in the fixed-length phase is no better or no worse than going first for Portugal.

An implicit assumption behind Proposition 1 is that a penalty-taker's probability of scoring is fixed and independent of the score difference he faces. This assumption leads to the result teams are indifferent between going first or going second. Denote as  $G$  the game we have analyzed in section 3, and denote as  $\tilde{G}$  as the game with the same two teams where  $B$  is the first mover and  $A$  is the second mover.

**Claim 2** *The ex ante probability of each team winning under strategy profile  $(l^A, l^B)$  is the same in game  $G$  and in game  $\tilde{G}$ .*

The above claim is counterfactual. Most teams choose to go first after they win the coin toss. The common reason given for this preference is that by going first and scoring a team is putting pressure on their opponents, which negatively affects their chances of scoring. Ronaldo may be still be right but his claim needs to be validated in a model where pressure plays a critical role. We turn to such a model in the next section.

## 4 Ronaldo May be Wrong

We say that “pressure” is on a penalty kicker if his team is trailing. That is, penalty-taker  $j$  from team  $A$  is under pressure if  $s_j^A \leq -1$ , and on penalty-taker  $j$  from team  $B$  if  $s_j^B \geq 1$ . By the rules of the shootout, penalty-takers of team  $A$  in the sudden-death phase never face

any pressure. In the fixed-length phase, from (1) pressure is on for penalty-taker  $l_j^A$  from team  $A$  with  $1 \leq j \leq F$ , if the score difference satisfies

$$\max\{-(j-1), -(F-j+1)\} \leq s_j^A \leq -1.$$

In particular, there is never any pressure for  $l_1^A$ , and pressure is on for the final penalty-taker  $l_F^A$  in the fixed-length phase only when  $s_F^A = -1$ . Similarly, for team  $B$ , from (2), pressure is on for penalty-taker  $l_j^B$  with  $1 \leq j \leq F$ , if the score difference satisfies

$$1 \leq s_j^B \leq \min\{j, F-j+1\}.$$

Unlike for team  $A$ , pressure is on for the first penalty-taker from  $B$  if  $l_1^A$  scored and pressure is on for  $l_F^B$  only when  $s_F^B = 1$ .

For convenience, we denote the probability that  $l_j^T$  scores under pressure as  $p_j^T$ . We assume that each penalty-taker is weakly less likely to score under pressure than under no pressure:

$$0 < p_j^T \leq q_j^T \tag{3}$$

for each  $T = A, B$  and  $j = 1, \dots, N$ .

We consider the same game of a penalty shootout, except that each of the  $N$  penalty-takers from either team is represented by two exogenous probabilities strictly between 0 and 1, instead of 1. As in the game without pressure in section 3, in this game with pressure each team is risk-neutral and cares only about winning.

## 4.1 Pressure in sudden death and transition

We first re-establish Lemma 3 for the first mover  $A$  under pressure. Even though in the sudden-death phase there is never any pressure for any of  $A$ 's penalty-takers,  $A$  can put  $B$ 's penalty-takers under pressure by scoring first in each round. This additional reason of pressuring the rival strengthens  $A$ 's incentive to assign its penalty-takers in decreasing order according to their scoring probabilities under no pressure.

**Lemma 4** *Under pressure, any strategy  $l^A$  with  $q_j^A < q_{j+1}^A$  for  $j$  such that  $F+1 \leq j \leq N-1$  is strictly dominated for  $A$  by a strategy that switches  $l_j^A$  and  $l_{j+1}^A$ .*

For the first mover  $A$  in two sudden-death rounds  $j$  and  $j + 1$ , switching the two penalty-takers  $l_j^A$  and  $l_{j+1}^A$  affects  $A$ 's probabilities of winning and losing in the two rounds. However, both winning and losing occur for the two switched penalty-takers under no pressure. This is no longer true for the second mover  $B$  considering switching penalty-takers in two adjacent sudden-death rounds. To re-establish Lemma 3 for the second mover  $B$ , we need to impose restrictions on how pressure negatively affects scoring probabilities beyond (3).

We say that pressure has “proportionally decreasing impact on scoring” by two penalty-takers, one with probabilities of scoring under no pressure and under pressure  $q$  and  $p$ , and another with probabilities  $\tilde{q}$  and  $\tilde{p}$ , if

$$q \geq \tilde{q} \implies \frac{q - p}{q} \leq \frac{\tilde{q} - \tilde{p}}{\tilde{q}}. \quad (4)$$

That is, if one penalty-taker has a greater probability of scoring than another penalty-taker under no pressure, then the decrease in the probability of scoring under pressure is weakly proportionally smaller for the first penalty-taker than for the second penalty-taker. Condition (4) is strictly satisfied if

$$q - p \leq \tilde{q} - \tilde{p},$$

that is, if pressure decreases the probability of scoring of the penalty-taker with a greater probability of scoring under no pressure by a weakly smaller amount than it decreases the probability of scoring of the penalty-taker with a smaller probability of scoring under no pressure. Further, (4) implies that pressure is order-preserving:  $q > \tilde{q}$  implies that  $p > \tilde{p}$ .

We say that pressure has “proportionally decreasing impact on missing” by two penalty-takers, with probabilities of scoring under no pressure and under pressure  $q$  and  $p$ , and  $\tilde{q}$  and  $\tilde{p}$  respectively, if

$$1 - \tilde{q} \geq 1 - q \implies \frac{(1 - \tilde{p}) - (1 - \tilde{q})}{1 - \tilde{q}} \leq \frac{(1 - p) - (1 - q)}{1 - q}. \quad (5)$$

That is, if one penalty-taker has a greater probability of missing than another penalty-taker under no pressure, then the increase in the probability of missing under pressure is weakly proportionally smaller for the first penalty-taker than for the second penalty-taker. If pressure increases the probability of missing for the penalty-taker with a greater probability

of missing by a smaller amount than it increases the probability of missing for the penalty-taker with a smaller probability of missing, that is, if

$$(1 - \tilde{p}) - (1 - \tilde{q}) \leq (1 - p) - (1 - q),$$

then (4) is strictly satisfied.

We say that pressure has “proportionally decreasing impact” on two penalty-takers, with probabilities of scoring under no pressure and under pressure  $q$  and  $p$ , and  $\tilde{q}$  and  $\tilde{p}$  respectively, if (4) and (5) both hold, with at least one of them holding strictly.

**Lemma 5** *Under pressure, any strategy  $l^B$  with  $q_j^B < q_{j+1}^B$  for  $j$  such that  $F + 1 \leq j \leq N - 1$  is strictly dominated for  $B$  by a strategy that switches  $l_j^B$  and  $l_{j+1}^B$ , if pressure has proportionally decreasing impact on  $l_j^B$  and  $l_{j+1}^B$ .*

We have defined proportionally decreasing impact of pressure separately for scoring and for missing, and specifically for a pair of penalty-takers. If pressure reduces the probability of scoring for all penalty-takers on a team by the same amount, then (4) and (5) hold for all pairs of penalty-takers on the team. The proof of Lemma 5 makes it clear that proportionally decreasing impact on scoring and proportionally decreasing impact on missing play different roles to ensure that the second mover  $B$  should assign its penalty-takers to the sudden-death phase in the decreasing order of their scoring probabilities under no pressure.

When the penalty-taker  $l_j^A$  from  $A$  scores in a sudden-death round  $j \geq F + 1$ , team  $B$  can still win the shootout in round  $j + 1$ , if  $l_j^B$  scores under pressure in round  $j$  and  $l_{j+1}^B$  scores under no pressure in round  $j + 1$  after  $l_{j+1}^A$  misses. Although both  $l_j^B$  and  $l_{j+1}^B$  score with a smaller probability under pressure than under no pressure, if pressure has proportionally decreasing impact on scoring by these two penalty-takers, to increase the chance of winning under pressure in the sudden-death phase, whoever has a greater probability of scoring under no pressure has a proportionally smaller decrease in probability of scoring under pressure and should go in round  $j$  under pressure instead of in round  $j + 1$  under no pressure. At the same time,  $B$  loses in rounds  $j$  and  $j + 1$ , if  $l_j^B$  misses under pressure in round  $j$  and  $l_{j+1}^B$  misses under pressure in round  $j + 1$  when given the opportunity. To decrease the chance of losing under pressure in the two rounds, whoever has a smaller probability of missing under

pressure should go in round  $j$  instead of round  $j + 1$ . Since pressure is order-preserving under the assumption of decreasing impact on scoring, the one that should go in round  $j$  is the one who has a smaller probability of missing under no pressure.

When the penalty-taker  $l_j^A$  misses in a sudden-death round  $j \geq F + 1$ , team  $B$  can still lose the shootout in round  $j + 1$ , if  $l_j^B$  misses under no pressure in round  $j$  and  $l_{j+1}^B$  misses under pressure in round  $j + 1$  after  $l_{j+1}^A$  scores. Both  $l_j^B$  and  $l_{j+1}^B$  miss with a greater probability under pressure than under no pressure. However, if pressure has proportionally decreasing impact on missing by these two penalty-takers, to decrease the chance of losing under no pressure in the sudden-death phase, the penalty-taker that has a smaller probability of missing under no pressure has a proportionally greater increase in probability of missing under pressure, and therefore should go in round  $j$  under no pressure instead of in round  $j + 1$  under pressure. At the same time,  $B$  wins in round  $j$  if  $l_j^B$  scores under no pressure, or in round  $j + 1$  if  $l_{j+1}^B$  scores under no pressure after  $l_{j+1}^A$  misses. To increase the chance of winning under no pressure in the two rounds  $j$  and  $j + 1$ , whoever has a greater probability of scoring under no pressure should go in round  $j$ .

The assumption of proportionally decreasing impact by pressure, conditions (4) and (5), also allows us to re-establish Lemma 2 for the transition round  $F$ . In particular, for the second mover  $B$ , the argument for switching  $l_F^B$  assigned to the transition round and  $l_{F+1}^B$  assigned to be the first in the sudden-death phase when  $q_F^B < q_{F+1}^B$ , is the same as for switching  $l_j^B$  and  $l_{j+1}^B$  for rounds  $j \geq F + 1$  and  $j + 1$  in the sudden-death phase when  $q_j^B < q_{j+1}^B$  in Lemma 5. This is because the two strategic situations are identical from  $B$ 's perspective. For the same reason, for the first mover  $A$ , the argument for switching  $l_F^A$  and  $l_{F+1}^A$  when  $q_F^A < q_{F+1}^A$  at  $s_F^A = 0$  is the same as for switching  $l_j^A$  and  $l_{j+1}^A$  for rounds  $j \geq F + 1$  and  $j + 1$  in the sudden-death phase when  $q_j^A < q_{j+1}^A$  in Lemma 4. Further, the argument for switching  $l_F^A$  and  $l_{F+1}^A$  when  $q_F^A < q_{F+1}^A$  at  $s_F^A = 1$  is the same as in Lemma 2 for switching the same two penalty-takers at  $s_F^A = 1$  under no pressure. Since  $l_F^A$  and  $l_{F+1}^A$  are never under pressure, switching does not decrease the chance of losing under no pressure; and since  $l_F^B$  is always under pressure, switching does not increase the chance of winning by pressuring the rival. We just need to add the argument for switching for  $A$  for the remaining case of  $s_F^A = -1$ .

**Lemma 6** *Under pressure, any strategy  $l^B$  with  $q_F^B < q_{F+1}^B$  is strictly dominated for  $B$  by a strategy that switches  $l_F^B$  and  $l_{F+1}^B$ , if pressure has proportionally decreasing impact on  $l_F^B$  and  $l_{F+1}^B$ ; any strategy  $l^A$  with  $q_F^A < q_{F+1}^A$  is strictly dominated for  $A$  by a strategy that switches  $l_F^A$  and  $l_{F+1}^A$ , if pressure has proportionally decreasing impact on scoring by  $l_F^A$  and  $l_{F+1}^A$ .*

When it turns out that the first mover  $A$  is behind in the last round  $F$  of the fixed-length phase,  $A$  can still win the shootout in the first round  $F + 1$  of the sudden-death phase. Winning requires  $l_F^A$  to score under pressure and  $l_{F+1}^A$  to score under no pressure when given by the opportunity. As in the case where  $B$  is behind in a sudden-death round in Lemma 5, under the assumption that pressure has proportionally decreasing impact on scoring by  $l_F^A$  and  $l_{F+1}^A$ , to increase the chance of winning under pressure, whoever has a greater probability of scoring under no pressure should go in round  $j$  under pressure instead of in round  $j + 1$  under no pressure. At the same time,  $A$  can lose in rounds  $F$  and  $F + 1$ : if  $l_{F+1}^B$  always misses then  $A$  loses only when  $l_F^A$  misses under pressure in round  $F$ , and if  $l_{F+1}^B$  always scores then  $A$  loses unless  $l_F^A$  scores under pressure and  $l_{F+1}^A$  scores under no pressure. In the first case, to decrease the chance of losing in rounds  $F$  and  $F + 1$ , whoever between  $l_F^A$  and  $l_{F+1}^A$  has a greater probability of scoring under pressure should go in round  $F$ , and that is the same penalty-taker with a greater probability of scoring under no pressure because proportionally decreasing impact is order-preserving. In the second case, again whoever has a greater probability of scoring under pressure should go in round  $F$ , because this maximizes the chance of scoring under pressure in round  $F$  and scoring under no pressure in round  $F + 1$  under the assumption of proportionally decreasing impact. Regardless of the probability of  $l_{F+1}^B$  scoring,  $A$  should assign the penalty-taker with a greater probability of scoring under no pressure to the transition round  $F$  instead of the first round in the sudden-death phase.

## 4.2 Pressure in fixed-length

Now we consider switching two penalty-takers in the fixed-length phase. Under no pressure, we have shown in Lemma 1 that the order of penalty-takers is irrelevant. Those assigned towards the end of the fixed-length phase may not get the opportunity to take a penalty but at the same time can be decisive. Under pressure, order of penalty-takers in the fixed-length

phase generally matters. This is because by scoring a penalty-taker can put his opponents under pressure or maintain the pressure, and by missing he misses these opportunities and can allow his next rival to erase or alleviate the pressure the rival team faces in later rounds, or even put his own next teammate under pressure.

We start by considering switching the second-to-last penalty-taker  $l_{F-1}^B$  and the last penalty-taker  $l_F^B$  of the second mover  $B$ . By (2), there are 4 possible score differences facing  $l_{F-1}^B$ ; they are  $s_{F-1}^B = -1, 0, 1, 2$ . At  $s_{F-1}^B = -1$ , penalty-taker  $l_{F-1}^B$  is decisive by scoring. Furthermore, when given the opportunity,  $l_F^A$  is always under pressure and  $l_F^B$  is never under pressure, regardless of how the shootout unfolds. As in Lemma 1, switching  $l_{F-1}^B$  and  $l_F^B$  has no impact on  $B$ 's chance of winning the shootout. Symmetrically, at  $s_{F-1}^B = 2$ , penalty-takers  $l_{F-1}^B$  and  $l_F^B$  (when given the opportunity) are always under pressure while  $l_F^A$  is never under pressure, and so switching  $l_{F-1}^B$  and  $l_F^B$  has no impact on  $B$ 's chance of winning the shootout. For the other two score differences  $s_{F-1}^B = 0$  or  $1$ , penalty-taker  $l_{F-1}^B$  is not decisive either way, but whether he scores or misses will determine whether or not  $l_F^A$  is under pressure, with impact on the latter's scoring probability. We note that Claim 1 applies to switching  $l_{F-1}^B$  and  $l_F^B$ , because there is a single score difference  $s_{F+1}^A$  if the shootout continues to the sudden-death phase.

**Lemma 7** *Under pressure, any strategy  $l^B$  with  $q_{F-1}^B < q_F^B$  is strictly dominated for  $B$  by a strategy that switches  $l_{F-1}^B$  and  $l_F^B$ , if pressure has proportionally decreasing impact on  $l_{F-1}^B$  and  $l_F^B$ .*

Lemma 7 shows that the order between the second-to-last penalty-taker  $l_{F-1}^B$  and the last penalty-taker  $l_F^B$  assigned to the fixed-length phase matters to the second mover  $B$ . Somewhat paradoxically, the order matters precisely when it turns out that  $l_{F-1}^B$  is not decisive either way, because whether he scores or misses in round  $F - 1$  determines whether the last remaining rival penalty-taker  $l_F^A$  in the fixed-length phase faces pressure or not.

When the scores are tied,  $l_{F-1}^B$  is not decisive, and is not under pressure. He can put the rival  $l_F^A$  under pressure by scoring, and by missing he leaves his teammate  $l_F^B$  under pressure when  $l_F^A$  scores. In contrast to the same case of tied-scores of Lemma 1, where we show that the order between  $l_{F-1}^B$  and  $l_F^B$  is irrelevant, if  $l_F^B$  has a greater of scoring under no

pressure than  $l_{F-1}^B$  under no pressure, then  $B$  should switch them, for two reasons. The first is pressuring the rival: as in Lemma 4 for the first mover  $A$ , so long as the rival  $l_F^A$  has a strictly smaller probability of scoring under pressure than under no pressure,  $B$ 's chance of winning before the sudden-death phase is increased by switching, because  $l_F^B$  is more likely to score and put  $l_F^A$  under pressure than  $l_{F-1}^B$ . The second reason is avoid losing under no pressure, as in Lemma 5 for the second mover  $B$  when scores are tied. Under the assumption of proportionally decreasing impact on missing, by having a greater probability of scoring under no pressure and hence a smaller probability of missing,  $l_F^B$  has a proportionally greater probability of missing under pressure. By switching  $l_F^B$  to round  $F-1$ , team  $B$  also decreases the chance of losing the shootout before the sudden-death phase from missing in round  $F-1$  under no pressure and missing in round  $F$  under pressure after  $l_F^A$  scores.

When  $B$  is behind by just 1,  $l_{F-1}^B$  is not decisive, but is now under pressure. Team  $B$  can lose the shootout before the sudden-death phase, but with just one rival penalty-taker  $l_F^A$  to go,  $B$  loses when  $l_{F-1}^B$  and  $l_F^B$  are both under pressure and  $l_F^A$  is never under pressure. As in Lemma 1, the order between  $l_{F-1}^B$  and  $l_F^B$  does not impact  $B$ 's probability of losing in round  $F$ . Team  $B$  can still win the shootout before sudden-death phase, but only if  $l_{F-1}^B$  scores under the pressure in round  $F-1$  so that his teammate  $l_F^B$  can score under no pressure when the rival  $l_F^A$  misses in round  $F$ . Under the assumption of proportionally decreasing impact on scoring, whoever between  $l_{F-1}^B$  and  $l_F^B$  has a greater probability of scoring under no pressure has a proportionally smaller decrease in the probability of scoring under pressure. In contrast to Lemma 1, where the order is irrelevant, that penalty-taker should go in round  $F-1$  to increase  $B$ 's chance of winning under pressure.

We now tackle the question of order between the second-to-last penalty-taker  $l_{F-1}^A$  and the last penalty-taker  $l_F^A$  of the first mover  $A$ . Again, Claim 1 applies to switching  $l_{F-1}^B$  and  $l_F^B$ , because there is a single score difference  $s_{F+1}^A$  if the shootout continues to the sudden-death phase. By (1), there are 5 possible score differences facing  $l_{F-1}^A$ ; they are  $s_{F-1}^A = -2, -1, 0, 1, 2$ . At  $s_{F-1}^A = -2$ , penalty-taker  $l_{F-1}^A$  is decisive by missing, and at  $s_F^A = 2$ ,  $l_{F-1}^A$  is decisive by scoring. As we have argued before Lemma 7, in both these two cases, whether the remaining penalty-takers before the sudden-death phase,  $l_{F-1}^B$ ,  $l_F^A$  and  $l_F^B$ , are under pressure or not does not depend on how the shootout unfolds. The same is true

at  $s_{F-1}^A = 1$ : penalty-taker  $l_{F-1}^A$  is not decisive either way, but  $l_{F-1}^A$  and  $l_F^A$  are never under pressure while  $l_{F-1}^B$  is always under pressure. As in Lemma 1, in all these cases, switching  $l_{F-1}^A$  and  $l_F^A$  has no impact on  $A$ 's chance of winning the shootout. The following lemma establishes the counterpart of Lemma 7 for the first mover  $A$ .

**Lemma 8** *Under pressure, any strategy  $l^A$  with  $q_{F-1}^A < q_F^A$  is strictly dominated for  $A$  by a strategy that switches  $l_{F-1}^A$  and  $l_F^A$ , if pressure has proportionally decreasing impact on  $l_{F-1}^A$  and  $l_F^A$ .*

As in the argument in Lemma 7 for the second mover  $B$ , order of the two last penalty-takers  $l_{F-1}^A$  and  $l_F^A$  in the fixed-length phase for the first mover  $A$  matters for the same two reasons, but in different combinations. The first reason that order between the two penalty-takers matters is that switching can increase the chance of winning under pressure and decrease the chance of losing under no pressure. For example, when team  $A$  is behind but the second-to-last penalty-taker  $l_{F-1}^A$  is not yet decisive by missing in the proof of Lemma 8, winning for the sudden-death phase requires both  $l_{F-1}^A$  and the last penalty-taker  $l_F^A$  to score. However,  $l_{F-1}^A$  has to score under pressure to pull even with  $B$ , and after  $l_{F-1}^B$  misses,  $l_F^A$  has to score under no pressure to win it in case  $l_F^B$  misses. To increase its chance of winning under pressure, team  $B$  should put whoever has a greater probability of scoring under no pressure in round  $F - 1$  rather than round  $F$ , because under proportionally decreasing impact on scoring, that penalty-taker has a proportionally smaller decrease in the probability of scoring under pressure. In the same case in the proof of Lemma 8, there is also change in pressure between  $l_{F-1}^A$  and  $l_F^A$  when  $A$  loses in round  $F$  because  $l_{F-1}^B$  misses and  $l_F^B$  scores. In contrast to Lemma 1, the order between penalty-takers matters to the sum of the probability of losing from  $l_{F-1}^A$  missing under pressure and  $l_F^A$  scoring under pressure and the probability of losing from  $l_{F-1}^A$  scoring under pressure and  $l_F^A$  missing under no pressure. To decrease the chance of losing under no pressure, team  $A$  again should put whoever has a greater probability of scoring under no pressure in round  $F - 1$  rather than round  $F$ .

The second reason that order of the two last penalty-takers  $l_{F-1}^A$  and  $l_F^A$  in the fixed-length phase matters for the first mover  $A$  is pressuring the rival  $l_{F-1}^B$  in between them: whether  $l_{F-1}^A$  scores or misses can mean whether there is pressure or not on the next rival penalty-taker

$l_{F-1}^B$ . This consideration is relevant only when the scores are tied. To increase its chance of winning in round  $F$ , whoever between  $l_{F-1}^A$  and  $l_F^A$  has a greater probability of scoring under no pressure should go in round  $F - 1$ , because winning means  $l_{F-1}^A$  and  $l_F^A$  face the same situation of no pressure. Putting the penalty-taker with a greater probability of scoring under no pressure also helps reducing  $A$ 's chance of losing in round  $F$ . Unlike winning, however, losing in this case also means change in pressure between  $l_{F-1}^A$  and  $l_F^A$ : while  $l_{F-1}^A$  never faces any pressure, losing can happen when  $l_F^A$  is put under pressure because  $l_{F-1}^A$  misses and  $l_{F-1}^B$  scores. Under the assumption that pressure has proportionally decreasing impact on missing, this change in pressure reinforces pressuring the rival and strengthens  $A$ 's incentive to put whoever has a greater probability of scoring under no pressure to round  $F - 1$ .

To extend Lemma 7 and 8 to the entire fixed-length phase, we need a preliminary result. When team  $T$  switches two adjacent penalty-takers  $l_j^T$  and  $l_{j+1}^T$  assigned before the two rounds in the fixed-length phase, with  $j \leq F - 2$ , the shootout can continue to round  $j + 2$  with difference score differences  $s_{j+2}^A$  satisfying (1). The following result establishes that, conditional on reach round  $j + 2$ , team  $A$ 's probability of winning of the shootout is strictly increasing in  $s_{j+2}^A$ , regardless of the strategies of the two teams.

**Claim 3** *Fix any strategy profile  $(l^A, l^B)$ . Conditional on reaching round  $j \leq F$ , team  $A$ 's probability of winning the shootout is strictly increasing in  $s_j^A$ .*

The feasible score differences  $s_F^A$  satisfying (1) for the last round  $F$  in the fixed-length phase are 1, 0 and  $-1$ . At  $s_F^A = 1$ , penalty-taker  $l_F^A$  is decisive by winning, and  $A$  cannot lose in round  $F$ ; at  $s_F^A = 0$ , by scoring penalty-taker  $l_F^A$  makes the latter decisive by missing, but by missing  $l_F^A$  makes  $l_F^B$  decisive by scoring; at  $s_F^A = -1$ , penalty-taker  $l_F^A$  is decisive by missing, and  $A$  cannot win in round  $F$ . Regardless of  $A$ 's continuation probability of winning in the sudden-death phase, and regardless of how the scoring probabilities of  $l_F^A$  and  $l_F^B$  under pressure compare with their respective probabilities under no pressure,  $A$ 's probability of winning the shootout conditional on reaching round  $F$  is strictly increasing in the score difference  $s_F^A$ . The proof of Claim 3 then proceeds by backward induction to the second round  $j = 2$  when there are different score differences  $s_2^A$  satisfying (1).

Claim 3 allows us to generalize the argument of switching beyond Claim 1. The following lemma extends Lemma 7 and 8 to the entire fixed-length phase. For the second-mover  $B$ , as in the proof of Lemma 7, in switching  $l_j^B$  and  $l_{j+1}^B$  for any  $j \leq F - 2$ , we need to consider score differences  $s_j^B = 0$  and  $s_j^B = 1$ . For all other score differences  $s_j^B \leq -1$  satisfying (2), penalty-takers  $l_j^B$  and  $l_{j+1}^B$  are never under pressure and  $l_{j+1}^A$  is always under pressure regardless of their realized scores. As in Lemma 1, switching  $l_j^B$  and  $l_{j+1}^B$  has no impact on  $B$ 's probability of winning the shootout conditional on reaching round  $j$  with the score difference  $s_j^B$ . The same argument holds for score difference  $s_j^B \geq 2$ . Similarly, for the first-mover  $A$ , we need consider only two cases in switching  $l_j^A$  and  $l_{j+1}^A$  for  $j \leq F - 2$ :  $s_j^A = -1$  and  $s_j^A = 0$ .

**Lemma 9** *Under pressure, any strategy  $l^T$ ,  $T = A, B$ , with  $q_j^T < q_{j+1}^T$  for  $j \leq F - 2$  is strictly dominated for  $T$  by a strategy that switches  $l_j^T$  and  $l_{j+1}^T$ , if pressure has proportionally decreasing impact on  $l_j^T$  and  $l_{j+1}^T$ .*

The proof of Lemma 9 is based on the same two underlying reasons for switching two penalty-takers  $l_j^T$  and  $l_{j+1}^T$  for team  $T$ : change in pressure and pressuring the rival. Unlike in Lemma 7 and Lemma 8, switching can result in different score differences  $s_{j+2}^A$  when the shootout continues to round  $j + 2$ . However, Claim 3 allows us to show that, if  $q_j^T < q_{j+1}^T$ , then conditional on reaching  $l_j^T$  at each possible score difference  $s_j^T$ , switching  $l_j^T$  and  $l_{j+1}^T$  increases  $T$ 's chance of winning under pressure and decreases its chance of losing under no pressure when pressure has proportionally decreasing impact on the two penalty-takers, and at the same time maximizing the probability of putting the rival penalty taker between  $l_j^T$  and  $l_{j+1}^T$  under pressure.

### 4.3 If Ronaldo was wrong

Ronaldo claimed that his position as the last penalty-taker in the fixed-five phase on the Portugal team, which was the second mover in the penalty shootout against Spain in the semi-final of Euro 2012, was irrelevant to Portugal's shoutout loss. Assuming that he was the best penalty-taker on the Portugal team, Lemma 7 already establishes his claim was wrong if pressure has proportionally decreasing impact on him and Bruno Alves, who was

the fourth penalty-taker on the Portugal team. Switching Ronaldo and Alves would have increased Portugal’s ex ante chance of winning the shootout.

Lemmas 4, 5, 6, 8, 7, and 9 together imply that any strategy  $l^T$ ,  $T = A, B$ , with  $q_j^T < q_{j+1}^T$  for  $j = 1, \dots, N - 1$  is strictly dominated for  $T$  by a strategy that switches  $l_j^T$  and  $l_{j+1}^T$ , if pressure has proportionally decreasing impact on  $l_j^T$  and  $l_{j+1}^T$ . We state the strongest implication of proportionally decreasing impact by pressure as the main result.

**Proposition 2** *Suppose that pressure has proportionally decreasing impact on all pairs of penalty-takers of a team. It is strictly dominant for the team to assign the penalty-takers in the decreasing order of their probabilities of scoring under no pressure.*

When all  $N$  penalty-takers on a team are strictly ranked in their probabilities of scoring under no pressure, there is a unique strictly dominant strategy for the team. If there are ties in the probabilities of scoring under no pressure, the assumption of proportionally decreasing impact by pressure, conditions (4) and (5), require their probabilities of scoring under pressure to be also equal. The order between two penalty-takers with the same probability of scoring under no pressure is therefore irrelevant, and Proposition 2 holds with minimum modification.

## 5 Discussion

We have made two implicit assumptions in the setting up the penalty shootout game with or without pressure. First, teams submit their list after learning which of them goes first in each round; and second, a team may not change the remaining part of the list according to how the shootout unfolds. According to Law 10 (“Determining the Outcome of a Match”) of the International Football Association Board’s (IFAB) *Laws of the Game*, the teams are not required to inform the referee of their list at any moment. Our results from the benchmark game without pressure, Proposition 1, and from the game with pressure, Proposition 2, are robust with respect to these two assumptions. This is because we have actually shown that for any team, regardless of whether it is the first mover or the second mover, conditional on reaching any round with any score difference, whether it is in the fixed-length phase

or in the sudden-death phase, the set of strictly dominant strategies among the remaining penalty-takers for that team is characterized by Proposition 1 in the game without pressure, and by Proposition 2 in the game with pressure.

We have presented the main result under the assumption that all pairs of penalty-takers of a team satisfy the assumption of proportionally decreasing impact by pressure. When the assumption fails for some penalty-takers, we can partially characterize equilibrium strategies, by carefully distinguishing pressure on scoring from pressure on missing, the first mover from the second mover, and the fixed-length phase from the sudden-death phase, and by applying our results from Lemmas 4, 5, 6, 8, 7 and 9 separately. For example, a superstar like Cristiano Ronaldo may well have the highest probability of scoring under no pressure and the same probability of scoring under pressure as under no pressure. In this case, pressure has decreasing impact on scoring by the superstar but decreasing impact on missing does not hold. If the superstar is on the first mover, then our results show that it is strictly dominant to assign him to the fixed-length phase: by Lemma 4, any strategy of the first mover that assigns the superstar to any position in the sudden-death phase other than the first one is strictly dominated; and by Lemma 6, the first mover should further switch him from the first position in the sudden-death phase to the last position in the fixed-length phase. We leave the full characterization of equilibrium to future work.

## References

- [1] J. Apesteguia and I. Palacios-Huerta, Psychological Pressure in Competitive Environments: Evidence from a Randomized Natural Experiment, *American Economic Review*, 100(5), Pages 2548-2564, 2010.

# Appendix

## Proof of Lemma 1

It suffices to show that switching two adjacent penalty-takers assigned to the fixed-length phase in each team does not affect the probability of winning for each team.

Fix any  $j$  such that  $1 \leq j \leq F - 1$ , and consider team  $A$  switching  $l_j^A$  and  $l_{j+1}^A$ . If the shootout does not reach  $l_j^A$ , switching has no effect. Conditional on the shootout reaching  $l_j^A$ , with a score difference  $s_j^A$  satisfying (1), there are three cases.

Case (A1):  $-(F - j - 1) \leq s_j^A \leq F - j - 1$ . In this case, the shootout will continue to  $l_{j+1}^A$  regardless of the success or failure of  $l_j^A$  and  $l_j^B$ . This is because neither  $l_j^A$  nor  $l_j^B$  is decisive by scoring or by missing. We argue that conditional on  $s_j^A$ , the probability of reaching each possible score difference  $s_{j+1}^B$  after penalty-taker  $l_{j+1}^A$ , which can be adding  $-1, 0, 1$  or  $2$  to  $s_j^A$ , is unaffected by the switching. This is because for any fixed score  $0$  or  $1$  achieved by  $l_j^B$ , the probability of reaching each possible score difference  $s_{j+1}^B$  depends only on the sum of the scores achieved by  $l_j^A$  and  $l_{j+1}^A$ , not on the order of their scores. Since in each continuation associated with some  $s_{j+1}^B$ , the probability of winning the shootout for each team is unaffected by the switching, the ex ante probability of winning is unaffected.

Case (A2):  $s_j^A = -(F - j)$  or  $s_j^A = F - j$ . We will only provide a proof for the first subcase of  $s_j^A = -(F - j)$ ; the other subcase is symmetric. In the first subcase,  $l_j^A$  himself is not decisive by missing, but by missing his kick he makes the opponent penalty-taker  $l_j^B$  decisive by scoring. Indeed, when  $l_j^B$  scores, team  $A$  loses the shootout if  $l_j^A$  misses or  $l_j^A$  scores and  $l_{j+1}^A$  misses. Conditional on  $s_j^A = -(F - j)$ , the probability of team  $A$  losing the shootout when  $l_j^B$  is decisive by scoring is therefore

$$(1 - q_j^A)q_j^B + q_j^A q_j^B (1 - q_{j+1}^A) = q_j^B (1 - q_j^A q_{j+1}^A),$$

which is unaffected by switching  $l_j^A$  and  $l_{j+1}^A$ . In any other outcome, the shootout continues to  $l_{j+1}^A$ , and furthermore, for any fixed score  $0$  or  $1$  achieved by  $l_j^B$ , the probability of reaching each possible score difference  $s_{j+1}^B$  after  $l_{j+1}^A$  depends only on the sum of the scores achieved by  $l_j^A$  and  $l_{j+1}^A$ , not on the order of their scores. It follows that in this case the ex ante probability of each team winning is unaffected by the switching.

Case (A3):  $s_j^A = -(F - j + 1)$  or  $s_j^A = F - j + 1$ . We will only provide a proof for the first subcase of  $s_j^A = -(F - j + 1)$ ; the other subcase is symmetric. In the first subcase,  $l_j^A$  is decisive by missing. Indeed, team  $A$  loses the shootout unless  $l_j^A$  scores,  $l_j^B$  misses and  $l_{j+1}^A$  scores. Conditional on  $s_j^A = -(F - j + 1)$ , the probability that  $A$  loses the shootout is

$$1 - q_j^A + q_j^A q_j^B + q_j^A (1 - q_j^B)(1 - q_{j+1}^A) = 1 - q_j^A (1 - q_j^B) q_{j+1}^A,$$

which is unchanged if  $A$  switches  $l_j^A$  and  $l_{j+1}^A$ . It follows that in this case the ex ante probability of each team winning is unaffected by the switching.

Now we show that the ex ante probability of each team winning the shootout is unchanged if  $B$  switches  $l_j^B$  and  $l_{j+1}^B$  for any  $j \leq F - 1$ . If the shootout does not reach  $l_j^B$ , switching has no effect. Conditional on the shootout reaching  $l_j^B$ , with a score difference  $s_j^B$  satisfying (2), there are at most three cases.

Case (B1):  $-(F - j - 2) \leq s_j^B \leq F - j - 1$ . In this case, the shootout will continue to  $l_{j+1}^B$  regardless of the success or failure of  $l_j^B$  and  $l_{j+1}^A$ , because neither  $l_j^B$  nor  $l_{j+1}^A$  is decisive by scoring or by missing. The argument for why switching  $l_j^B$  and  $l_{j+1}^B$  has no effect on each team's winning probability is the same as in case (A1) above.

Case (B2):  $s_j^B = -(F - j - 1)$  or  $s_j^B = F - j$ . In the first subcase of  $s_j^B = -(F - j - 1)$ ,  $l_j^B$  is not decisive but by scoring he makes  $l_{j+1}^A$  decisive by missing. Conditional on  $s_j^B = -(F - j - 1)$ , the probability of team  $B$  winning the shootout when  $l_{j+1}^A$  is decisive by missing is

$$q_j^B (1 - q_{j+1}^A) + (1 - q_j^B)(1 - q_{j+1}^A) q_{j+1}^B,$$

which is unchanged if  $B$  switches  $l_j^B$  and  $l_{j+1}^B$ . In any other outcome, the shootout continues to  $l_{j+1}^B$ , and for any fixed score 0 or 1 achieved by  $l_{j+1}^A$ , the probability of reaching each possible score difference  $s_{j+2}^A$  after  $l_{j+1}^B$  does not depend on the order of the scores achieved by  $l_j^B$  and  $l_{j+1}^B$ . It follows that in this case the ex ante probability of each team winning is unaffected by the switching. The argument for the second subcase of  $s_j^B = F - j$  is symmetric.

Case (B3):  $s_j^B = -(F - j)$  or  $s_j^B = F - j + 1$ . In the first subcase of  $s_j^B = -(F - j)$ ,  $l_j^B$  is decisive by scoring. Team  $B$  wins the shootout unless  $l_j^B$  misses,  $l_{j+1}^A$  scores and  $l_{j+1}^B$  misses.

Conditional on  $s_j^B = -(F - j)$ , the probability that shootout continues to  $l_{j+2}^A$  is

$$(1 - q_j^B)q_{j+1}^A(1 - q_{j+1}^B),$$

which is unchanged if  $B$  switches  $l_j^B$  and  $l_{j+1}^B$ . It follows that in this case the ex ante probability of each team winning is unaffected by the switching. The argument for the second subcase of  $s_j^B = F - j + 1$  is symmetric.

## Proof of Claim 1

Let  $w_{j,j+1}^T$  be  $T$ 's probability of winning in rounds  $j$  and  $j + 1$ , conditional on the shootout reaching  $l_j^T$ , and let  $W_{j+2}^T$  be  $T$ 's probability of winning conditional on reaching round  $j + 2$ . We have that  $W_{j+2}^T$  is strictly between 0 and 1 because all scoring probabilities are strictly between 0 and 1. Conditional on reaching  $l_j^A$ , team  $A$ 's probability of winning the shootout is given by

$$w_{j,j+1}^A + (1 - w_{j,j+1}^A - w_{j,j+1}^B)W_{j+2}^A.$$

Consider switching  $l_j^A$  and  $l_{j+1}^A$ . Suppose that after the switch  $A$ 's probability of winning in rounds  $j$  and  $j + 1$  becomes  $\tilde{w}_{j,j+1}^A \geq w_{j,j+1}^A$ , and  $A$ 's probability of losing becomes  $\tilde{w}_{j,j+1}^B \leq w_{j,j+1}^B$ , with at least one strict inequality. The switch does not affect  $W_{j+2}^A$ , because there is a single score difference  $s_{j+2}^A = 0$  conditional on reaching round  $j + 2$ . Then, conditional on reaching  $l_j^A$ , after switching  $A$ 's probability of winning becomes

$$\begin{aligned} & \tilde{w}_{j,j+1}^A + (1 - \tilde{w}_{j,j+1}^A - \tilde{w}_{j,j+1}^B)W_{j+2}^A \\ &= w_{j,j+1}^A + (1 - w_{j,j+1}^A - w_{j,j+1}^B)W_{j+2}^A + (\tilde{w}_{j,j+1}^A - w_{j,j+1}^A)(1 - W_{j+2}^A) + (w_{j,j+1}^B - \tilde{w}_{j,j+1}^B)W_{j+2}^A \\ &> w_{j,j+1}^A + (1 - w_{j,j+1}^A - w_{j,j+1}^B)W_{j+2}^A, \end{aligned}$$

because  $\tilde{w}_{j,j+1}^A \geq w_{j,j+1}^A$  and  $\tilde{w}_{j,j+1}^B \leq w_{j,j+1}^B$ , with at least one strict inequality, and  $0 < W_{j+2}^A < 1$ .

## Proof of Lemma 2

Fix any  $l^B$  and suppose that  $l^A$  is such that  $q_F^A < q_{F+1}^A$ . Consider switching  $l_F^A$  and  $l_{F+1}^A$ . Conditional on any realized  $s_F^A$  that violates (1) for  $j = F$ , the switch does not affect each team's winning probability. There are three cases assuming that  $s_F^A$  satisfies (1).

Case (A1):  $s_F^A = 0$ . Penalty-taker  $l_F^A$  is not decisive either way, but if he scores he makes  $l_F^B$  decisive by missing, and if he misses he makes  $l_F^B$  decisive by scoring. Team  $A$  wins the shootout before it reaches round  $F + 2$ , when  $l_F^A$  scores and  $l_F^B$  misses, or when both  $l_F^A$  and  $l_F^B$  score or both miss, followed by  $l_{F+1}^A$  scoring and  $l_{F+1}^B$  missing. Conditional on  $s_F^A = 0$ , the probability of this winning scenario is

$$q_F^A(1 - q_F^B) + (q_F^A q_F^B + (1 - q_F^A)(1 - q_F^B)) q_{F+1}^A(1 - q_{F+1}^B).$$

Since  $q_F^A < q_{F+1}^A$ , the above is increased if we switch  $l_F^A$  and  $l_{F+1}^A$ . In the symmetric losing scenario, team  $A$  loses the shootout before it reaches round  $F + 2$ , when  $l_F^A$  misses and  $l_F^B$  scores, or when both  $l_F^A$  and  $l_F^B$  score or both miss, followed by  $l_{F+1}^A$  missing and  $l_{F+1}^B$  scoring. Conditional on  $s_F^A = 0$ , the probability of the losing scenario is

$$(1 - q_F^A)q_F^B + (q_F^A q_F^B + (1 - q_F^A)(1 - q_F^B)) (1 - q_{F+1}^A)q_{F+1}^B.$$

Since  $q_F^A < q_{F+1}^A$ , the above is decreased if we switch  $l_F^A$  and  $l_{F+1}^A$ . It follows from Claim 1 that switching strictly increases the ex ante probability that  $A$  wins the shootout in this case.

Case (A2):  $s_F^A = -1$ . Penalty-taker  $l_F^A$  is decisive by missing, and if he scores  $l_F^B$  becomes decisive by scoring. Team  $A$  wins the shootout before it reaches round  $F + 2$ , when  $l_F^A$  scores,  $l_F^B$  misses,  $l_{F+1}^A$  scores and  $l_{F+1}^B$  misses. Conditional on  $s_F^A = -1$ , the probability of this winning scenario is

$$q_F^A(1 - q_F^B)q_{F+1}^A(1 - q_{F+1}^B).$$

The above is unchanged if  $A$  switches  $l_F^A$  and  $l_{F+1}^A$ . Team  $A$  loses the shootout before it reaches round  $F + 2$ , when  $l_F^A$  misses, or when both  $l_F^A$  and  $l_F^B$  score, or when  $l_F^A$  scores and  $l_F^B$  misses, followed by  $l_{F+1}^A$  scoring and  $l_{F+1}^B$  missing. Conditional on  $s_F^A = -1$ , the probability of this losing scenario is

$$1 - q_F^A + q_F^A q_F^B + q_F^A(1 - q_F^B)q_{F+1}^A(1 - q_{F+1}^B).$$

Since  $q_F^A < q_{F+1}^A$ , the above becomes strictly smaller if we switch  $l_F^A$  and  $l_{F+1}^A$ . By Claim 1, switching strictly increases the ex ante probability that  $A$  wins the shootout in this case.

Case (A3):  $s_F^A = 1$ . Penalty-taker  $l_F^A$  is decisive by scoring, and if he misses,  $l_F^B$  becomes decisive by missing. Team  $A$  wins the shootout before it reaches round  $F+2$ , when  $l_F^A$  scores, or when both  $l_F^A$  and  $l_F^B$  miss, or when  $l_F^A$  misses and  $l_F^B$  scores, followed by  $l_{F+1}^A$  scoring and  $l_{F+1}^B$  missing. Conditional on  $s_F^A = 1$ , the probability of this winning scenario is

$$q_F^A + (1 - q_F^A)(1 - q_F^B) + (1 - q_F^A)q_F^B q_{F+1}^A (1 - q_{F+1}^B).$$

Since  $q_F^A < q_{F+1}^A$ , the above becomes strictly greater if we switch  $l_F^A$  and  $l_{F+1}^A$ . Team  $A$  loses the shootout before it reaches round  $F+2$ , when  $l_F^A$  misses,  $l_F^B$  scores,  $l_{F+1}^A$  misses and  $l_{F+1}^B$  scores. Conditional on  $s_F^A = 1$ , the probability of this losing scenario is

$$(1 - q_F^A)q_F^B(1 - q_{F+1}^A)q_{F+1}^B.$$

The above is unchanged if  $A$  switches  $l_F^A$  and  $l_{F+1}^A$ . It follows from Claim 1 that switching strictly increases the ex ante probability that  $A$  wins the shootout in this case.

Now we show that the ex ante probability of each team winning the shootout is strictly increased if  $B$  switches  $l_F^B$  and  $l_{F+1}^B$  when  $q_F^B < q_{F+1}^B$ . Conditional on any realized  $s_F^B$  that violates (2) for  $j = F$ , the switch does not affect each team's winning probability. There are only two cases assuming that  $s_F^B$  satisfies (2).

Case (B1):  $s_F^B = 0$ . Penalty-taker  $l_F^B$  is decisive by scoring. Team  $B$  wins before reaching round  $F+2$ , when  $l_F^B$  scores, or when both  $l_F^B$  and  $l_{F+1}^A$  miss but  $l_{F+1}^B$  scores. Conditional on  $s_F^B = 0$ , the probability of this winning scenario is

$$q_F^B + (1 - q_F^B)(1 - q_{F+1}^A)q_{F+1}^B,$$

which is strictly increased when  $B$  switches  $l_F^B$  and  $l_{F+1}^B$ . Team  $B$  loses before reaching round  $F+2$ , when  $l_F^B$  misses,  $l_{F+1}^A$  scores and  $l_{F+1}^B$  misses. Conditional on  $s_F^B = 0$ , the probability of this losing scenario is

$$(1 - q_F^B)q_{F+1}^A(1 - q_{F+1}^B),$$

which is unaffected by switching  $l_F^B$  and  $l_{F+1}^B$ . By Claim 1, switching strictly increases the ex ante probability that  $B$  wins the shootout in this case.

Case (B2):  $s_F^B = 1$ . Penalty-taker  $l_F^B$  is decisive by missing. Team  $B$  wins before reaching round  $F+2$ , when  $l_F^B$  scores,  $l_{F+1}^A$  misses and  $l_{F+1}^B$  scores. Conditional on  $s_F^B = 1$ , the

probability of this winning scenario is

$$q_F^B(1 - q_{F+1}^A)q_{F+1}^B,$$

which is unaffected when  $B$  switches  $l_F^B$  and  $l_{F+1}^B$ . Team  $B$  loses before reaching round  $F+2$ , when  $l_F^B$  misses, or when both  $l_F^B$  and  $l_{F+1}^A$  score, followed by  $l_{F+1}^B$  missing. Conditional on  $s_F^B = 1$ , the probability of this losing scenario is

$$1 - q_F^B + q_F^B q_{F+1}^A(1 - q_{F+1}^B),$$

which is strictly decreased when  $B$  switches  $l_F^B$  and  $l_{F+1}^B$  because  $q_F^B < q_{F+1}^B$ . It follows from Claim 1 that switching strictly increases the ex ante probability that  $B$  wins the shootout in this case.

## Proof of Lemma 2

In game  $\tilde{G}$ , we continue to refer to the score difference facing penalty-taker  $l_j^B$  assigned to the  $j$ -th position by team  $B$  as the difference between the number of successful kicks by penalty-takers  $l_1^A, \dots, l_{j-1}^A$  and the number of successful kicks by  $l_1^B, \dots, l_{j-1}^B$ , and denote it as  $\tilde{s}_j^B$ . The score difference  $\tilde{s}_j^A$  facing penalty-taker  $l_j^A$  assigned to the  $j$ -th position by team  $A$  is the difference between the number of successful kicks by penalty-takers  $l_1^A, \dots, l_{j-1}^A$  and the number of successful kicks by  $l_1^B, \dots, l_j^B$ .

Conditional on game  $G$  reaching  $l_N^A$ , the score difference  $s_N^A$  can only be 0, and the same is true for  $\tilde{s}_N^B$  in game  $\tilde{G}$  conditional on it reaching  $l_N^B$ . Team  $A$  wins the shootout in  $G$  if  $l_N^A$  scores and  $l_N^B$  misses, or if  $A$  wins the coin toss after both  $l_N^A$  and  $l_N^B$  score or both miss. These are the same events for  $A$  winning in  $\tilde{G}$ . Thus, the probability  $W_N^A$  that  $A$  wins the shootout in  $G$  conditional on reaching  $l_N^A$  is equal to the probability  $\tilde{W}_N^A$  that  $A$  wins the shootout in  $\tilde{G}$  conditional on reaching  $l_N^B$ .

By backward induction, for any round  $j \geq F+1$  in the sudden-death phase, the probability  $W_j^A$  of  $A$  winning the shootout in game  $G$  conditional on reaching  $l_j^A$  with the score difference  $s_j^A = 0$ , is equal to the probability  $\tilde{W}_j^A$  of  $A$  winning the shootout in game  $\tilde{G}$  conditional on reaching  $l_j^B$  with the score difference  $s_j^B = 0$ .

By (1), conditional on game  $G$  reaching  $l_F^A$ , the score difference  $s_F^A$  can be 1, 0, or  $-1$ , and the same is true for  $\tilde{s}_F^B$  in game  $\tilde{G}$  conditional on reaching  $l_F^B$ . Conditional on each possible score difference  $s_F^A$ , the events that lead to team  $A$  winning the shootout in round  $F$  of  $G$  are the same that lead to  $A$  winning in round  $F$  of  $\tilde{G}$  for the score difference  $\tilde{s}_F^B = s_F^A$ . The events that lead to shootout continuing to round  $F + 1$  in  $G$ , with score difference  $s_{F+1}^A = 0$ , are also the same that lead to round  $F + 1$  in  $\tilde{G}$  with  $\tilde{s}_{F+1}^B = 0$  for the score difference  $\tilde{s}_F^B = s_F^A$ . Since  $\tilde{W}_{F+1}^A = W_{F+1}^A$ , the probability  $W_F^A(s_F^A)$  that  $A$  wins the shootout in  $G$  conditional on reaching  $l_F^A$  with score difference  $s_F^A$  is equal to the probability  $\tilde{W}_F^A(\tilde{s}_F^B)$  that  $A$  wins the shootout in  $\tilde{G}$  conditional on reaching  $l_j^B$  with  $\tilde{s}_F^B = s_F^A$ .

For any round  $j \leq F - 1$  in the fixed-length phase, conditional on game  $G$  reaching  $l_j^A$ , the set of score differences  $s_j^A$  satisfying (1) are the same as  $\tilde{s}_j^B$  conditional on reaching  $l_j^B$  in  $\tilde{G}$ . Conditional on each possible score difference  $s_j^A$ , the events that lead to team  $A$  winning the shootout in round  $j$  of  $G$  are the same that lead to  $A$  winning in round  $F$  of  $\tilde{G}$  for the score difference  $\tilde{s}_j^B = s_j^A$ . The events that lead to shootout continuing to round  $j + 1$  in  $G$  with score difference  $s_{j+1}^A = s_j^A + \delta$ ,  $\delta = -1, 0, 1$ , are also the same that lead to round  $j + 1$  with  $\tilde{s}_{j+1}^B = s_{j+1}^A$  in  $\tilde{G}$  for the score difference  $\tilde{s}_j^B = s_j^A$ . Since  $\tilde{W}_{j+1}^A(\tilde{s}_{j+1}^B) = W_{j+1}^A(\tilde{s}_{j+1}^B)$  by backward induction, the probability  $W_j^A(s_j^A)$  that  $A$  wins the shootout in  $G$  conditional on reaching  $l_j^A$  with score difference  $s_j^A$  is equal to the probability  $\tilde{W}_F^A(\tilde{s}_j^B)$  that  $A$  wins the shootout in  $\tilde{G}$  conditional on reaching  $l_j^B$  with  $\tilde{s}_j^B = s_j^A$ .

The lemma follows immediately by backward induction because  $s_1^A = 0$  in  $G$  and  $\tilde{s}_1^B = 0$  in  $\tilde{G}$ .

## Proof of Lemma 4

Fix any  $l^B$  and suppose that  $l^A$  is such that  $q_j^A < q_{j+1}^A$  for some  $j$  with  $F + 1 \leq j \leq N - 1$ . Consider switching  $l_j^A$  and  $l_{j+1}^A$ . The switch does not affect each team's winning probability unless  $s_j^A = 0$ . Conditional on  $s_j^A = 0$ , team  $A$  wins the shootout before it reaches  $l_{j+2}^A$ , when  $l_j^A$  scores under no pressure and  $l_j^B$  misses under pressure, or when  $l_j^A$  scores under no pressure and  $l_j^B$  scores under pressure or both miss under no pressure, followed by  $l_{j+1}^A$  scoring under no pressure and  $l_{j+1}^B$  missing under pressure. Conditional on  $s_j^A = 0$ , the probability of this

winning scenario is

$$q_j^A(1 - p_j^B) + (q_j^A p_j^B + (1 - q_j^A)(1 - q_j^B)) q_{j+1}^A(1 - p_{j+1}^B).$$

When  $A$  switches  $l_j^A$  and  $l_{j+1}^A$ , the above is changed by

$$(q_{j+1}^A - q_j^A)(1 - p_j^B) - (q_{j+1}^A - q_j^A)(1 - q_j^B)(1 - p_{j+1}^B) \geq (q_{j+1}^A - q_j^A)(1 - q_j^B)p_{j+1}^B > 0.$$

where the first inequality follows because  $p_j^B \leq q_j^B$ , and the second follows because  $q_{j+1}^A > q_j^A$ . Team  $A$  loses the shootout before it reaches  $l_{j+2}^A$ , when  $l_j^A$  misses under no pressure and  $l_j^B$  scores under no pressure, or when  $l_j^A$  scores under no pressure and  $l_j^B$  scores under pressure or both miss under no pressure, followed by  $l_{j+1}^A$  missing under no pressure and  $l_{j+1}^B$  scoring under no pressure. Conditional on  $s_j^A = 0$ , the probability of this losing scenario is

$$(1 - q_j^A)q_j^B + (q_j^A p_j^B + (1 - q_j^A)(1 - q_j^B)) (1 - q_{j+1}^A)q_{j+1}^B.$$

When  $A$  switches  $l_j^A$  and  $l_{j+1}^A$ , the above is changed by

$$-(q_{j+1}^A - q_j^A)(q_j^B - p_j^B q_{j+1}^B) \leq -(q_{j+1}^A - q_j^A)p_j^B(1 - q_{j+1}^B) < 0,$$

where the first inequality follows because  $q_j^B \geq p_j^B$ , and the second follows because  $q_j^A < q_{j+1}^A$ . By Claim 1, switching strictly increases the ex ante probability that team  $A$  wins the shootout in this case.

## Proof of Lemma 5

Fix any  $l^A$  and suppose that  $l^B$  is such that  $q_j^B < q_{j+1}^B$  for some  $j$  with  $F + 1 \leq j \leq N - 1$ . Consider switching  $l_j^B$  and  $l_{j+1}^B$ . There are two cases.

Case (B1):  $s_j^B = 1$ . Team  $B$  wins the shootout before it reaches  $l_{j+2}^A$ , when  $l_j^B$  scores under pressure,  $l_{j+1}^A$  misses under no pressure and then  $l_{j+1}^B$  scores under no pressure. Conditional on  $s_j^B = 1$ , the probability of this winning scenario is

$$p_j^B(1 - q_{j+1}^A)q_{j+1}^B.$$

Since  $q_j^B < q_{j+1}^B$ , (4) implies that the above is weakly increased when  $B$  switches  $l_j^B$  and  $l_{j+1}^B$ . Team  $B$  loses the shootout before it reaches  $l_{j+2}^A$ , either when  $l_j^B$  misses under pressure, or

when  $l_j^B$  scores under pressure, but  $l_{j+1}^A$  scores under no pressure and then  $l_{j+1}^B$  misses under pressure. Conditional on  $s_j^B = 1$ , the probability of this losing scenario is

$$(1 - p_j^B) + p_j^B q_{j+1}^A (1 - p_{j+1}^B).$$

By (4),  $q_j^B < q_{j+1}^B$  implies  $p_j^B < p_{j+1}^B$ , and so the above is strictly decreased when  $B$  switches  $l_j^B$  and  $l_{j+1}^B$ . It follows from Claim 1 that switching strictly increases the ex ante probability that team  $B$  wins the shootout in this case.

Case (B2):  $s_j^B = 0$ . Team  $B$  wins the shootout before it reaches  $l_{j+2}^A$ , when either  $l_j^B$  scores under no pressure, or  $l_j^B$  misses under no pressure,  $l_{j+1}^A$  misses under no pressure and then  $l_{j+1}^B$  scores under no pressure. Conditional on  $s_j^B = 0$ , the probability of this winning scenario is

$$q_j^B + (1 - q_j^B)(1 - q_{j+1}^A)q_{j+1}^B.$$

When  $B$  switches  $l_j^B$  and  $l_{j+1}^B$ , the above is strictly increased because  $q_j^B < q_{j+1}^B$ . Team  $B$  loses the shootout before it reaches  $l_{j+2}^A$ , when  $l_j^B$  misses under no pressure,  $l_{j+1}^A$  scores under no pressure and  $l_{j+1}^B$  misses under pressure. Conditional on  $s_j^B = 0$ , the probability of this losing scenario is

$$(1 - q_j^B)q_{j+1}^A(1 - p_{j+1}^B).$$

Since  $q_j^B < q_{j+1}^B$ , (5) implies that the above is weakly decreased when  $B$  switches  $l_j^B$  and  $l_{j+1}^B$ . The lemma follows immediately.

## Proof of Lemma 6

Consider switching by  $A$  of  $l_F^A$  and  $l_{F+1}^A$  when  $q_F^A < q_{F+1}^A$ , for the remaining case of  $s_F^A = -1$ . Team  $A$  wins the shootout before it reaches  $l_{F+2}^A$ , when  $l_F^A$  scores under pressure,  $l_F^B$  misses under no pressure,  $l_{F+1}^A$  scores under no pressure and  $l_{F+1}^B$  misses under pressure. Conditional on  $s_F^A = -1$ , the probability of this winning scenario is

$$p_F^A(1 - q_F^B)q_{F+1}^A(1 - p_{F+1}^B).$$

By (4), the above is weakly increased when  $A$  switches  $l_F^A$  and  $l_{F+1}^A$ . Team  $A$  loses the shootout before it reaches  $l_{F+2}^A$ , when  $l_F^A$  misses under pressure, or when  $l_F^A$  scores under

pressure and  $l_F^B$  scores under no pressure, or when  $l_F^A$  scores under pressure,  $l_F^B$  misses under no pressure,  $l_{F+1}^A$  misses under no pressure and  $l_{F+1}^B$  scores under pressure. Conditional on  $s_F^A = -1$ , the probability of this losing scenario is

$$(1 - p_F^A) + p_F^A q_F^B + p_F^A (1 - q_F^B) (1 - q_{F+1}^A) q_{F+1}^B.$$

When  $A$  switches  $l_F^A$  and  $l_{F+1}^A$ , the change in the above probability is

$$(1 - q_F^B) \left( (1 - q_{F+1}^B) (p_F^A - p_{F+1}^A) + q_{F+1}^B (p_F^A q_{F+1}^A - p_{F+1}^A q_F^A) \right).$$

Since  $q_F^A < q_{F+1}^A$ , condition (4) implies that  $p_{F+1}^A q_F^A \geq p_F^A q_{F+1}^A$  and  $p_{F+1}^A > p_F^A$ . Thus, switching strictly decreases the ex ante probability of the losing scenario. It follows from Claim 1 that switching strictly increases the ex ante probability that team  $A$  wins the shootout in this case.

## Proof of Lemma 7

Fix any  $l^A$  and suppose that  $l^B$  is such that  $q_{F-1}^B < q_F^B$ . Consider switching  $l_{F-1}^B$  and  $l_F^B$ . There are two remaining cases.

Case (B1):  $s_{F-1}^B = 0$ . Team  $B$  wins the shootout in round  $F$ , when  $l_{F-1}^B$  scores under no pressure and  $l_F^A$  misses under pressure, or when  $l_F^B$  scores under no pressure after both  $l_{F-1}^B$  and  $l_F^A$  miss under no pressure. Conditional on  $s_{F-1}^B = 0$ , the probability of this winning scenario is

$$q_{F-1}^B (1 - p_F^A) + (1 - q_{F-1}^B) (1 - q_F^A) q_F^B.$$

When  $B$  switches  $l_{F-1}^B$  and  $l_F^B$ , the above is changed by

$$(q_F^B - q_{F-1}^B) (q_F^A - p_F^A),$$

which is non-negative because  $q_F^B > q_{F-1}^B$  and  $q_F^A \geq p_F^A$ . Team  $B$  loses the shootout in round  $F$  when  $l_{F-1}^B$  misses under no pressure,  $l_F^A$  scores under no pressure and  $l_F^B$  misses under pressure. Conditional on  $s_{F-1}^B = 0$ , the probability of this losing scenario is

$$(1 - q_{F-1}^B) q_F^A (1 - p_F^B).$$

Since  $q_{F-1}^B < q_F^B$ , (5) implies that the above is weakly decreased when  $B$  switches  $l_{F-1}^B$  and  $l_F^B$ . It follows from Claim 1 that  $B$ 's ex ante probability of winning is weakly increased by switching  $l_{F-1}^B$  and  $l_F^B$ .

Case (B2):  $s_{F-1}^B = 1$ . Team  $B$  wins the shootout in round  $F$  when  $l_{F-1}^B$  scores under pressure,  $l_F^A$  misses under no pressure, and  $l_F^B$  scores under no pressure. Conditional on  $s_{F-1}^B = 1$ , the probability of this winning scenario is

$$p_{F-1}^B(1 - q_F^A)q_F^B.$$

Since  $q_{F-1}^B < q_F^B$ , (4) implies that the above is weakly increased when  $B$  switches  $l_{F-1}^B$  and  $l_F^B$ . Team  $B$  loses the shootout in round  $F$ , when  $l_{F-1}^B$  misses under pressure and  $l_F^A$  scores under no pressure, or when  $l_{F-1}^B$  misses under pressure,  $l_F^A$  misses under no pressure and  $l_F^B$  misses under pressure, or when  $l_{F-1}^B$  misses under pressure,  $l_F^A$  scores under no pressure and  $l_F^B$  misses under pressure, or when  $l_{F-1}^B$  scores under pressure,  $l_F^A$  scores under no pressure and  $l_F^B$  misses under pressure. Conditional on  $s_{F-1}^B = 1$ , the probability of this losing scenario is

$$(1 - p_{F-1}^B)(q_F^A + (1 - q_F^A)(1 - p_F^B)) + p_{F-1}^B q_F^A(1 - p_F^B).$$

The above is unchanged when  $B$  switches  $l_{F-1}^B$  and  $l_F^B$ . It follows from Claim 1 that  $B$ 's ex ante probability of winning is weakly increased by switching  $l_{F-1}^B$  and  $l_F^B$  in this case.

Since at least one of the inequalities (4) and (5) holds with a strict inequality, switching  $l_{F-1}^B$  and  $l_F^B$  strictly increases  $B$ 's ex ante probability of winning. The lemma follows immediately.

## Proof of Lemma 8

Fix any  $l^B$  and suppose that  $l^A$  is such that  $q_{F-1}^A < q_F^A$ . Consider  $A$  switching  $l_{F-1}^A$  and  $l_F^A$ . There are two remaining cases.

Case (A1):  $s_{F-1}^A = -1$ . Team  $A$  wins the shootout in round  $F$ , when  $l_{F-1}^A$  scores under pressure,  $l_{F-1}^B$  misses under no pressure,  $l_F^A$  scores under no pressure and  $l_F^B$  misses under pressure. Conditional on  $s_{F-1}^A = -1$ , the probability of this winning scenario is

$$p_{F-1}^A(1 - q_{F-1}^B)q_F^A(1 - p_F^B).$$

Since  $q_{F-1}^A < q_F^A$ , when  $A$  switches  $l_{F-1}^A$  and  $l_F^A$ , the change to the above is non-negative by (4). Team  $A$  can lose the shootout in round  $F$ , with  $l_{F-1}^A$  always under pressure and  $l_{F-1}^B$  and  $l_F^B$  always under no pressure, but  $l_F^A$  can be either under pressure or under no pressure depending on the sequence of scores. Losing happens in round  $F-1$  when  $l_{F-1}^A$  misses and  $l_{F-1}^B$  scores, or in round  $F$  when  $l_{F-1}^A$  and  $l_F^B$  both score or both miss, followed by either  $l_F^A$  missing under pressure or  $l_F^A$  scoring under pressure and  $l_F^B$  scoring, or in round  $F$  when  $l_{F-1}^A$  scores and  $l_{F-1}^B$  misses, followed by  $l_F^A$  missing under no pressure and  $l_F^B$  scores. Conditional on  $s_{F-1}^A = -1$ , the probability of this losing scenario is

$$(1 - p_{F-1}^A)q_{F-1}^B + (p_{F-1}^A q_{F-1}^B + (1 - p_{F-1}^A)(1 - q_{F-1}^B))(1 - p_F^A + p_F^A q_F^B) + p_{F-1}^A(1 - q_{F-1}^B)(1 - q_F^A)q_F^B.$$

When  $A$  switches  $l_{F-1}^A$  and  $l_F^A$ , the only affected parts of the above probability are

$$(1 - p_{F-1}^A)(1 - q_{F-1}^B)p_F^A q_F^B + p_{F-1}^A(1 - q_{F-1}^B)(1 - q_F^A)q_F^B.$$

Since  $q_{F-1}^A < q_F^A$ , (4) implies that the above is weakly decreased when  $A$  switches  $l_{F-1}^A$  and  $l_F^A$ . It follows from Claim 1 that  $A$ 's ex ante probability of winning is strictly increased by switching  $l_{F-1}^A$  and  $l_F^A$ .

Case (A2):  $s_{F-1}^A = 0$ . Team  $A$  can win the shootout before the sudden-death phase, but only in round  $F$ . This happens when  $l_{F-1}^A$  and  $l_{F-1}^B$  both score or both miss in round  $F-1$ , followed by  $l_F^A$  scoring under no pressure and  $l_F^B$  missing under pressure, or when  $l_{F-1}^A$  scores under no pressure and  $l_{F-1}^B$  missing under pressure, followed by either  $l_F^A$  scoring again under no pressure or both  $l_F^A$  and  $l_F^B$  missing. Conditional on  $s_{F-1}^A = 0$ , the probability of this winning scenario is

$$(q_{F-1}^A p_{F-1}^B + (1 - q_{F-1}^A)(1 - q_{F-1}^B))q_F^A(1 - p_F^B) + q_{F-1}^A(1 - p_{F-1}^B)(q_F^A + (1 - q_F^A)(1 - p_F^B)).$$

When  $A$  switches  $l_{F-1}^A$  and  $l_F^A$ , the only affected parts of the above probability are

$$(1 - q_{F-1}^A)(1 - q_{F-1}^B)q_F^A(1 - p_F^B) + q_{F-1}^A(1 - p_{F-1}^B)(1 - q_F^A)(1 - p_F^B)$$

Since  $q_F^A > q_{F-1}^A$  and  $q_{F-1}^B \geq p_{F-1}^B$ , switching weakly increases the probability of winning scenario. Team  $A$  can also lose the shootout before the sudden-death phase, but again only in round  $F$ . This happens when  $l_{F-1}^A$  and  $l_{F-1}^B$  both score or both miss in round  $F-1$ ,

followed by  $l_F^A$  missing under no pressure and  $l_F^B$  scoring under no pressure, or when  $l_{F-1}^A$  misses under no pressure and  $l_{F-1}^B$  scoring under no pressure, followed by either  $l_F^A$  missing again but under pressure or both  $l_F^A$  and  $l_F^B$  scoring. Conditional on  $s_{F-1}^A = 0$ , the probability of this losing scenario is

$$(q_{F-1}^A p_{F-1}^B + (1 - q_{F-1}^A)(1 - q_{F-1}^B))(1 - q_F^A)q_F^B + (1 - q_{F-1}^A)q_{F-1}^B(1 - p_F^A + p_F^A q_F^B).$$

When  $A$  switches  $l_{F-1}^A$  and  $l_F^A$ , the change in the probability of the losing scenario is

$$\begin{aligned} & (q_F^A - q_{F-1}^A)p_{F-1}^B q_F^B - ((1 - q_{F-1}^A)(1 - p_F^A + p_F^A q_F^B) - (1 - q_F^A)(1 - p_{F-1}^A + p_{F-1}^A q_F^B)) q_{F-1}^B \\ & \leq ((1 - q_F^A)(1 - p_{F-1}^A) - (1 - q_{F-1}^A)(1 - p_F^A)) (1 - q_F^B) q_{F-1}^B \\ & \leq 0, \end{aligned}$$

where the first inequality follows because  $p_{F-1}^B \leq q_{F-1}^B$ , and the second inequality follows from (5) because  $q_F^A > q_{F-1}^A$ . Thus, switching weakly decreases the probability of losing scenario. It follows from Claim 1 that  $A$ 's ex ante probability of winning is increased by switching  $l_{F-1}^A$  and  $l_F^A$ .

### Proof of Claim 3

Fix any strategy profile  $(l^A, l^B)$ . For any  $j \leq F$ , let  $W_j^T(s_j^A)$ ,  $T = A, B$ , be the probability of  $T$  winning the shootout conditional on reaching  $s_j^A$  satisfying (1). Let  $W^T$  be  $T$ 's probability of winning conditional on reaching the first round  $F + 1$  of the sudden-death phase. We use backward induction.

The feasible score differences  $s_F^A$  satisfying (1) are  $-1, 0$  and  $1$ . We have

$$\begin{aligned} W_F^A(1) &= q_F^A + (1 - q_F^A)(1 - p_F^B) + (1 - q_F^A)p_F^B W^A \\ W_F^A(0) &= q_F^A(1 - p_F^B) + (q_F^A p_F^B + (1 - q_F^A)(1 - q_F^B))W^A \\ W_F^A(-1) &= q_F^A(1 - q_F^B)W^A. \end{aligned}$$

Thus,

$$\begin{aligned} W_F^A(1) - W_F^A(0) &= (q_F^A p_F^B + (1 - q_F^A)(1 - p_F^B))(1 - W^A) + (1 - q_F^A)p_F^B W^A > 0 \\ W_F^A(0) - W_F^A(-1) &= q_F^A(1 - p_F^B)(1 - W^A) + (q_F^A p_F^B + (1 - q_F^A)(1 - q_F^B))W^A > 0. \end{aligned}$$

Suppose that  $W_k^A(s_k^A) < W_k^A(s_k^A + 1)$  for each round  $k = j + 1, \dots, F$  such that  $s_k^A$  and  $s_k^A + 1$  satisfy (1) for round  $k$ . We want to show that  $W_j^A(s_j^A) < W_j^A(s_j^A + 1)$  for  $s_j^A$  and  $s_j^A + 1$  satisfying (1).

First, consider  $s_j^A = F - j$ . Penalty-taker  $l_j^A$  is decisive by winning at the score difference  $s_j^A + 1$ , and so

$$\begin{aligned} W_j^A(s_j^A + 1) &= q_j^A + (1 - q_j^A)(1 - p_j^B) + (1 - q_j^A)p_j^B W_{j+1}^A(s_j^A) \\ &> W_j^A(s_j^A) = q_j^A(1 - p_j^B) + (q_j^A p_j^B + (1 - q_j^A)(1 - p_j^B))W_{j+1}^A(s_j^A) + (1 - q_j^A)p_j^B W_{j+1}^A(s_j^A - 1), \end{aligned}$$

because  $W_{j+1}^A(s_j^A) > W_{j+1}^A(s_j^A - 1)$  by the induction hypothesis.

Next, consider  $0 \leq s_j^A \leq F - j - 1$ . We have

$$\begin{aligned} W_j^A(s_j^A + 1) &= q_j^A(1 - p_j^B)W_{j+1}^A(s_j^A + 2) + (q_j^A p_j^B + (1 - q_j^A)(1 - p_j^B))W_{j+1}^A(s_j^A + 1) \\ &\quad + (1 - q_j^A)p_j^B W_{j+1}^A(s_j^A), \end{aligned}$$

where  $W_{j+1}^A(s_j^A + 2) = 1$  if  $s_j^A = F - j - 1$ , and

$$\begin{aligned} W_j^A(s_j^A) &= q_j^A(1 - p_j^B)W_{j+1}^A(s_j^A + 1) + (q_j^A p_j^B + (1 - q_j^A)(1 - r_j^B))W_{j+1}^A(s_j^A) \\ &\quad + (1 - q_j^A)r_j^B W_{j+1}^A(s_j^A - 1), \end{aligned}$$

where  $r_j^B = q_j^B$  if  $s_j^A = 0$  and  $r_j^B = p_j^B$  if  $s_j^A \geq 1$ , because  $l_j^B$  is under pressure if and only if  $l_j^A$  scores at  $s_j^A = 0$ , and is never under pressure at  $s_j^A \geq 1$ . Thus,

$$\begin{aligned} &W_j^A(s_j^A + 1) - W_j^A(s_j^A) \\ &= q_j^A(1 - p_j^B)(W_{j+1}^A(s_j^A + 2) - W_{j+1}^A(s_j^A + 1)) + (1 - q_j^A)r_j^B(W_{j+1}^A(s_j^A) - W_{j+1}^A(s_j^A - 1)) \\ &\quad + (q_j^A p_j^B + (1 - q_j^A)(1 - p_j^B))(W_{j+1}^A(s_j^A + 1) - W_{j+1}^A(s_j^A)). \end{aligned}$$

The above is strictly positive by the induction hypothesis.

Now, consider  $-(F - j) \leq s_j^A \leq -1$ . We have

$$\begin{aligned} W_j^A(s_j^A + 1) &= r_j^A(1 - r_j^B)W_{j+1}^A(s_j^A + 2) + (r_j^A r_j^B + (1 - r_j^A)(1 - q_j^B))W_{j+1}^A(s_j^A + 1) \\ &\quad + (1 - r_j^A)q_j^B W_{j+1}^A(s_j^A), \end{aligned}$$

where  $r_j^A = p_j^A$  and  $r_j^B = q_j^B$  if  $s_j^A \leq -2$  (because  $l_j^A$  is always under pressure and  $l_j^B$  is not under pressure even after  $l_j^A$  scores at  $s_j^A + 1 \leq -1$ ), and  $r_j^A = q_j^A$  and  $r_j^B = p_j^B$  if

$s_j^A = -1$  (because  $l_j^A$  is under no pressure and  $l_j^B$  is under pressure if and only if  $l_j^A$  scores at  $s_j^A + 1 = 0$ ), and

$$\begin{aligned} W_j^A(s_j^A) &= p_j^A(1 - q_j^B)W_{j+1}^A(s_j^A + 1) + (p_j^A q_j^B + (1 - p_j^A)(1 - q_j^B))W_{j+1}^A(s_j^A) \\ &\quad + (1 - p_j^A)q_j^B W_{j+1}^A(s_j^A - 1). \end{aligned}$$

Taking the difference, we have

$$\begin{aligned} &W_j^A(s_j^A + 1) - W_j^A(s_j^A) \\ &= r_j^A(1 - r_j^B)(W_{j+1}^A(s_j^A + 2) - W_{j+1}^A(s_j^A + 1)) + (1 - p_j^A)q_j^B(W_{j+1}^A(s_j^A) - W_{j+1}^A(s_j^A - 1)) \\ &\quad + (r_j^A q_j^B + (1 - p_j^A)(1 - q_j^B))(W_{j+1}^A(s_j^A + 1) - W_{j+1}^A(s_j^A)). \end{aligned}$$

The above is strictly positive by the induction hypothesis.

Finally, consider  $s_j^A = -(F - j + 1)$ . Penalty-taker  $s_j^A$  is decisive by missing, and so

$$W_j^A(s_j^A + 1) = p_j^A(1 - q_j^B)W_{j+1}^A(s_j^A) > W_j^A(s_j^A) = 0.$$

## Proof of Lemma 9

Fix any  $l^A$  and suppose that  $l^B$  is such that  $q_j^B < q_{j+1}^B$  for some  $j \leq F - 2$ . Consider switching  $l_j^B$  and  $l_{j+1}^B$ . There are two remaining cases.

Case (B1):  $s_j^B = 0$ . Since  $j \leq F - 2$ , the shoutout cannot end in round  $j$  or  $j + 1$ , and will continue to round  $j + 2$  with score difference  $s_j^B + \delta$ ,  $\delta = -2, -1, 0, 1$ . Penalty-taker  $l_j^B$  is not under pressure, and can put  $l_{j+1}^A$  under pressure by scoring in round  $j$ ; penalty-taker  $l_{j+1}^B$  is not under pressure unless  $l_j^B$  misses and  $l_{j+1}^A$  scores. Conditional on  $s_j^B = 0$ , the probability of  $B$  winning the shootout is

$$\begin{aligned} &q_j^B(1 - p_{j+1}^A)q_{j+1}^B W_{j+2}^B(-2) \\ &+ (q_j^B p_{j+1}^A q_{j+1}^B + q_j^B(1 - p_{j+1}^A)(1 - q_{j+1}^B) + (1 - q_j^B)(1 - q_{j+1}^A)q_{j+1}^B) W_{j+2}^B(-1) \\ &+ (q_j^B p_{j+1}^A(1 - q_{j+1}^B) + (1 - q_j^B)q_{j+1}^A p_{j+1}^B + (1 - q_j^B)(1 - q_{j+1}^A)(1 - q_{j+1}^B)) W_{j+2}^B(0) \\ &+ (1 - q_j^B)q_{j+1}^A(1 - p_{j+1}^B)W_{j+2}^B(1). \end{aligned}$$

When  $B$  switches  $l_j^B$  and  $l_{j+1}^B$ , the first term in the above expression is unchanged; the second term is changed by

$$(q_{j+1}^A - p_{j+1}^A)(q_{j+1}^B - q_j^B)W_{j+2}^B(-1) > 0,$$

which is strictly positive; and the fourth term is changed by

$$q_{j+1}^A((1 - q_{j+1}^B)(1 - p_j^B) - (1 - q_j^B)(1 - p_{j+1}^B))W_{j+2}^B(1),$$

which is non-positive by (5) because  $q_j^B < q_{j+1}^B$ . Thus, the total change in  $B$ 's probability of winning the shootout is

$$\begin{aligned} & (q_{j+1}^A - p_{j+1}^A)(q_{j+1}^B - q_j^B)(W_{j+2}^B(-1) - W_{j+2}^B(0)) \\ & + q_{j+1}^A((1 - q_{j+1}^B)(1 - p_j^B) - (1 - q_j^B)(1 - p_{j+1}^B))(W_{j+2}^B(1) - W_{j+2}^B(0)), \end{aligned}$$

which is positive by Claim 3. It follows that switching increases  $B$ 's probability of winning.

Case (B2):  $s_j^B = 1$ . The shootout will continue to round  $j + 2$  with score difference  $s_j^B + \delta$ ,  $\delta = -2, -1, 0, 1$ , unless  $j = F - 2$ , in which case  $B$  loses the shootout in round  $j + 1$  with  $\delta = 1$ . Penalty-taker  $l_j^B$  is under pressure and  $l_{j+1}^A$  is never under pressure; penalty-taker  $l_{j+1}^B$  is under pressure unless  $l_j^B$  scores and  $l_{j+1}^A$  misses. Conditional on  $s_j^B = 1$ , the probability of  $B$  winning the shootout is

$$\begin{aligned} & p_j^B(1 - q_{j+1}^A)q_{j+1}^B W_{j+2}^B(-1) \\ & + (p_j^B q_{j+1}^A p_{j+1}^B + p_j^B(1 - q_{j+1}^A)(1 - q_{j+1}^B) + (1 - p_j^B)(1 - q_{j+1}^A)p_{j+1}^B) W_{j+2}^B(0) \\ & + (p_j^B q_{j+1}^A(1 - p_{j+1}^B) + (1 - p_j^B)q_{j+1}^A p_{j+1}^B + (1 - p_j^B)(1 - q_{j+1}^A)(1 - p_{j+1}^B)) W_{j+2}^B(1) \\ & + (1 - p_j^B)q_{j+1}^A(1 - p_{j+1}^B)W_{j+2}^B(2), \end{aligned}$$

where  $W_{j+2}^B(2) = 0$  when  $j = F - 2$ . When  $B$  switches  $l_j^B$  and  $l_{j+1}^B$ , the first term in the above expression is changed by

$$(1 - q_{j+1}^A)(p_{j+1}^B q_j^B - p_j^B q_{j+1}^B)W_{j+2}^B(-1),$$

which is non-negative by (4) because  $q_j^B < q_{j+1}^B$ ; and both the third term and the fourth term are unchanged. Thus, the total change in  $B$ 's probability of winning the shootout is

$$(1 - q_{j+1}^A)(p_{j+1}^B q_j^B - p_j^B q_{j+1}^B)(W_{j+2}^B(-1) - W_{j+2}^B(0)),$$

which is non-negative by Claim 3. It follows that in this case switching weakly increases  $B$ 's probability of winning.

Now, fix any  $l^B$  and suppose that  $l^A$  is such that  $q_j^A < q_{j+1}^A$  for some  $j \leq F-2$ . Consider switching  $l_j^A$  and  $l_{j+1}^A$ . There are two remaining cases. In each case below, conditional on reaching round  $l_j^A$  with score difference  $s_j^A$ , the shootout will always continue to round  $j+2$  with  $s_{j+2}^A = s_j^A + \delta$ ,  $\delta = 2, 1, 0, -1, -2$ .

Case (A1):  $s_j^A = -1$ . Penalty-taker  $l_j^A$  is under pressure and  $l_j^B$  is never under pressure; penalty-taker  $l_{j+1}^A$  is under pressure unless  $l_j^A$  scores and  $l_j^B$  misses; and penalty-taker  $l_{j+1}^B$  is not under pressure unless  $l_j^A$  and  $l_{j+1}^A$  both score and  $l_j^B$  misses. Conditional on  $s_j^A = -1$ , the probability of  $B$  winning the shootout is

$$\begin{aligned}
& p_j^A(1 - q_j^B)q_{j+1}^A(1 - p_{j+1}^B)W_{j+2}^A(1) \\
& + (p_j^A q_j^B p_{j+1}^A(1 - q_{j+1}^B) + p_j^A(1 - q_j^B)(q_{j+1}^A p_{j+1}^B + (1 - q_{j+1}^A)(1 - q_{j+1}^B))) \\
& \quad + (1 - p_j^A)(1 - q_j^B)p_{j+1}^A(1 - q_{j+1}^B)) W_{j+2}^A(0) \\
& + ((p_j^A q_j^B + (1 - p_j^A)(1 - q_j^B))(p_{j+1}^A q_{j+1}^B + (1 - p_{j+1}^A)(1 - q_{j+1}^B))) \\
& \quad + p_j^A(1 - q_j^B)(1 - q_{j+1}^A)q_{j+1}^B + (1 - p_j^A)q_j^B p_{j+1}^A(1 - q_{j+1}^B)) W_{j+2}^A(-1) \\
& + (p_j^A q_j^B(1 - p_{j+1}^A)q_{j+1}^B + (1 - p_j^A)q_j^B(p_{j+1}^A q_{j+1}^B + (1 - p_{j+1}^A)(1 - q_{j+1}^B))) \\
& \quad + (1 - p_j^A)(1 - q_j^B)(1 - p_{j+1}^A)q_{j+1}^B) W_{j+2}^A(-2) \\
& + (1 - p_j^A)q_j^B(1 - p_{j+1}^A)q_{j+1}^B W_{j+2}^A(-3).
\end{aligned}$$

When  $A$  switches  $l_j^A$  and  $l_{j+1}^A$ , the first term in the above expression is changed by

$$(1 - q_j^B)(1 - p_{j+1}^B)(p_{j+1}^A q_j^A - p_j^A q_{j+1}^A)W_{j+2}^A(1),$$

which is non-negative by (4) because  $q_j^A < q_{j+1}^A$ ; the third term is changed by

$$-(1 - q_j^B)q_{j+1}^B(p_{j+1}^A q_j^A - p_j^A q_{j+1}^A)W_{j+2}^A(-1),$$

which is non-positive by (4) because  $q_j^A < q_{j+1}^A$ ; and the fourth and the fifth terms are unchanged. Thus, the total change in  $A$ 's probability of winning the shootout is

$$(1 - q_j^B)(p_{j+1}^A q_j^A - p_j^A q_{j+1}^A) \left( (1 - p_{j+1}^B)(W_{j+2}^A(1) - W_{j+2}^A(0)) + q_{j+1}^B(W_{j+2}^A(0) - W_{j+2}^A(-1)) \right),$$

which is non-negative by Claim 3.

Case (A2):  $s_j^A = 0$ . Penalty-taker  $l_j^A$  is not under pressure, and can put  $l_j^B$  under pressure by scoring; penalty-taker  $l_{j+1}^A$  is not under pressure unless  $l_j^A$  misses and  $l_j^B$  scores; and  $l_{j+1}^B$

is under pressure if  $l_j^A$  and  $l_{j+1}^A$  score at least one more than  $l_j^B$ . Conditional on  $s_j^A = 0$ , the probability of  $A$  winning the shootout is

$$\begin{aligned}
& q_j^A(1-p_j^B)q_{j+1}^A(1-p_{j+1}^B)W_{j+2}^A(2) \\
& + (q_j^A p_j^B q_{j+1}^A(1-p_{j+1}^B) + q_j^A(1-p_j^B)(q_{j+1}^A p_{j+1}^B + (1-q_{j+1}^A)(1-p_{j+1}^B)) \\
& \quad + (1-q_j^A)(1-q_j^B)q_{j+1}^A(1-p_{j+1}^B)) W_{j+2}^A(1) \\
& + ((q_j^A p_j^B + (1-q_j^A)(1-q_j^B))(q_{j+1}^A p_{j+1}^B + (1-q_{j+1}^A)(1-q_{j+1}^B)) \\
& \quad + q_j^A(1-p_j^B)(1-q_{j+1}^A)p_{j+1}^B + (1-q_j^A)q_j^B p_{j+1}^A(1-q_{j+1}^B)) W_{j+2}^A(0) \\
& + (q_j^A p_j^B(1-q_{j+1}^A)q_{j+1}^B + (1-q_j^A)q_j^B(p_{j+1}^A q_{j+1}^B + (1-p_{j+1}^A)(1-q_{j+1}^B)) \\
& \quad + (1-q_j^A)(1-q_j^B)(1-q_{j+1}^A)q_{j+1}^B) W_{j+2}^A(-1) \\
& + (1-q_j^A)q_j^B(1-p_{j+1}^A)q_{j+1}^B W_{j+2}^A(-2).
\end{aligned}$$

When  $A$  switches  $l_j^A$  and  $l_{j+1}^A$ , the first term in the above expression is unchanged; the second term is changed by

$$(q_j^B - p_j^B)(q_{j+1}^A - q_j^A)(1-p_{j+1}^B)W_{j+2}^A(1);$$

the third term is changed by

$$\begin{aligned}
& ((q_j^B - p_j^B)(q_{j+1}^A - q_j^A)(p_{j+1}^B - (1-q_{j+1}^B)) \\
& + q_j^B((1-q_j^A)(1-p_{j+1}^A) - (1-q_{j+1}^A)(1-p_j^A))(1-q_{j+1}^B)) W_{j+2}^A(0);
\end{aligned}$$

and the fifth term is changed by

$$-q_j^B((1-q_j^A)(1-p_{j+1}^A) - (1-q_{j+1}^A)(1-p_j^A))q_{j+1}^B W_{j+2}^A(-2).$$

Thus, the total change in  $A$ 's probability of winning the shootout is

$$\begin{aligned}
& (q_j^B - p_j^B)(q_{j+1}^A - q_j^A) \\
& \cdot ((1-p_{j+1}^B)(W_{j+2}^A(1) - W_{j+2}^A(-1)) + (p_{j+1}^B - (1-q_{j+1}^B))(W_{j+2}^A(0) - W_{j+2}^A(-1))) \\
& + q_j^B((1-q_j^A)(1-p_{j+1}^A) - (1-q_{j+1}^A)(1-p_j^A)) \cdot \\
& \cdot ((1-q_{j+1}^B)(W_{j+2}^A(0) - W_{j+2}^A(-1)) + q_{j+1}^B(W_{j+2}^A(-1) - W_{j+2}^A(-2))).
\end{aligned}$$

By Claim 3,  $W_{j+2}^A(1) - W_{j+2}^A(-1) > W_{j+2}^A(0) - W_{j+2}^A(-1) > 0$ , and thus the total change is strictly positive.