Stochastic Sequential Screening*

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Abstract

We study when and how randomization can help improve the seller's revenue in the sequential screening setting. In a model with discrete ex ante types and a continuum of ex post valuations, the standard approach based on solving a relaxed problem that keeps only local downward incentive compatibility constraints often fails. Under a strengthening of first-order stochastic dominance ordering on the valuation distribution functions of ex ante types, we introduce and solve a modified relaxed problem by retaining all local incentive compatibility constraints, provide necessary and sufficient conditions for optimal mechanisms to be stochastic, and characterize optimal stochastic contracts. Our analysis mostly focuses on the case of three ex ante types, but our methodology of solving the modified problem, as well as the necessary and sufficient conditions for randomization to be optimal, can be extended to any finite number of ex ante types.

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1 Introduction

Random allocations through rationing and lotteries are common for selling event tickets, material inputs, or consumer products (see Gilbert and Klemperer (2000) for a list of examples). For the static environments of monopoly pricing or auctions, the literature of mechanism design (see Myerson (1981), Riley and Zeckhauser (1983), and Bulow and Roberts (1989), among others) has established when and how randomization can help alleviate incentive problems. In particular, Riley and Zeckhauser (1983) prove that a posted price is always revenue-maximizing when the seller can fully commit to a selling mechanism for a single buyer. They interpret random allocations as "haggling," and show that they do not help the seller to price discriminate. In this paper, we will show why and how random allocations can help dynamic price discrimination, and characterize optimal stochastic mechanisms.

Relatively little is known about random allocations in dynamic environments. Almost all the dynamic mechanism design literature adopts the standard approach which forms a relaxed problem by keeping only local downward incentive compatibility constraints and then imposes strong conditions under which the deterministic solution to the relaxed problem also solves the original problem. Consider the formulation of the two-period sequential screening problem first introduced by Courty and Li (2000) where a seller of an indivisible good designs a selling mechanism for a buyer who knows which distribution that the valuation of the good is drawn from in period one (his ex ante type) but his valuation is only realized in period two after agreeing to the mechanism. With discrete ex ante types ranked by first order stochastic dominance, the standard approach forms a relaxed problem by keeping only local downward incentive compatibility constraints and the individual rationality constraint of the lowest ex ante type. If the solution to the relaxed problem, found through point-wise maximization, can be represented by cutoff valuations that are monotone in types, then this solution satisfies all dropped local upward and non-local incentive compatibility constraints and hence it corresponds to an optimal mechanism. Moreover, this mechanism is deterministic, implementable by a menu of option contracts.

The standard approach fails if point-wise maximization leads to allocations that are not in a cutoff form for some ex ante types, or if allocations are in a cutoff form for all types, but the cutoffs are not monotone with respect to type. The existing literature on dynamic mechanism design is silent on how to characterize the optimal mechanism in this case.

The goal of this paper is to characterize optimal dynamic mechanisms when the standard approach of point-wise maximization approach fails and shed light on the role of randomization in optimal mechanisms. Our approach is based on a modified relaxed problem. We impose the same binding local downward incentive compatibility constraints and the individual rationality constraint for the lowest type to arrive at the same objective function as in the standard approach. However, we retain local upward incentive compatibility constraints, as well as monotonicity of the allocation with respect to ex post valuation for each type.[1] By imposing a strengthening of first order stochastic dominance, we show that any solution to our modified relaxed problem corresponds to an optimal mechanism because it satisfies all dropped constraints in the original problem.

Our analysis focuses mostly on the sequential screening problem with three ex ante types, although it can be generalized to any finite number of types ranked by first order stochastic dominance. We need a minimum of three types for the standard approach to fail and for stochastic mechanisms to be optimal.[2] The modified relaxed problem is to choose non-decreasing allocations of the stochastically dominated low type and the middle-ranked type to maximize the sum of the expected dynamic virtual surpluses of the two types, subject to the local upward incentive compatibility constraint, which requires a weighted average of the middle type's allocation to be greater than or equal to the average of the low type with the same weights. If the solution to the modified relaxed problem is deterministic and satisfies the local upward incentive compatibility constraint with slack, then it solves the original problem – this is when the standard approach works. When the standard approach fails, the solution can still be deterministic with a binding local upward incentive compatibility constraint, as it can be optimal for the two types to have the same cutoff allocation as a

[1]Adding these constraints to the standard relaxed problem means we do not distinguish whether the standard approach fails because the point-wise maximizer is in a cutoff form but the cutoffs are not monotone in type, or because after ironing the cutoffs are not monotone.

[2]With only two types the upward incentive compatibility constraint never binds at the solution to the relaxed problem. The allocation to the stochastically dominant high type is efficient. By Riley and Zeckhauser (1983), there is always a cutoff solution to maximizing the dynamic virtual surplus of the dominated low type among all non-decreasing allocations, and by first order stochastic dominance ranking, the cutoff is inefficiently high.

compromise between maximizing the sum of dynamic virtual surpluses and satisfying the local upward incentive compatibility constraint.

We identify the necessary and sufficient conditions for optimality of stochastic mechanisms with a perturbation argument. Starting from a deterministic solution with a common cutoff for the low and the middle types, we ask whether it is possible to increase the dynamic virtual surplus of either type by replacing the cutoff allocation for that type with a stochastic allocation that leaves the local upward incentive compatibility constraint binding. We show that the necessary and sufficient conditions for randomization can be stated as a comparison between the ratio of the average dynamic virtual surplus of either the low type or the middle type to the average slack in the local upward incentive compatibility constraint for any interval below the common cutoff and the ratio for any interval above the cutoff. If the former is always smaller than the latter for both types, then any optimal mechanism is deterministic; conversely, if the condition fails for either the low type or the middle type, or both, then any optimal mechanism is stochastic. These ratio conditions are straightforward to verify under additional assumptions on the shape of the average ratio and the point ratio. The same conditions allow us to provide a full characterization of the optimal mechanisms, whether they are stochastic or deterministic. Both our sufficient and necessary conditions for stochastic mechanisms to be optimal and our characterization of optimal stochastic mechanisms have their counterparts with more than three types.

There is an extensive literature on "ironing" in static mechanism design problems when various regularity conditions fail, starting from Myerson (1981) and Riley and Zeckhauser (1983). The techniques are well presented in, e.g., Fudenberg and Tirole (1991), and have also been extended to multi-dimensional screening problems (see, e.g., Rochet and Chone, 1998). Although dynamic mechanism design in general, and sequential screening in particular, is closely related to multi-dimensional screening,[3] there has not been much progress made in the existing literature in characterizing stochastic dynamic mechanisms when the standard approach of point-wise maximization fails. Courty and Li (2000) primarily focus

[3]Krähmer and Strausz (2017) establish the equivalence between the sequential screening model and a static screening model with two-dimensional private information, by endowing the seller with a payoff that is a function of the buyer's ex ante type and ex post valuation. They use the equivalence to provide a sufficient condition for deterministic mechanisms to be optimal in the dynamic setting.

on deterministic mechanisms, but they provide an example of stochastic mechanism at the end of their paper, which serves as the starting point of the present paper. In a dynamic non-linear pricing setting of Mussa and Rosen (1978) with a Markovian information structure, Battaglini and Lamba (2019) argue that the first-order approach generally fails. They provide an example with three payoff types and two time periods, and show that "dynamic pooling" is generally optimal. In particular, after some report in the first period, two different second-period types receive the same quantities. Their general analysis is quite different from ours, and mostly focus on providing approximation results. Krasikov and Lamba (2021) study a sequential screening problem where a buyer's valuation follows a Poisson renewal process. At any instant between the time of contracting and a terminal time, the buyer's valuation either stays the same or, when Poisson shock occurs, is redrawn from an exogenous continuous distribution. They show that the standard approach fails, and focus on characterizing the optimal deterministic dynamic mechanism.

Bergemann, Castro, and Weintraub (2020) study a sequential screening model with ex post individual rationality constraints, and provide necessary and sufficient conditions for optimal sequential screening to be stochastic. Our model differs from Bergemann, Castro, and Weintraub (2020) because we impose ex ante rather than ex post individual rationality constraints. In their benchmark model with two ex ante types, the only relevant incentive compatibility constraint in Bergemann, Castro, and Weintraub (2020) is downward and local. In contrast, even with three ex ante types, our model has upward and global incentive compatibility constraints. Correspondingly, we impose a stronger condition than first order stochastic dominance on ex ante types to construct the relaxed problem with only local downward incentive compatibility constraints. Our surplus-to-slack ratio is inspired by the profit-to-rent ratio defined in Bergemann, Castro, and Weintraub (2020). Although our ratio arises from the dynamic virtual surplus and an upward incentive compatibility constraint while theirs is static with only a downward constraint , the two ratios play a similar role in establishing necessary and sufficient conditions for stochastic mechanisms to be optimal in the respective problems.[4]

[4] A similar ratio also appears in a sequential delegation setting of Krähmer and Kováč (2016), and is crucial to determine whether it is optimal to screen the agent's initial information. Their model share similar information structure as our model, but their analysis is quite different because there are no transfers.

This paper is organized as follows. In Section 2, we present the main model of sequential screening with three ex ante types. After introducing the standard relaxed problem and our modified relaxed problem, we use a numerical example to show that the standard approach based on the relaxed problem can fail; the optimal mechanism can still be deterministic when the standard approach fails; and random allocations can be optimal. In Section 3, we introduce average and point surplus-to-slack ratios, and use them to characterize necessary and sufficient conditions for stochastic mechanisms. In Section 4, we show how to construct optimal stochastic mechanisms using the modified relaxed problem. Section 5 offers extensions of our main results to sequential screening with more than three types. We show how to generalize the necessary and sufficient conditions for stochastic mechanisms. Our approach based on solving the modified relaxed problem is local, and requires a strengthening of first order stochastic dominance ranking to provide the sufficient conditions for randomization and to characterize optimal stochastic mechanisms. In Section 6, we provide an alternative global approach and show that the insights based on the local approach remain largely intact without strengthening the first order stochastic dominance ranking. In Appendix A we further develop the illustrative example and present a class of analytical examples with exponential distributions to illustrate how to construct the optimal stochastic mechanism with three ex ante types, and how to generalize the construction to more than three types.

2 The Model

A seller has one object for sale to a potential buyer. There are two periods. The seller and the buyer are risk-neutral, and do not discount. The buyer's value $\omega \in \Omega \equiv [\underline{\omega}, \overline{\omega}]$ for the good is unknown to both the buyer and the seller in period one. We allow for the possibility that $\overline{\omega} = \infty$. The seller's reservation value is known to be $c < \overline{\omega}$.

In period one, the buyer privately observes a signal $\theta \in \Theta$ about ω, which we refer to as his type. We assume that $\Theta = \{H, M, L\}$, with probability ϕ_θ for each $\theta = H, M, L$ and $\sum_\theta \phi_\theta = 1$. For each $\theta \in \Theta$, let $F_\theta(\cdot)$ be the conditional distribution function over Ω, and we assume that $F_\theta(\cdot)$ has positive and finite density $f_\theta(\cdot)$. We assume that type H is higher than M, which is in turn higher than L in first order stochastic dominance, that is,

$F_H(\omega) \leq F_M(\omega) \leq F_L(\omega)$ for all ω, with strict inequalities for a positive measure of ω. In period two, the buyer observes his value ω. The non-participation payoff of the buyer is normalized to 0 regardless of his ex ante type.

The seller chooses a direct revelation mechanism $(x_\theta(\omega), t_\theta(\omega))$, where $x_\theta(\omega)$ is the allocation rule and $t_\theta(\omega)$ is payment rule for reported type θ in period one and reported value ω in period two. The objective function of the seller's optimization problem is

$$\max_{(x_\theta, t_\theta)} \sum_{\theta=H,M,L} \phi_\theta \int_{\underline{\omega}}^{\overline{\omega}} (t_\theta(\omega) - c x_\theta(\omega)) f_\theta(\omega) d\omega \tag{P}$$

subject to four sets of constraints. First, the incentive compatibility constraints in period two: for each $\theta = H, M, L$, and for all $\omega, \omega' \in [\underline{\omega}, \overline{\omega}]$,

$$\omega x_\theta(\omega) - t_\theta(\omega) \geq \omega x_\theta(\omega') - t_\theta(\omega'). \tag{IC$_\theta$}$$

Second, the individual rationality constraints in period one: for each $\theta = H, M, L$,

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega x_\theta(\omega) - t_\theta(\omega)) f_\theta(\omega) d\omega \geq 0. \tag{IR$_\theta$}$$

Third, the IC constraints in period one: for each $\theta = H, M, L$,

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega x_\theta(\omega) - t_\theta(\omega)) f_\theta(\omega) d\omega \geq \int_{\underline{\omega}}^{\overline{\omega}} (\omega x_{\theta'}(\omega) - t_{\theta'}(\omega)) f_\theta(\omega) d\omega, \tag{IC$_{\theta\theta'}$}$$

for all $\theta' \neq \theta = H, M, L$. Fourth, the feasibility constraints on allocations x_θ, $\theta = H, M, L$:

$$0 \leq x_\theta(\omega) \leq 1 \tag{FE$_\theta$}$$

for all $\omega \in [\underline{\omega}, \overline{\omega}]$. A solution to (P) is an "optimal mechanism."

2.1 A modified relaxed problem

A standard result in mechanism states that allocation monotonicity with respect to valuation together with an envelope condition is both necessary and sufficient for incentive compatibility in period two. That is, for each $\theta = H, M, L$, IC_θ holds if and only if $x_\theta(\omega)$ is non-decreasing in ω, and

$$\omega x_\theta(\omega) - t_\theta(\omega) = u_\theta(\underline{\omega}) + \int_{\underline{\omega}}^{\omega} x_\theta(s)\,ds$$

for all ω, where $u_\theta(\underline{\omega}) = \underline{\omega} x_\theta(\underline{\omega}) - t_\theta(\underline{\omega})$. Through integration by parts, we can rewrite individual rationality constraint IR_θ for each $\theta = H, M, L$ as

$$u_\theta(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_\theta(\omega)(1 - F_\theta(\omega))\,d\omega \geq 0,$$

and incentive compatibility constraint $\text{IC}_{\theta\theta'}$ for each pair $\theta \neq \theta' = H, M, L$ as

$$u_\theta(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_\theta(\omega)(1 - F_\theta(\omega))\,d\omega \geq u_{\theta'}(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_{\theta'}(\omega)(1 - F_\theta(\omega))\,d\omega.$$

The standard "relaxed problem" is derived by binding the two local downward period one IC constraints, IC_{ML} and IC_{HM}, and the individual rationality constraint for the lowest type, IR_L. The three binding constraints can be used to solve for $u_\theta(\underline{\omega})$ for each $\theta = H, M, L$. Define the dynamic virtual surplus function $\delta_\theta(\omega)$, $\theta = H, M, L$, as the difference between the trade surplus with type θ and the information rent paid to all types higher than θ per unit of allocation of the good to type θ, given by

$$\begin{aligned}
\delta_H(\omega) &= \omega - c, \\
\delta_M(\omega) &= \omega - c - \frac{\phi_H(F_M(\omega) - F_H(\omega))}{\phi_M f_M(\omega)}, \\
\delta_L(\omega) &= \omega - c - \frac{(\phi_M + \phi_H)(F_L(\omega) - F_M(\omega))}{\phi_L f_L(\omega)}.
\end{aligned}$$

The objective function in the relaxed problem becomes

$$\max_{(x_\theta)} \sum_{\theta=H,M,L} \int_{\underline{\omega}}^{\overline{\omega}} x_\theta(\omega)\phi_\theta\delta_\theta(\omega) f_\theta(\omega)d\omega. \tag{RP}$$

The standard approach is to solve (RP) by point-wise maximization, subject to only the feasibility constraints (FE_θ). The allocation for type H is efficient with $x_H(\omega) = \mathbb{1}_{\omega \geq c}$.

When $\delta_M(\omega)$ and $\delta_L(\omega)$ both cross 0 only once and only from below, and when the crossing point of δ_M is smaller than or equal that of δ_L, the solutions to the relaxed problem are deterministic in a cutoff form, and satisfy all remaining period one IC constraints as well as individual rationality constraints. The conditions on the distribution functions $\{F_\theta\}_{\theta=H,M,L}$ that ensure both single-crossing of $\delta_M(\omega)$ and $\delta_L(\omega)$, and the "right" order of the crossing points are known as "regularity conditions." When these conditions hold, the deterministic allocations given by the crossing points of $\delta_M(\omega)$ and $\delta_L(\omega)$ correspond to a solution to (P). When one of these conditions fails, however, the standard approach fails and is silent about how to find a solution to (P).

Now we introduce the following modified relaxed problem and use it characterize potentially stochastic solutions to (P) when the standard approach fails. The objective function and the choice variables are the same as those in the standard relaxed problem, but instead of unconstrained point-wise maximization, we retain two sets of constraints: the local *upward* IC constraint IC_{LM} as well as the local *downward* IC constraints, and the monotonicity constraints as well as the feasibility constraints on the allocations.[5] With $x_H(\omega) = \mathbb{1}_{\omega \geq c}$, our modified relaxed problem is given by

$$\max_{x_M(\omega), x_L(\omega)} \int_{\underline{\omega}}^{\overline{\omega}} x_M(\omega)\phi_M\delta_M(\omega)f_M(\omega)d\omega + \int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega)\phi_L\delta_L(\omega)f_L(\omega)d\omega. \tag{MRP}$$

subject to $x_M(\omega)$ and $x_L(\omega)$ being non-decreasing functions with values on $[0,1]$, and IC_{LM}, which by the binding IC_{ML} is equivalent to

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_M(\omega) - x_L(\omega)\right)\left(F_L(\omega) - F_M(\omega)\right) d\omega \geq 0. \tag{IC$'_{LM}$}$$

Note that IC'_{LM} requires that a weighted average of type M's allocation x_M is greater than the average of type L's allocation x_L with the same weights.

By imposing IC'_{LM}, we allow the solution to (MRP) to be either deterministic or stochastic. If the solution is deterministic, then as in the standard approach, it satisfies all dropped IC constraints and therefore corresponds a solution to (P). It is important to note, however,

[5] The other local upward IC constraint IC_{MH} is also dropped because we will show later that it is never binding at any solution to (P) in the main model with only three ex ante types.

that the solution to (MRP) could be deterministic even though the retained constraint IC'_{LM} is binding. In other words, our approach allows us to potentially identify conditions under which the regularity conditions fail — because IC'_{LM} is binding — and yet solutions to (P) remain deterministic (Proposition 1 in the next section). This is one insight we can obtain with our approach of including IC'_{LM} in (MRP).

It can be easily verified that any solution to (MRP) satisfies IR_H and IR_M. A solution to (MRP) then solves the seller's problem if it also satisfies the dropped period one IC constraints IC_{HL}, IC_{MH} and IC_{LH}. Using the expressions of $u_\theta(\underline{\omega})$, $\theta = L, M, H$, from binding IR_L, IC_{ML} and IC_{HM}, we find that IC_{HL}, IC_{MH} and IC_{LH} are equivalent to, respectively,

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_M(\omega) - x_L(\omega)\right)\left(F_M(\omega) - F_H(\omega)\right) d\omega \geq 0, \tag{IC'_{HL}}$$

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_H(\omega) - x_M(\omega)\right)\left(F_M(\omega) - F_H(\omega)\right) d\omega \geq 0, \tag{IC'_{MH}}$$

$$\int_{\underline{\omega}}^{\overline{\omega}} (x_H(\omega) - x_L(\omega))(F_L(\omega) - F_M(\omega))d\omega + \int_{\underline{\omega}}^{\overline{\omega}} (x_H(\omega) - x_M(\omega))(F_M(\omega) - F_H(\omega))d\omega \geq 0. \tag{IC'_{LH}}$$

For future reference, we summarize the above observation in the following lemma. In the remainder of the paper, by stating that a solution (x_L, x_M) to (MRP) "corresponds to" a solution to (P), we mean that (x_L, x_M) together with $x_H(\omega) = \mathbb{1}_{\omega \geq c}$ and t_θ, $\theta = H, M, L$, derived from binding IC_{ML} and IC_{HM}, solve (P).

Lemma 1 *Any solution to (MRP) that satisfies conditions IC'_{HL}, IC'_{MH} and IC'_{LH} corresponds to a solution to (P).*

From now on, we will focus our analysis on (MRP). By using the information we garner from the solutions to (MRP), we will be able to provide conditions to ensure that a solution to (MRP) satisfies IC'_{HL}, IC'_{MH}, and IC'_{LH}, and therefore corresponds to a solution to (P).

2.2 An illustrating example

Suppose $c = 1$, and the conditional distributions are, for $\omega \in [0, \infty)$,

$$F_L(\omega) = 1 - e^{-\omega},\ F_H(\omega) = 1 - e^{-0.7\omega},\ F_M(\omega) = 1 - 0.4e^{-\omega} - 0.6e^{-0.7\omega}.$$

It is easy to verify that both $\delta_L(\omega)$ and $\delta_M(\omega)$ cross 0 from below only once. Denote the crossing point as $\hat{k}_L$ and $\hat{k}_M$ respectively. Let $\phi_L = 0.4$; we have $\hat{k}_L \approx 1.52$. There are three cases, depending on the value of ϕ_H.

(i) For $\phi_H = 0.4$, we have $\hat{k}_M \approx 1.40 < \hat{k}_L$. This is a regular case in the existing literature. Deterministic allocations $x^*_\theta(\omega) = \mathbb{1}_{\omega \geq \hat{k}_\theta}$, $\theta = M, L$, solve the standard relaxed problem, and corresponds to a solution to (P).

(ii) For $\phi_H = 0.5$, we have $\hat{k}_M \approx 3.09 > \hat{k}_L$. The existing approach of point-wise maximization fails, because these deterministic allocations violate the upward period one incentive compatibility constraint, IC_{LM}, or equivalently, IC'_{LM}. The deterministic solution to (MRP) forces a common threshold $\hat{k} \approx 1.76$ between $\hat{k}_L$ and $\hat{k}_M$ on the allocations of types M and L. Proposition 1 shows this deterministic solution, $x^*_\theta(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$, $\theta = M, L$, is in fact optimal.

(iii) For $\phi_H = 0.55$, we have we have $\hat{k}_M \approx 10.68 > \hat{k}_L$. Again, point-wise maximization fails. The deterministic solution to (MRP) has a common threshold of $\hat{k} \approx 5.20$, between $\hat{k}_L$ and $\hat{k}_M$. Proposition 2 constructs stochastic allocations that improve upon this deterministic solution. We keep type M's allocation at $x_M(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$, and choose an interval $[a, b] \ni \hat{k}$ for type L so that type L's allocation $x_L(\omega)$ remains 0 for $\omega < a$ and 1 for $\omega > b$, but is changed to a constant $\chi_L \in (0, 1)$ for $\omega \in [a, b]$, with the constant level χ_L binding IC'_{LM}:

$$\chi_L \int_a^{\hat{k}} (F_L(\omega) - F_M(\omega))d\omega = (1 - \chi_L) \int_{\hat{k}}^b (F_L(\omega) - F_M(\omega))d\omega.$$

The change to (MRP) is then

$$\chi_L \int_a^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega - (1 - \chi_L) \int_{\hat{k}}^b \phi_L \delta_L(\omega) f_L(\omega) d\omega.$$

For $a = 2.08$, and $b = 10$, the change is is positive. Thus, the stochastic solution (x_M, x_L) improves upon the deterministic solution with the common threshold $\hat{k}$ in (MRP). Proposition 2 makes assumptions on $\{F_\theta(\omega)\}_{\theta=H,M,L}$ so that (x_M, x_L) is feasible in the original problem, implying that randomization is optimal. Proposition 3 further shows that for these parameter values, in a solution to (P) type M's allocation $x^*_M(\omega)$ is deterministic, with threshold $k_M \approx 2.25$, while type L's allocation $x^*_L(\omega)$ is stochastic, with support $[a^*_L, b^*_L]$ where $a^*_L \approx 2.08$ and $b^*_L = \infty$.

3 Deterministic versus Stochastic Mechanisms

In this section, we characterize when a stochastic mechanism, as opposed to a deterministic one, is optimal in sequential screening. We use (MRP) defined in Section 2.1 to characterize the "optimal deterministic mechanism," which is profit-maximizing among mechanisms with deterministic allocation rules. If randomization can strictly improve the seller's profit upon the optimal deterministic mechanism, then any solution to (P) is stochastic; otherwise solutions to (P) are deterministic.

3.1 Optimality of deterministic mechanisms

A deterministic mechanism is given by an allocation rule x_θ and transfer rule t_θ, $\theta = H, M, L$, such that there is a threshold k_θ for each θ with $x_\theta(\omega) = \mathbb{1}_{\omega \geq k_\theta}$. We say that $x_\theta(\omega) = \mathbb{1}_{\omega \geq k_\theta^*}$, $\theta = M, L$, is a "deterministic solution" to (MRP), if k_M^* and k_L^* maximize

$$S_M(k_M) + S_L(k_L)$$

subject to $k_M \leq k_L$, where

$$S_\theta(k) \equiv \int_k^{\overline{\omega}} \phi_\theta \delta_\theta(\omega) f_\theta(\omega) d\omega$$

for each $\theta = M, L$. An optimal deterministic mechanism is a deterministic solution (x_θ, t_θ), $\theta = H, M, L$, to (P). The following result is straightforward and the proof is omitted.

Lemma 2 *Any deterministic solution to (MRP) corresponds to an optimal deterministic mechanism.*

We assume throughout that, for each $\theta = M, L$, there is a unique maximizer $\hat{k}_\theta$ of $S_\theta(k)$. If it is interior, $\hat{k}_\theta$ satisfies the first order necessary condition of $\delta_\theta(\hat{k}_\theta) = 0$. If $\hat{k}_M \leq \hat{k}_L$, then the constraint $k_M \leq k_L$ is not binding, and $\hat{k}_M$ and $\hat{k}_L$ are the deterministic solution to (MRP), and by Lemma 2, correspond to an optimal deterministic mechanism. In fact, this is the regular case in the existing literature, and $x_\theta(\omega) = \mathbb{1}_{\omega \geq \hat{k}_\theta}$, $\theta = M, L$, is optimal overall. A necessary condition for a stochastic mechanism to be optimal is thus $\hat{k}_M > \hat{k}_L$. When $\hat{k}_M > \hat{k}_L$, the constraint $k_M \leq k_L$ binds at any deterministic solution to (MRP), and the solution is given by $k_M = k_L = \hat{k}$ for some $\hat{k} \in [\hat{k}_L, \hat{k}_M]$.[6] When $\hat{k}$ is interior, it satisfies the first order necessary condition

$$\phi_L \delta_L(\hat{k}) f_L(\hat{k}) + \phi_M \delta_M(\hat{k}) f_M(\hat{k}) = 0. \tag{FOC}$$

As illustrated in Section 2.2, solutions to (P) may still be deterministic when $\hat{k}_M > \hat{k}_L$. Intuitively, when the solution to (MRP) without IC'_{LM} violates IC'_{LM}, the deterministic mechanism with common threshold $\hat{k}$ may be optimal because it may be better to bring the two thresholds together instead of introducing randomization for one or both types.

To formally characterize conditions under which the deterministic mechanism with common threshold $\hat{k}$ is optimal, we introduce the average "surplus-to-slack" ratio for type $\theta = M, L$ over any interval $[a, b] \subseteq [\underline{\omega}, \overline{\omega}]$ as

$$R_\theta(a, b) = \frac{\int_a^b \phi_\theta \delta_\theta(\omega) f_\theta(\omega) d\omega}{\int_a^b (F_L(\omega) - F_M(\omega)) d\omega}.$$

The numerator of $R_\theta(a, b)$ is the total dynamic virtual surplus generated from type θ by setting $x_\theta(\omega) = 1$ for $\omega \in [a, b]$, which can be positive or negative. The denominator of $R_\theta(a, b)$ has two different interpretations depending on θ.[7] For R_L, it is the total incentive cost of setting $x_L(\omega) = 1$ for $\omega \in [a, b]$, which arises because this allocation to type L makes it harder to satisfy IC'_{LM}. For R_M, the denominator represents the total incentive benefit of setting $x_M(\omega) = 1$ for all $\omega \in [a, b]$, which arises because this allocation to type M makes it easier to satisfy IC'_{LM}. In either case, the denominator is always positive and corresponds to the change in the slack in IC'_{LM}. Since IC'_{LM} is equivalent to IC_{LM} given that IC_{ML} binds in (MRP), the denominator of R_θ, and hence the ratio itself, reflects the fact in a dynamic mechanism design problem, local downward constraints are necessary but not sufficient for

[6] If $\hat{k} > \hat{k}_M$, the value of the objective function could be increased by lowering the threshold for type M from $\hat{k}$ to $\hat{k}_M$ without violating IC'_{LM}; if $\hat{k} < \hat{k}_L$, the value of the objective function could be increased by raising the threshold for type L from $\hat{k}$ to $\hat{k}_L$ without violating IC'_{LM}. In either case we have a contradiction to the optimality of $\hat{k}$.

[7] When there are more than three ex ante types, or when we take a global approach that incorporates IC'_{HL} as well as IC'_{LM}, the denominator of the relevant surplus-to-slack ratio, or indeed whether any such ratio is useful at all, depends on which IC constraints are binding. See Section 5 and 6 respectively for more detailed discussion.

incentive compatibility.[8]

The point surplus-to-slack ratio at any $\omega \in [a, b]$ is defined as

$$r_\theta(\omega) = \frac{\phi_\theta \delta_\theta(\omega) f_\theta(\omega)}{F_L(\omega) - F_M(\omega)}.$$

The point ratio at ω is the common limit of the average ratio $R_\theta(a, \omega)$ from the left and the average ratio $R_\theta(\omega, b)$ from the right:

$$r_\theta(\omega) = \lim_{a \uparrow \omega} R_\theta(a, \omega) = \lim_{b \downarrow \omega} R_\theta(\omega, b).$$

In reverse, we can write $R_\theta(a, b)$ as a weighted average of $r_\theta(\omega)$ over $\omega \in [a, b]$

$$R_\theta(a, b) = \frac{\int_a^b r_\theta(\omega)(F_L(\omega) - F_M(\omega))dw}{\int_a^b (F_L(\omega) - F_M(\omega))d\omega}.$$

Now we use the surplus-to-slack ratio to state our first result: when $\hat{k}_M > \hat{k}_L$, if for both types $\theta = M, L$,

$$\max_{a \leq \hat{k}} R_\theta(a, \hat{k}) \leq \min_{b \geq \hat{k}} R_\theta(\hat{k}, b), \tag{DET$_\theta$}$$

then the deterministic mechanism with common threshold $\hat{k}$ is optimal.[9] We establish the claim by the method of Lagrangian relaxation. Let $\lambda \geq 0$ be the multiplier associated with IC'_{LM} in (MRP), and write the Lagrangian as

$$\begin{aligned} \mathcal{L}(x_M, x_L; \lambda) = &\int_{\underline{\omega}}^{\overline{\omega}} x_M(\omega) \left(\phi_M f_M(\omega) \delta_M(\omega) + \lambda(F_L(\omega) - F_M(\omega))\right) d\omega \\ &+ \int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega) \left(\phi_L f_L(\omega) \delta_L(\omega) - \lambda(F_L(\omega) - F_M(\omega))\right) d\omega. \end{aligned}$$

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[8]Thus, our surplus-to-slack ratio has no counterparts in a static mechanism design problem where an allocation to a type is a real number rather than a function. It is also conceptually different from the profit-to-rent ratio in Bergemann, Castro, and Weintraub (2020). In their two-type model of sequential screening with ex post individual rationality constraints, the question is whether a stochastic allocation schedule for the low type is more profitable than charging the optimal price ex post. There is a single downward incentive compatibility constraint. The denominator in their ratio is the information rent to the high type, while the numerator is the static virtual surplus.

[9]Since $\max_{\omega \leq \hat{k}} R_\theta(\omega, \hat{k}) \geq r_\theta(\hat{k}) \geq \min_{\omega \geq \hat{k}} R_\theta(\hat{k}, \omega)$, condition (DET$_\theta$) holds for type θ if and only if $\max_{\omega \leq \hat{k}} R_\theta(\omega, \hat{k}) = r_\theta(\hat{k}) = \min_{\omega \geq \hat{k}} R_\theta(\hat{k}, \omega)$. This explains where our guess for the multiplier $\hat{\lambda}$ in the proof of Proposition 1 comes from.

We choose a particular non-negative value $\hat{\lambda}$ for λ, and show that $\mathcal{L}(x_M, x_L; \hat{\lambda})$ is maximized by $x^*_\theta(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$ for each $\theta = M, L$, among all weakly increasing functions $x_\theta(\omega)$.[10] Since $\hat{\lambda} \geq 0$, the maximal value $\mathcal{L}(x^*_M, x^*_L; \hat{\lambda})$ is an upper bound of the objective function of (MRP) for any (x_M, x_L) that satisfies IC'_{LM}, and since the maximizers (x^*_M, x^*_L) bind IC'_{LM}, $\mathcal{L}(x^*_M, x^*_L; \hat{\lambda})$ is just the value of the objective function evaluated at (x^*_M, x^*_L). Therefore, the deterministic solution given by $\hat{k}$ solves (MRP). Being deterministic, it satisfies all dropped constraints IC'_{HL}, IC'_{MH} and IC'_{LH}. By Lemma 1, it corresponds to a solution to (P).

Proposition 1 *Suppose $\hat{k}_M > \hat{k}_L$ and $\hat{k}$ is interior. If condition (DET$_\theta$) holds for both types $\theta = M, L$, then the deterministic mechanism with common threshold $\hat{k}$ is optimal.*

Proof. Define $\hat{\lambda} = r_L(\hat{k})$. We first prove by contradiction that $\hat{\lambda} \geq 0$. Since $\hat{k}$ satisfies (FOC), if $\hat{\lambda} < 0$, then we have $r_M(\hat{k}) > 0 > r_L(\hat{k})$. By continuity, there exists $w' < \hat{k}$ such that $r_M(\omega) > 0$ for all $\omega \in [w', \hat{k}]$, and there exists $w'' > \hat{k}$ such that $r_L(\omega) < 0$ for all $\omega \in [\hat{k}, w''] > 0$. The value of the objective function of (MRP) can be improved by changing the threshold for type M from $\hat{k}$ to w' and the threshold for type L from $\hat{k}$ to w''. Such changes satisfy IC'_{LM}, contradicting the optimality of $\hat{k}$ as the deterministic solution.

Consider type L part of $\mathcal{L}(x_M, x_L; \hat{\lambda})$. By Riley and Zeckhauser (1983), it has a deterministic maximizer in a weakly increasing function $x_L(\omega)$ with values in $[0, 1]$. Thus, we only need to show that

$$\int_{w'}^{\overline{\omega}} \left(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\right) d\omega \leq \int_{\hat{k}}^{\overline{\omega}} \left(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\right) d\omega$$

for all $w' \in [\underline{\omega}, \overline{\omega}]$. The above is the same as

$$R_L(a, \hat{k}) \leq \hat{\lambda} \leq R_L(\hat{k}, b)$$

for all $a \leq \hat{k}$ and $b \geq \hat{k}$, which is exactly condition (DET$_L$).

[10] Riley and Zeckhauser (1983) study a monopoly pricing problem, and show that there is always a deterministic solution as an optimal mechanism. See also Myerson (1981) for the same conclusion in an optimal auction problem when there is a single bidder. These conclusions are a special case of a general result that there is always a deterministic solution in maximizing a linear functional of a weakly increasing function. This result is used in a similar way by Bergemann, Castro, and Weintraub (2020) in their analysis of randomization in sequential screening with ex post individual rationality constraints.

From the first order necessary condition for $\hat{k}$, we have $\hat{\lambda} = -r_M(\hat{k})$. A symmetric argument establishes that the type M part of $\mathcal{L}(x_M, x_L; \hat{\lambda})$ is maximized by $x^*_M(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$ among all weakly increasing functions $x_M(\omega)$. The proposition then follows Lagrangian relaxation and Lemma 1. ■

Proposition 1 provides a sufficient condition for deterministic mechanisms to remain optimal when the regularity condition in the literature fails, that is, when $\hat{k}_M > \hat{k}_L$. This condition is derived from pairwise comparisons of average ratios of dynamic virtual surplus to information rent associated with IC'_{LM}. Each pair of ratios are evaluated at an interval below and an interval above the common threshold $\hat{k}$ of types M and L when the optimal deterministic mechanism binds IC'_{LM}. In particular, solutions to (P) remain deterministic and is given by $\hat{k}$ if for both types the average ratio below $\hat{k}$ is always lower than the point ratio at $\hat{k}$ which in turn always exceeds the average ratio above $\hat{k}$.

3.2 Optimality of stochastic mechanisms

Now we establish sufficient conditions for solutions to (P) to be stochastic. Proposition 1 implies that a necessary condition for randomization is $\hat{k}_M > \hat{k}_L$ and a failure of condition (DET_θ) for either type $\theta = M, L$. It turns out that this necessary condition is also sufficient for randomization under mild assumptions on the distributions.

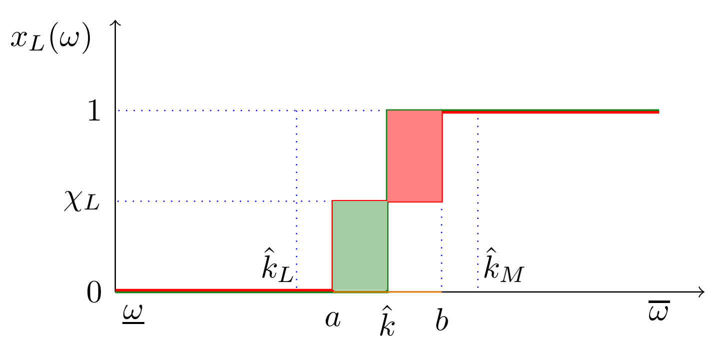

Figure 1

We first show that, if condition (DET_θ) fails for type θ, we can perturb the allocation rule for type θ around $\hat{k}$ to form a stochastic one that does strictly better than the optimal deterministic mechanism given by $\hat{k}$ in (MRP). This is illustrated in Figure 1. It can be understood as constructing a particular class of perturbations to the deterministic solution to (MRP) represented by the common threshold $\hat{k}$ for types M and L. We pick an interval $[a, b]$ containing $\hat{k}$ and replace $x_L(\omega)$, for all $\omega \in [a, b]$, by $\chi_L \in (0, 1)$ which binds IC'_{LM}. The profitability of any such perturbation over the deterministic solution $\hat{k}$, represented by the reversal of condition (DET_L), is then sufficient for randomization to be optimal.

Next, we provide sufficient conditions for the stochastic allocation resulted from the above perturbation to satisfy the dropped IC constraints IC'_{HL}, IC'_{MH}, and IC'_{LH}, and hence be feasible in the seller's original problem. These conditions are on the distribution functions $\{F_\theta\}_{\theta=H,M,L}$. By first order stochastic dominance, there is a unique "weighting" function $\tau(\omega)$ for F_M that maps $[\underline{\omega}, \overline{\omega}]$ to $[0, 1]$ such that

$$F_M(\omega) = \tau(\omega)F_L(\omega) + (1 - \tau(\omega))F_H(\omega)$$

We assume that τ is either non-decreasing or non-increasing, depending on whether (DET_θ) fails for type L or type H. The following lemma demonstrates how we use the monotonicity condition. The proof is straightforward and omitted.

Lemma 3 *If τ is non-decreasing (non-increasing), then for all $a < k < b$,*

$$\frac{\int_a^k (F_M(\omega) - F_H(\omega))d\omega}{\int_a^k (F_L(\omega) - F_M(\omega))d\omega} \leq (\geq) \frac{\int_k^b (F_M(\omega) - F_H(\omega))d\omega}{\int_k^b (F_L(\omega) - F_M(\omega))d\omega}.$$

Now we are ready to present sufficient conditions for randomization to be part of a solution to (P).

Proposition 2 *Suppose $\hat{k}_M > \hat{k}_L$ and $\hat{k}$ is interior. (i) If condition (DET_L) fails, then any solution to (MRP) is stochastic, and if further $\tau(\omega)$ is non-decreasing, any solution to (P) is stochastic. (ii) If condition (DET_M) fails, then any solution to (MRP) is stochastic, and if further $\tau(\omega)$ is non-increasing, any solution to (P) is stochastic.*

Proof. (i) Since $\hat{k}_M > \hat{k}_L$, the deterministic solution to (MRP) is $x_M(\omega) = x_L(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$. By Lemma 2, this deterministic solution is an optimal deterministic mechanism.

Since $R_L(\omega, \hat{k})$ and $R_L(\hat{k}, \omega)$ are continuous in ω, the maximum and the minimum in condition (DET_L) are attained. Let w' and w'' attain the maximum and the minimum respectively. Then, $w' \leq \hat{k} \leq w''$, with at least one strict inequality. By continuity of $R_L(\omega, \hat{k})$ and $R_L(\hat{k}, \omega)$ in ω, there exist an interval $[a, b] \ni \hat{k}$ such that $R_L(a, \hat{k}) > R_L(\hat{k}, b)$. Now, starting with the deterministic allocation $x_M(\omega) = x_L(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$, we keep $x_M(\omega)$ for type M but change allocation for type L to $\hat{x}_L(\omega)$ as

$$\hat{x}_L(\omega) = \begin{cases} \chi_L & \text{if} \quad \omega \in [a, b] \\ x_L(\omega) & \text{if} \quad \omega \notin [a, b] \end{cases}$$

where

$$\chi_L \equiv \frac{\int_{\hat{k}}^{b} (F_L(\omega) - F_M(\omega)) d\omega}{\int_{a}^{b} (F_L(\omega) - F_M(\omega)) d\omega}.$$

is chosen to bind IC'_{LM}. Since $a < \hat{k} < b$, we have $\chi_L \in (0, 1)$. The change in the value of the objective function in (MRP) is

$$\chi_L \int_a^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega - (1 - \chi_L) \int_{\hat{k}}^{b} \phi_L \delta_L(\omega) f_L(\omega) d\omega.$$

With the expression of χ_L, the above has the same sign as $R_L(a, \hat{k}) - R_L(\hat{k}, b)$, which is positive. Thus, $(x_M, \hat{x}_L)$ is a stochastic allocation that gives a greater value for the objective function of (MRP) than the deterministic solution $x_M(\omega) = x_L(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$. It follows that any solution to (MRP) is stochastic.

Given that $x_H(\omega) = \mathbb{1}_{\omega \geq c}$ and $x_M(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$ with $\hat{k} > \hat{k}_L > c$, $(\hat{x}_L, x_M)$ satisfies IC'_{MH}. Further, since $R_L(w', \hat{k}) \geq r_L(\hat{k}) > 0$, we can always choose a such that $a > c$, implying that $(\hat{x}_L, x_M)$ also satisfies IC'_{LH}. Finally, for IC'_{HL}, since $(\hat{x}_L, x_M)$ binds IC'_{LM}, we have

$$\chi_L \int_a^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega = (1 - \chi_L) \int_{\hat{k}}^{b} (F_L(\omega) - F_M(\omega)) d\omega.$$

By Lemma 3, the above implies IC'_{HL}. Thus, $(x_M, \hat{x}_L)$ corresponds to a feasible mechanism in (P). Since it a generates a strictly higher revenue than the optimal deterministic mechanism corresponding to (x_M, x_L), we conclude that solutions to (P) must be stochastic.

(ii) The proof to (ii) is symmetric and is omitted. ■

Combining Proposition 1 and Proposition 2, we have established the necessary and sufficient conditions for randomization to be optimal. Given that $\hat{k}_M > \hat{k}_L$ so that the standard approach of point-wise maximization fails, (DET$_\theta$) for $\theta = M, L$ prove to be the critical conditions that determine whether a solution to (P) involves stochastic allocations.

In proving the sufficiency of the failure of condition (DET$_\theta$) for randomization, we use a particular perturbation of the optimal deterministic mechanism by introducing randomization in the allocation for one type over an internal $[a, b] \subseteq [\underline{\omega}, \overline{\omega}]$. These particular perturbations are by themselves generally suboptimal. However, in next section we will show that other members of the simple class of stochastic mechanisms that the perturbations belong to are in fact optimal.

4 Optimal Randomization

In this section, we will use (MRP) to characterize randomization in the solution to (P). The analysis adapts ironing techniques used in standard mechanism design problems (e.g., Fudenberg and Tirole (1991)). As we aim for a characterization of solutions to (P) under additional assumptions on the design problem, which may or may not involve randomization, we will not directly connect these assumptions to the necessary and sufficient conditions for randomization in Propositions 1 and 2 until after we present the main result in this section.

We first show that there is always a solution to (MRP) with at most one level of stochastic allocation for types M and L. That is, for each type $\theta = M, L$, if $x_\theta(w), x_\theta(w') \in (0, 1)$ then $x_\theta(w) = x_\theta(w')$. This is because both the objective function and the constraint IC'_{LM} are linear functionals of non-decreasing schedules x_θ. Similarly, there is always a solution to (MRP) where randomization occurs only for one of the two types M and L, because there is a single constraint IC'_{LM} for two non-decreasing allocation functions x_M and x_L. The proof is a standard application of Theorem 1 of Luenberger (1969) (p. 217) to (MRP), which states that, if $(x^*_L(\omega), x^*_M(\omega))$ solves for modified relaxed problem, then there exists a multiplier $\lambda \geq 0$ for IC'_{LM} with complementary slackness, such that for each $\theta = M, L$, $x^*_\theta(\omega)$ maximizes $\mathcal{L}(x_M, x_L; \lambda)$ among all weakly increasing $x_\theta(\omega)$ with values in $[0, 1]$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$.[11]

[11] To apply the Luenberger Theorem, we need to show that the feasible set in (MRP) contains some

Lemma 4 *There is a solution* $(x_L^*(\omega), x_M^*(\omega))$ *to (MRP) such that: (i) for each* $\theta = M, L$*,*

$$x_\theta^*(\omega) = \begin{cases} 0 & if \quad \omega \in [\underline{\omega}, a) \\ \chi & if \quad \omega \in [a, b) \\ 1 & if \quad \omega \in [b, \overline{\omega}] \end{cases}$$

for some $\underline{\omega} \leq a \leq b \leq \overline{\omega}$ *and* $\chi \in (0, 1)$*; (ii) for* $\theta = L$ *or* $\theta = M$*, or both,* $x_\theta^*(\omega) = 0$ *or 1 for all* $\omega \in [\underline{\omega}, \overline{\omega}]$*.*

Using Lemma 4, and slightly abusing notation, we denote the allocation $x_\theta^*(\omega)$ in a solution to (MRP) as $\chi_\theta^{[a,b]}$. This notation includes deterministic allocations for type θ as special cases, with $a = b$, or $\chi_\theta = 0, 1$. Lemma 4 implies that, if randomization occurs in a solution to (MRP), then there is always a solution (x_L^*, x_M^*) that randomization occurs for only one type and for only one non-degenerate interval $[a, b]$.

4.1 A characterization of optimal mechanisms

We start by characterizing possible candidates of solution (x_L^*, x_M^*) to (MRP). Each x_θ^*, $\theta = M, L$, is considered separately, allowing deterministic allocations $x_\theta^*(\omega) = \mathbb{1}_{\omega \geq k}$ for some $k \in (\underline{\omega}, \overline{\omega})$ and stochastic allocations $\chi^{[a,b]}$ for some $[a, b] \subseteq [\underline{\omega}, \overline{\omega}]$ and $\chi \in (0, 1)$. The possible candidates x_θ^* depend on the shape of the point surplus-to-slack ratio r_θ.

Suppose that $x_\theta^*(\omega) = \chi^{[a,b]}$ with some $[a, b] \subseteq [\underline{\omega}, \overline{\omega}]$ and $\chi \in (0, 1)$ is part of a solution to (MRP). By the Luenberger Theorem, since we can increase or decrease χ to affect the Lagrangian $\mathcal{L}(x_M^*, x_L^*; \lambda)$ without violating the monotonicity constraint on x_θ^*, we have, depending on $\theta = L$ or $\theta = M$,

$$R_L(a, b) = \lambda \text{ or } R_M(a, b) = -\lambda.$$

Next, we can always increase a to affect the Lagrangian, and if $a > \underline{\omega}$, we can also decrease

(x_M, x_L) that satisfies IC'_{LM} strictly. This is clearly true.

it. Similarly, we can always decrease b, and if $b < \overline{\omega}$, we can also increase it. As a result,

$$r_L(a) \geq \lambda \geq r_L(b) \quad \text{or} \quad r_M(a) \geq -\lambda \geq r_M(b),$$

with $r_\theta(a) = R_\theta(a,b)$ if $a > \underline{\omega}$ and $r_\theta(b) = R_\theta(a,b)$ if $b < \overline{\omega}$.

The above conditions for $x^*_\theta(\omega) = \chi^{[a,b]}$ to be part of a solution to (MRP) cannot be satisfied if $r_\theta(\omega)$ is strictly increasing in $[a,b]$. Indeed, if $r_\theta(\omega)$ is strictly increasing for all $\omega \in [\underline{\omega}, \overline{\omega}]$, then $x^*_\theta(\omega)$ is deterministic in any solution (x^*_L, x^*_M) to (MRP). For simplicity we impose a restriction on the shape of r_θ that will be shown to imply a unique candidate for the interval $[a,b]$ at which $x^*_\theta(\omega)$ is strictly between 0 and 1 as part of solution (x^*_L, x^*_M) to (MRP). We assume that r_θ is either strictly increasing in the entire support, or "single dipped," in that it has a "peak" at ω^p_θ and a "trough" at ω^t_θ satisfying $\underline{\omega} \leq \omega^p_\theta < \omega^t_\theta \leq \overline{\omega}$, such that r_θ is strictly increasing in $[\underline{\omega}, \omega^p_\theta]$ and in $[\omega^t_\theta, \overline{\omega}]$, and strictly decreasing in $(\omega^p_\theta, \omega^t_\theta)$. See Figure 2 for an illustration where both ω^p_θ and ω^t_θ are interior. For any $r \in [r_\theta(\omega^t_\theta), r_\theta(\omega^p_\theta)]$, let $\alpha_\theta(r) \in [\underline{\omega}, \omega^p_\theta]$ such that $r_\theta(\alpha_\theta(r)) = r$, and set $\alpha_\theta(r) = \underline{\omega}$ if $r_\theta(\underline{\omega}) > r$. This is the inverse of r_θ on the strictly increasing interval $[\underline{\omega}, \omega^p_\theta]$. Symmetrically, let $\beta_\theta(r) \in [\omega^t_\theta, \overline{\omega}]$ be the inverse of r_θ on the strictly increasing interval $[\omega^t_\theta, \overline{\omega}]$, satisfying $r_\theta(\beta_\theta(r)) = r$ and $\beta_\theta(r) = \overline{\omega}$ if $r_\theta(\overline{\omega}) < r$. By definition, $\alpha_\theta(\omega^p_\theta) = \omega^p_\theta$, and $\beta_\theta(\omega^t_\theta) = \omega^t_\theta$. The following result is straightforward and the proof is omitted.

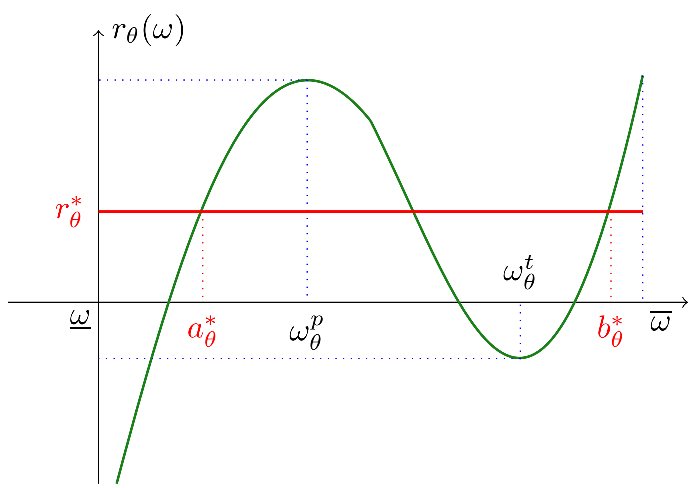

Figure 2

Lemma 5 *Suppose that $r_\theta(\omega)$ is single dipped. The difference $R_\theta(\alpha_\theta(r), \beta_\theta(r)) - r$ crosses 0 once and from above on the interval $r \in [r_\theta(\omega_\theta^t), r_\theta(\omega_\theta^p)]$.*

By Lemma 5, there is a unique $r_\theta^* \in (r_\theta(\omega_\theta^t), r_\theta(\omega_\theta^p))$, together with $a_\theta^* \equiv \alpha_\theta(r_\theta^*) \in [\underline{\omega}, \omega_\theta^p]$ and $b_\theta^* \equiv \beta_\theta(r_\theta^*) \in [\omega_\theta^t, \overline{\omega}]$, such that

$$r_\theta(a_\theta^*) \geq R_\theta(a_\theta^*, b_\theta^*) = r_\theta^* \geq r_\theta(b_\theta^*),$$

with $r_\theta(a_\theta^*) = r_\theta^*$ if $a_\theta^* > \underline{\omega}$, and $r_\theta(b_\theta^*) = r_\theta^*$ if $b_\theta^* < \overline{\omega}$. See Figure 2. It follows from the argument at the beginning of this subsection that if part of a solution to (MRP) is $x_\theta^*(\omega) = \chi^{[a,b]}$, then $a = a_\theta^*$ and $b = b_\theta^*$.

Conversely, if part of a solution to (MRP) is $x_\theta^*(\omega) = \mathbb{1}_{\omega \geq k}$ for some interior k, we argue by contradiction that $k \notin (a_\theta^*, b_\theta^*)$. By the Luenberger Theorem, since we can increase or decrease k to affect the Lagrangian $\mathcal{L}(x_M^*, x_L^*; \lambda)$ without violating the monotonicity constraint on x_θ^*,

$$r_L(k) = \lambda \;\text{ or }\; r_M(k) = -\lambda.$$

Given this, for any $k \in (a_\theta^*, b_\theta^*)$, we can always find $a < k$ with $R_L(a, k) > r_L(k)$ or $b > k$ with $R_L(k, b) < r_L(b)$, so that changing x_θ^* from $\mathbb{1}_{\omega \geq k}$ to either $\mathbb{1}_{\omega \geq a}$ or to $\mathbb{1}_{\omega \geq b}$ increases the Lagrangian, contradicting the Luenberger Theorem. The existence of such a and b is immediate for $k \in (\omega_\theta^p, \omega_\theta^t)$, as $r_\theta(\omega)$ is strictly decreasing in $(\omega_\theta^p, \omega_\theta^t)$. By Lemma 5, for $k \in (a_\theta^*, \omega_\theta^p]$, we can choose $b = \beta_\theta(r_\theta(k))$ as $r_\theta(k) > r_\theta^*$, and for $k \in [\omega_\theta^t, b_\theta^*)$, we can choose $a = \alpha_\theta(r_\theta(k))$ as $r_\theta(k) < r_\theta^*$.

Now we are ready to present our main characterization result on optimal stochastic mechanisms. We make two symmetric sets of assumptions on r_M and r_L in order to apply Lemma 4 and Lemma 5 to rule out randomization for one type and possibly rule in randomization for the other. When randomization occurs for type L, we denote the solution as $(x_L^*, x_M^*) = (\chi_L^{[a_L^*, b_L^*]}, \mathbb{1}_{\omega \geq k_M})$, where $[a_L^*, b_L^*]$ is the unique candidate interval implied by Lemma 5 for type L, and χ_L binds IC'_{LM}. The assumptions we make on r_M and r_L introduce cross-type restrictions that allow us to use Lagrangian relaxation and the Riley-Zeckhouser result in a similar way as in the proof of Proposition 1, and show that the candidate allocations lead to a solution to (MRP). Finally, the monotonicity restriction on the weighting function τ ensures that the solution satisfies the dropped IC constraints in (P) in the same way as in the proof of Proposition 2.

Proposition 3 *(i) Suppose that $r_M(\omega)$ is strictly increasing and $r_L(\omega)$ is single dipped. If $R_L(a_L^*, b_L^*) > 0$ and there exists some $k_M \in (a_L^*, b_L^*)$ such that $r_M(k_M) = -R_L(a_L^*, b_L^*)$, then $(x_L^*, x_M^*) = (\chi_L^{[a_L^*, b_L^*]}, \mathbb{1}_{\omega \geq k_M})$ solves (MRP); otherwise, any solution to (MRP) is deterministic. Further, if $\tau(\omega)$ is non-decreasing, any solution to (MRP) corresponds to a solution to (P). (ii) Suppose r_L is strictly increasing and r_M is single dipped. If $R_M(a_L^*, b_L^*) < 0$, and there is $k_L \in (a_M^*, b_M^*)$ such that $r_L(k_L) = -R_M(a_L^*, b_L^*)$, then $(x_L^*, x_M^*) = (\mathbb{1}_{\omega \geq k_L}, \chi_M^{[a_M^*, b_M^*]})$ solves (MRP); otherwise, any solution to (MRP) is deterministic. Further, if $\tau(\omega)$ is non-increasing, any solution to (MRP) corresponds to a solution to (P).*

Proof. (i) We will consider separately each type $\theta = M, L$ part of $\mathcal{L}(x_M, x_L; \hat{\lambda})$, with $\hat{\lambda} = -r_M(k_M)$. By Riley and Zeckhouser (1983), each type θ part has a deterministic maximizer among all weakly increasing $x_\theta(\omega)$ with the range in $[0, 1]$. It suffices to show that the proposed x_θ^* generates a greater value for type θ part of $\mathcal{L}(x_M, x_L; \hat{\lambda})$ than any $\mathbb{1}_{\omega \geq k}$.

For type M, for all $k \in [\underline{\omega}, \overline{\omega}]$, since $r_M(\omega)$ is strictly increasing, we have

$$\int_k^{\overline{\omega}} \left(\phi_M f_M(\omega)\delta_M(\omega) + \hat{\lambda}(F_L(\omega) - F_M(\omega))\right) d\omega \leq \int_{k_M}^{\overline{\omega}} \left(r_M(\omega) - r_M(k_M)\right) \left(F_L(\omega) - F_M(\omega)\right) d\omega.$$

The right-hand side of the above inequality is precisely type M part of $\mathcal{L}(x_M^*, x_L^*; \hat{\lambda})$.

For type L, consider first $k < a_L^*$. This is relevant only if $a_L^* > \underline{\omega}$. Since r_L is single dipped, $r_L(k) < r_L(a_L^*)$ for all $k < a_L^*$, and by Lemma 5, $r_L(a_L^*) = R_L(a_L^*, b_L^*)$. By assumption, $\hat{\lambda} = R_L(a_L^*, b_L^*)$. Then,

$$\int_k^{\overline{\omega}} \left(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\right) d\omega < \int_{a_L^*}^{\overline{\omega}} \left(r_L(\omega) - r_L(a_L^*)\right) \left(F_L(\omega) - F_M(\omega)\right) d\omega.$$

The right-hand side of the above inequality is precisely type L part of $\mathcal{L}(x_M^*, x_L^*; \hat{\lambda})$. A symmetric argument applies to all $k > b_L^*$. For the remaining case of $k \in [a_L^*, b_L^*]$, since $r_L(\omega)$ is single dipped, by Lemma 5, we have $R_L(a_L^*, b_L^*) = r_L^*$, and $r_L(k) - r_L^*$ crosses 0 from above exactly once as k goes from a_L^* to b_L^*. Given that $\hat{\lambda} = R_L(a_L^*, b_L^*)$,

$$\int_k^{\overline{\omega}} \left(\phi_L \delta_L(\omega) f_L(\omega) - \hat{\lambda}(F_L(\omega) - F_M(\omega))\right) d\omega = \int_k^{\overline{\omega}} \left(r_L(\omega) - r_L^*\right)\left(F_L(\omega) - F_M(\omega)\right) d\omega$$

is maximized by $k = a_L^*$ or by $k = b_L^*$, either of which yields type L part of $\mathcal{L}(x_M^*, x_L^*; \hat{\lambda})$.

Since $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$ $x_L^*(\omega) = \chi_L^{[a_L^*, b_L^*]}$ and maximize $\mathcal{L}(x_M, x_L; \hat{\lambda})$ among all weakly decreasing x_M and x_L, as in the proof of Proposition 1, (x_M^*, x_L^*) solves (MRP) by Lagrangian relaxation. As in the proof of Proposition 2, (x_M^*, x_L^*) satisfies IC'_{HL} and IC'_{MH}. By Lemma 3, it also satisfies IC'_{LH} because $\tau(\omega)$ is non-decreasing. By Lemma 1, it corresponds to a solution to (P).

Now suppose that there is a stochastic solution $(\tilde{x}_M, \tilde{x}_L)$ to (MRP). By Lemma 5, we have $\tilde{x}_M(\omega) = \mathbb{1}_{\omega \geq \tilde{k}_M}$ for some $\tilde{k}_M$, and $\tilde{x}_L(\omega) = \tilde{\chi}_L^{[a_L^*, b_L^*]}$ for some $\tilde{\chi}_L \in (0,1)$. By Luenberger's Theorem, there is a multiplier $\lambda \geq 0$ for IC'_{LM} with complementary slackness, such that among all non-decreasing functions with values on $[0,1]$, $(\tilde{x}_L(\omega), \tilde{x}_M(\omega))$ maximizes $\mathcal{L}(x_M, x_L; \lambda)$. This is impossible if $\lambda = 0$, as we would have $\tilde{x}_\theta(\omega) = \mathbb{1}_{\omega \geq k_\theta}$ for each $\theta = M, L$, which is deterministic instead of stochastic. Thus, $\lambda > 0$ and by complementary slackness, IC'_{LM} binds, implying that $a_L^* < \tilde{k}_M < b_L^*$. Since $\tilde{k}_M$ is interior to $[a_L^*, b_L^*]$, and hence can be increased or decreased while still maintaining the monotonicity constraint of $\tilde{x}_M(\omega)$, Luenberger's Theorem implies that $r_M(\tilde{k}_M) = -\lambda$. Similarly, since $\tilde{\chi}_L \in (0,1)$, and hence can be increased or decreased while still maintaining the monotonicity constraint of $\tilde{x}_L(\omega)$, Luenberger's Theorem implies that $R_L(a_L^*, b_L^*) = \lambda$. Thus, when the conditions stated in the proposition fail, the solution to the modified simplified problem is deterministic. Since a deterministic solution satisfies IC'_{HL}, IC'_{MH} and IC'_{LH}, the proposition immediately follows from Lemma 1.

(ii) The proof is symmetric to that of part (i) and is omitted. ■

Our main characterization result Proposition 3 can be extended to the case where both r_M and r_L are single dipped. By Lemma 4, there is always a solution to (MRP) with randomization for only one of the two types, but Lemma 5 implies that there are two candidate intervals $[a_L^*, b_L^*]$ and $[a_M^*, b_M^*]$ for stochastic allocation to take place. To establish that randomization for type L is optimal in (MRP), for example, we need to use Lemma 5 to rule in

$[a_L^*, b_L^*]$ as well as to rule out $[a_M^*, b_M^*]$. To go from a solution (x_L^*, x_M^*), whether stochastic or deterministic, to a solution to (P), we make the simplifying assumption that $\tau(\omega)$ is constant for all $\omega \in [\underline{\omega}, \overline{\omega}]$. The proof of the following corollary to Proposition 3 is straightforward and omitted.

Corollary 1 *Suppose that r_M and r_L are both single dipped. If $R_L(\underline{\omega}, \overline{\omega}) > 0$ and there is $k_M \in (a_L^*, b_L^*) \setminus (a_M^*, b_M^*)$ such that $r_M(k_M) = -R_L(\underline{\omega}, \overline{\omega})$, then $(x_L^*, x_M^*) = (\chi_L^{[a_L^*, b_L^*]}, \mathbb{1}_{\omega \geq k_M})$ solves (MRP); if $R_M(\underline{\omega}, \overline{\omega}) < 0$ and there exits some $k_L \in (a_M^*, b_M^*) \setminus (a_L^*, b_L^*)$ such that $r_L(k_L) = -R_M(\underline{\omega}, \overline{\omega})$, then $(x_L^*, x_M^*) = (\mathbb{1}_{\omega \geq k_L}, \chi_M^{[a_M^*, b_M^*]})$ solves (MRP); otherwise, any solution to (MRP) is deterministic. Further, if $\tau(\omega)$ is constant for all $\omega \in [\underline{\omega}, \overline{\omega}]$, any solution to (MRP) corresponds to a solution to (P).*

Under the assumptions on the shapes of r_L and r_M, we have provided necessary and sufficient conditions for stochastic solutions to (MRP), and hence to (P) under the additional monotonicity assumption on τ. We now argue that Proposition 3 is consistent with Proposition 1 and Proposition 2. In particular, if $R_L(a_L^*, b_L^*) > 0$, r_M is strictly increasing, and there is $k_M \in (a_L^*, b_L^*)$ such that $r_M(k_M) = -R_L(a_L^*, b_L^*)$, then $\hat{k}_M > \hat{k}_L$ and condition (DET$_L$) is violated for type L. For simplicity, we assume that $a_L^* > \underline{\omega}$ and $b_L^* < \overline{\omega}$.

To see that $\hat{k}_M > \hat{k}_L$, note that since $r_M(k_M) < 0$, r_M is strictly increasing and $k_M > a_L^*$, we have $\hat{k}_M > k_M > a_L^*$. We claim that $\hat{k}_L < a_L^*$. Write the surplus of type L as a function of the threshold as

$$S_L(k) = \int_k^{\overline{\omega}} r_L(\omega)(F_L(\omega) - F_M(\omega))d\omega.$$

Assume that there exist $a < a_L^*$ and $b \in [\omega_L^t, b_L^*)$ such that $r_L(a) = r_L(b) = 0$, where ω_L^t is the interior trough of r_L, so that a and b are the only two local maximizers of $S_L(k)$.[12] This means that $\alpha_L(0) = a$ and $\beta_L(0) = b$. Since $r_L^* = R_L(a_L^*, b_L^*) > 0$, Lemma 5 implies $R_L(a, b) > 0$. Then, $S_L(a) - S_L(b)$ is given by

$$\int_a^b r_L(\omega)(F_L(\omega) - F_M(\omega))d\omega = R_L(a, b) \int_a^b (F_L(\omega) - F_M(\omega))d\omega > 0.$$

[12]Since $r_L^*(\omega)$ is strictly decreasing on $[\omega_L^p, \omega_L^t]$, no local maximizer of S_L can be located on the interval. If $r_L(\omega_L^t) > 0$, then immediately we have the claim that $\hat{k}_L < a_L^*$.

It follows that $\hat{k}_L = a < a^*_L < \hat{k}_M$.

To see that condition (DET_L) fails, note that since $r_L(\omega)$ is increasing for any $\omega \le a^*_L$, $k_M > a^*_L$ with $r_M(k_M) = -r_L(a^*_L)$, and r_M is strictly increasing, we have

$$r_L(\omega) + r_M(\omega) < r_L(a^*_L) + r_M(a^*_L) = -r_M(k_M) + r_M(a^*_L) < 0, \ \forall \omega \le a^*_L$$
$$r_L(\omega) + r_M(\omega) > r_L(b^*_L) + r_M(b^*_L) = -r_M(k_M) + r_M(b^*_L) > 0, \ \forall \omega \ge b^*_L.$$

By (FOC), we have $\hat{k} \in (a^*_L, b^*_L)$. If $\hat{k} \in (\omega^p_L, \omega^t_L)$, then ($\text{DET}_L$) is clearly violated since $r_L(\omega)$ is strictly decreasing in $\omega \in (\omega^p_L, \omega^t_L)$. If $\hat{k} \in (a^*_L, \omega^p_L]$, then Lemma 5 implies that $r_L(\hat{k}) > R_L(\hat{k}, \beta_L(r_L(\hat{k})))$ because $r_L(\hat{k}) > r^*_L$. By continuity, (DET_L) fails. Symmetrically, (DET_L) fails for $\hat{k} \in (\omega^t_L, b^*_L]$.

4.2 Randomization for the low type

In order to apply Proposition 3 to analyze specific examples, it is often useful to provide sufficient conditions under which optimal randomization occurs only for type L. To do so, we make two additional assumptions. Specifically, we assume that (i) $\tau(\omega) = \tau$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$; and (ii) $f_H(\omega)/f_L(\omega)$ is strictly increasing in ω. We refer to the combination of conditions (i) and (ii) as "strong alignment."

By condition (i) we have

$$R_M(w, w') = \frac{\int_w^{w'} \phi_M(\omega - c) f_M(\omega) d\omega}{\int_w^{w'} (F_L(\omega) - F_M(\omega)) d\omega} - \frac{\tau \phi_H}{1 - \tau},$$
$$R_L(w, w') = \frac{\int_w^{w'} \phi_L(\omega - c) f_L(\omega) d\omega}{\int_w^{w'} (F_L(\omega) - F_M(\omega)) d\omega} - (\phi_M + \phi_H),$$

for any $w \le w'$. By condition (ii),

$$\frac{f_M(\omega)}{f_L(\omega)} < \frac{f_M(\hat{w})}{f_L(\hat{w})} < \frac{f_M(\omega')}{f_L(\omega')}$$

for any $\omega < \hat{w} < \omega'$. Thus, if $R_M(w, \hat{w}) > R_M(\hat{w}, w')$ for some $c < w < \hat{w} < w'$, then $R_L(w, \hat{w}) > R_L(\hat{w}, w')$. It follows that for sufficiently small c, whenever the sufficient condition for randomization is satisfied for type M, that is, condition (DET_M) fails, it is also satisfied for type L. The following result shows that, in fact, for $c \leq \underline{\omega}$, whenever a solution to (P) involves randomization for type M, it must also involve randomization for type L.

Proposition 4 *Suppose $c \leq \underline{\omega}$. Under strong alignment, if no deterministic mechanism is optimal, then it is optimal to randomize the allocation for type L only.*

Proof. Suppose that there is no deterministic mechanism that is optimal, but that there is a solution to (P) with randomization for type M only. By strong alignment, there is no deterministic solution to (MRP), and there is a solution (x_M^*, x_L^*) where x_M^* is random but x_L^* is deterministic. By part (i) of Lemma 4, we can assume that $x_M^*(\omega) = \chi_M^{[a,b]}$ and $x_L^*(\omega) = \mathbb{1}_{\omega \geq k_L}$. By Luenberger's Theorem, $\left(\chi_M^{[a,b]}, \mathbb{1}_{\omega \geq k_L}\right)$ maximizes $\mathcal{L}(x_M, x_L; \lambda)$ for some $\lambda \geq 0$ among all weakly increasing allocations for type M and type L.

First, we claim that $\lambda > 0$. Suppose instead $\lambda = 0$. Then, since $\chi_M \in (0,1)$, we have

$$\int_a^b \phi_M \delta_M(\omega) f_M(\omega) d\omega = 0.$$

It follows that replacing $x_M^*(\omega) = \chi_M^{[a,b]}$ with $\mathbb{1}_{\omega \geq a}$ does not change the value of the objective function in (MRP). Since the allocation for type M is weakly increased for all valuations, IC'_{LM} remains satisfied. This contradicts the optimality of (x_M^*, x_L^*), and establishes that $\lambda > 0$. By complementary slackness, IC'_{LM} binds, which implies that $k_L \in (a,b)$ and that χ_M is given by

$$\chi_M = \frac{\int_{k_L}^b (F_L(\omega) - F_M(\omega)) d\omega}{\int_a^b (F_L(\omega) - F_M(\omega)) d\omega}.$$

Next, we claim that $R_M(a, k_L) \geq R_M(k_L, b)$. Suppose not. Then by replacing $x_M^*(\omega)$ with $\mathbb{1}_{\omega \geq k_L}$, we continue to bind IC'_{LM}, and the total change in the objective function of (MRP) is given by

$$-\chi_M \int_a^{k_L} \phi_M \delta_M(\omega) f_M(\omega) d\omega + (1 - \chi_M) \int_{k_L}^b \phi_M \delta_M(\omega) f_M(\omega) d\omega,$$

which is strictly positive because $R_M(a, k_L) < R_M(k_L, b)$.

Since $c \leq \underline{\omega} \leq a$, under condition (ii) of strong alignment, $R_M(a, k_L) \geq R_M(k_L, b)$ implies $R_L(a, k_L) > R_L(k_L, b)$. Then, by replacing $x_L^*(\omega)$ with $\chi_M(a, b)$, we continue to bind IC'_{LM}, and the total change in the objective function of (MRP) is given by

$$\chi_M \int_a^{k_L} \phi_L \delta_L(\omega) f_L(\omega) d\omega - (1-\chi_M) \int_{k_L}^b \phi_L \delta_L(\omega) f_L(\omega) d\omega,$$

which is strictly positive because $R_L(a, k_L) > R_L(k_L, b)$. This contradicts the assumption that (x_M^*, x_L^*) is a solution to (MRP). ■

Proposition 4 does not rule out the possibility that there is a solution to (P) with randomization for both type M and type L. By part (ii) of Lemma 4, in this case there is another solution to (P) with randomization for at most one of the two types. Proposition 4 then implies that these other solutions necessarily involve randomization for type L only.

5 An Extension

We extend our analysis of stochastic sequential screening mechanisms to more than three ex ante types. Let $\Theta = \{1, \ldots, I\}$ with $I \geq 3$, be the ex ante type space, ranked by first order stochastic dominance, with type 1 being the lowest type, $f_i(\cdot)$ and $F_i(\cdot)$ being the conditional density and conditional distribution of valuations respectively, $i \in \Theta$. Let $\phi_i > 0$ being the fraction of type i, with $\sum_{i\in\Theta} \phi_i = 1$.

As in Section 2.1, we write the seller's problem as choosing a non-decreasing allocation rule $x_i(\omega)$, with values on $[0,1]$, and an integration constant $u_i(\underline{\omega})$ for each $i \in \Theta$, to solve

$$\max_{(x_i(\cdot), u_i(\underline{\omega}))} \sum_{i\in\Theta} \phi_i \left(\int_{\underline{\omega}}^{\overline{\omega}} x_i(\omega) \left((\omega - c) f_i(\omega) - (1 - F_i(\omega)) \right) d\omega - u_i(\underline{\omega}) \right), \tag{P$_\Theta$}$$

subject to, for each $i \in \Theta$, and each $j \in \Theta$ and $j \neq i$

$$u_i(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_i(\omega)(1 - F_i(\omega)) d\omega \geq 0, \tag{IR$_i$}$$

$$u_i(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_i(\omega)(1 - F_i(\omega)) d\omega \geq u_j(\underline{\omega}) + \int_{\underline{\omega}}^{\overline{\omega}} x_j(\omega)(1 - F_i(\omega)) d\omega, \tag{IC$_{i,j}$}$$

$$0 \leq x_i(\omega) \leq 1. \tag{FE$_i$}$$

The standard approach is to assume that the binding constraints are the lowest individual rationality constraint IR_1 and each local downward incentive compatibility constraint $\mathrm{IC}_{i,i-1}$, $i = 2, \ldots, I$. By substitutions one can then remove each $u_i(\underline{\omega})$ as a choice variable, and form an unconstrained relaxed problem of choosing allocation $x_i(\omega)$, with values on $[0, 1]$, for each $i \in \Theta$, to solve

$$\max_{(x_i(\cdot))} \sum_{i \in \Theta} \int_{\underline{\omega}}^{\overline{\omega}} x_i(\omega)\phi_i\delta_i(\omega)f_i(\omega)d\omega, \qquad (\mathrm{RP}_\Theta)$$

where $\delta_i(\omega)$ is the dynamic virtual surplus function of type $i = 1, \ldots, I-1$, given by

$$\delta_i(\omega) = \omega - c - \frac{\sum_{j=i+1}^{I} \phi_j}{\phi_i} \frac{F_i(\omega) - F_{i+1}(\omega)}{f_i(\omega)},$$

with $\delta_I(\omega) = \omega - c$. Point-wise maximization leads to deterministic allocations $x_i(\omega)$, but this approach fails if $x_i(\omega)$ fails to be increasing for some i, or if $x_i(\omega) < x_j(\omega)$ for some $i > j$ and some ω.

5.1 Necessary and sufficient conditions for randomization

As in the main model of three types, our approach is instead based on a modified relaxed problem. We drop all non-local IC constraints in period one, that is, all $\mathrm{IC}_{i,j}$ with $|i-j| \geq 2$. Using first order stochastic dominance ranking, it is then straightforward to show that IR_1 and each local downward $\mathrm{IC}_{i,i-1}$ for $i = 2, \ldots, I$ bind. First, IR_1 binds. If not, we can lower $u_i(0)$ for all $i = 1, \ldots, I$ by the same marginal amount. None of the local downward or upward IC's are affected. IR_1 still holds because it was slack; IR_2 is still slack because it is implied by $\mathrm{IC}_{2,1}$ and IR_1; and by induction IR_i for each higher i is still slack. This is a contradiction, establishing that IR_1 binds at any solution, and also IR_i is slack for all $i \geq 2$. Second, each local downward $\mathrm{IC}_{i,i-1}$ binds. Start with the highest type I. If $\mathrm{IC}_{I,I-1}$ is slack, we can reduce $u_I(0)$ by a marginal amount. This only impacts IR_I and $\mathrm{IC}_{I,I-1}$. Since IR_I is slack by the first step, we have a contradiction. Now suppose $\mathrm{IC}_{j,j-1}$ binds for all $j = i+1, \ldots, I$, and $\mathrm{IC}_{i,i-1}$ is slack. We can reduce $u_i(0)$ and all $u_j(0)$, $j = i+1, \ldots, I$, by the same marginal amount. All local downward and upward IC's continue to hold. Since IR's are all strict except for type 1, we have a contradiction, establishing that each $\mathrm{IC}_{i,i-1}$ binds binds at any solution to (P_Θ).

Using the binding constraints $\text{IC}_{i,i-1}$, we can formulate the modified relaxed problem, with the same objective and choice variables as in (P_Θ) but with two sets of constraints. With $x_I(\omega) = \mathbb{1}_{\omega \geq c}$, the modified relaxed problem is choosing $x_i(\omega)$, $i = 1, \dots, I-1$, to maximize

$$\sum_{i \leq I-1} \int_{\underline{\omega}}^{\overline{\omega}} x_i(\omega)\phi_i\delta_i(\omega)f_i(\omega)d\omega, \tag{MRP_Θ}$$

subject to each $x_i(\cdot)$, $i = 1, \dots, I-1$, is non-decreasing with the range of $[0, 1]$, and each local upward $\text{IC}_{i,i+1}$, which by the binding $\text{IC}_{i+1,i}$, is equivalent to

$$\int_{\underline{\omega}}^{\overline{\omega}} (x_{i+1}(\omega) - x_i(\omega))(F_i(\omega) - F_{i+1}(\omega))d\omega \geq 0. \tag{$\text{IC}'_{i,i+1}$}$$

The counterpart of Lemma 1 holds: any solution to (MRP_Θ) solves the original problem if it satisfies the dropped non-local period one IC constraints. With more than three types, we extend condition (i) of the strong alignment assumption introduced earlier in Section 4.2:

$$f_i(\omega) = (1 - \tau_i)f_I(\omega) + \tau_i f_1(\omega) \tag{A}$$

with $1 = \tau_1 > \tau_2 > \ldots > \tau_I = 0$. Under condition (A), any non-local incentive compatibility constraint is implied by a chain of local ones in the same direction and a single constraint in the opposite direction: for all $i \geq j+2$, the downward incentive compatibility constraint $\text{IC}_{i,j}$ is implied $\text{IC}_{i,i-1}$, ... , $\text{IC}_{j+1,j}$, and $\text{IC}_{j,j+1}$, for all $i \leq j-2$, the upward incentive compatibility constraint $\text{IC}_{i,j}$ is implied $\text{IC}_{i,i+1}$, ... , $\text{IC}_{j-1,j}$, and $\text{IC}_{j,j-1}$. Thus, under (A), any solution to (MRP_Θ) solves (P). Moreover, we can rewrite the objective of (MRP_Θ) and all $\text{IC}'_{i,i+1}$ using a single function $F_1(\omega) - F_I(\omega)$. Define the average ratio of surplus-to-slack for $i = 1, \dots, I-1$ as

$$R_i(w, w') = \frac{\int_w^{w'} \phi_i\delta_i(\omega)f_i(\omega)d\omega}{\int_w^{w'} F_1(\omega) - F_I(\omega)d\omega}$$

for all $w < w'$, and the corresponding point ratio as

$$r_i(\omega) = \frac{\phi_i\delta_i(\omega)f_i(\omega)}{F_1(\omega) - F_I(\omega)}.$$

Let $(\hat{w}_i)_{i\in\Theta}$ be the deterministic solution to (MRP_Θ):

$$\max_{(w_i)_{i\in\Theta}} \sum_{i\in\Theta} \int_{w_i}^{\overline{\omega}} \phi_i \delta_i(\omega) f_i(\omega) d\omega,$$

subject to that w_i is weakly decreasing in i. The following generalizes Propositions 1 and 2.

Proposition 5 *Suppose that condition (A) holds. (i) If*

$$\max_{\omega\leq\hat{w}_i} R_i(\omega, \hat{w}_i) \leq \min_{\omega\geq\hat{w}_i} R_i(\hat{w}_i, \omega), \tag{DET$_i$}$$

for all $i \in \Theta$, then $(\hat{w}_i)_{i\in\Theta}$ corresponds to a solution to (P). (ii) If (DET$_i$) fails for any i, then any solution to (P) is stochastic.

We provide a sketch of the proof here. To establish the necessary conditions for randomization, suppose that (DET_i) holds for all i. Since $(\hat{w}_i)_{i\in\Theta}$ is a deterministic solution to (MRP_Θ), there exist multipliers $\hat{\lambda}_{i,i+1} \geq 0$, each for $w_i \geq w_{i+1}$, $i = 1, \ldots, I-1$, satisfying complementary slackness, such that the first order necessary conditions hold:

$$r_i(\hat{w}_i) = \hat{\lambda}_{i,i+1} - \hat{\lambda}_{i-1,i}. \tag{FOC$_i$}$$

Define the following Lagrangian using $\hat{\lambda}_{i,i+1} \geq 0$ as the multiplier associated with $\text{IC}'_{i,i+1}$ in (MRP_Θ) for each $i = 1, \ldots, I-1$:

$$\mathcal{L}((x_i);(\hat{\lambda}_{i,i+1})) = \sum_{i\in\Theta} \int_{\underline{\omega}}^{\overline{\omega}} x_i(\omega)\Big(\phi_i\delta_i(\omega)f_i(\omega) - (\hat{\lambda}_{i,i+1} - \hat{\lambda}_{i-1,i})(F_1(\omega) - F_I(\omega))\Big)d\omega.$$

When condition (DET_i) holds for each i, by continuity and (FOC_i),

$$R_i(a, \hat{w}_i) \leq r_i(\hat{w}_i) = \hat{\lambda}_{i,i+1} - \hat{\lambda}_{i-1,i} \leq R_i(\hat{w}_i, b)$$

for all $a \leq \hat{w}_i \leq b$. Following the same steps of the proof of Proposition 1, we can show that the above inequalities imply that $x_i^*(\omega) = \mathbb{1}_{\omega\geq\hat{w}_i}$, $i \in \Theta$, maximizes $\mathcal{L}((x_i);(\hat{\lambda}_{i,i+1}))$ among all non-decreasing functions with values on $[0,1]$. Since $(x_i^*)_{i\in\Theta}$ is deterministic, by Lagrangian relaxation, it solves (P_Θ). Thus, a necessary condition for randomization is that condition

(DET$_i$) fails for some i.

For the sufficiency for randomization when (DET$_i$) fails for some i,[13] we can find any $a < \hat{w}_i < b$ such that $R_i(a, \hat{w}_i) > R_i(\hat{w}_i, b)$, and replace $\mathbb{1}_{\omega \geq \hat{w}_i}$ with $\chi_i^{[a,b]}$, where χ_i satisfies

$$\chi_i \int_a^{\hat{w}_i} (F_1(\omega) - F_I(\omega))d\omega = (1 - \chi_i) \int_{\hat{w}_i}^b (F_1(\omega) - F_I(\omega))d\omega.$$

Due to condition (A), neither $\text{IC}'_{i,i+1}$ nor $\text{IC}'_{i-1,i}$ is affected, but the change in the value of type i part of the objective function of (MRP$_\Theta$) is

$$\chi_i \int_a^{\hat{w}_i} \phi_i \delta_i(\omega) f_i(\omega) d\omega - (1 - \chi_i) \int_{\hat{w}_i}^b \phi_i \delta_i(\omega) f_i(\omega) d\omega,$$

which is strictly positive since $R_i(a, \hat{w}_i) > R_i(\hat{w}_i, b)$. Thus, the solution to (MRP$_\Theta$) is stochastic. Under condition (A), any solution to (P$_\Theta$) must also be stochastic.

Without condition (A), in general the necessary and sufficient conditions for randomization cannot be expressed in terms of a surplus-to-slack ratio. Specifically, the counterpart of (DET$_i$) can be shown to be

$$\max_{a \leq w_i} \int_a^{\hat{w}_i} \Big(\phi_i \delta_i(\omega) f_i(\omega) d\omega - \hat{\lambda}_{i,i+1}(F_i(\omega) - F_{i+1}(\omega)) + \hat{\lambda}_{i-1,i}(F_{i-1}(\omega) - F_i(\omega))\Big) d\omega \leq 0,$$
$$\min_{b \geq \hat{w}_i} \int_{\hat{w}_i}^b \Big(\phi_i \delta_i(\omega) f_i(\omega) d\omega - \hat{\lambda}_{i,i+1}(F_i(\omega) - F_{i+1}(\omega)) + \hat{\lambda}_{i-1.i}(F_{i-1}(\omega) - F_i(\omega))\Big) d\omega \geq 0.$$

If $\hat{w}_{i-1} > \hat{w}_i = \hat{w}_{i+1}$ so that $\hat{\lambda}_{i-1,i} = 0$, the above two conditions can be combined and expressed in terms of a surplus-to-slack ratio for type i, using $F_i - F_{i+1}$ to measure the rent; the result is precisely condition (DET$_L$) in the main model with three types. If $\hat{w}_{i-1} = \hat{w}_i > \hat{w}_{i+1}$, then $\hat{\lambda}_{i,i+1} = 0$, and the above condition can also be expressed in a surplus-to-slack ratio for type i, using $F_{i-1} - F_i$ instead; this is precisely condition (DET$_M$) in the main model. However, when $\lambda_{i,i+1}, \lambda_{i-1,i} > 0$, which implies $\hat{w}_{i-1} = \hat{w}_i = \hat{w}_{i+1}$, it is not possible to state the above two conditions for type i in terms of a single surplus-to-slack ratio.

[13] Consistent with Proposition 2, this cannot happen if neither of the two constraints $\hat{w}_{i+1} \leq \hat{w}_i \leq \hat{w}_{i-1}$ is binding at the solution. To see this claim, note that if neither constraint is binding, then $S_i(\hat{w}_i) \geq S_i(w_i)$ for all w_i. For all $a \leq \hat{w}_i$ we have $S_i(a) - S(\hat{w}_i) = R(a, \hat{w}_i) \int_a^{\hat{w}_i} (F_1(\omega) - F_I(\omega)) d\omega$. This implies that $R(a, \hat{w}_i) \leq 0$ for all $a \leq \hat{w}_i$. Similarly, $R(\hat{w}_i, b) \geq 0$ for all $b \geq \hat{w}_i$. The claim follows.

5.2 Optimal mechanisms

With more than three types and more than a single upward incentive compatibility constraint, characterizing optimal stochastic mechanisms becomes more involved, but most of our characterization results generalize, at least partially, to provide restrictions we can use to construct optimal stochastic mechanisms. Part (i) of Lemma 4 continues to hold: randomization occurs at more than one level strictly between 0 and 1 for each type $i \in \Theta$, and so without loss we can write the solution to (MRP$_\Theta$) as $\left(\chi_i^{[a_i^*,b_i^*]}\right)_{i\in\Theta}$.[14] Part (ii) of Lemma 4 does not generally hold, as randomization can occur for more than one type in any solution to (MRP$_\Theta$); indeed we will construct one such example in Appendix A.[15]

As in the main model of three types, no solution to (MRP$_\Theta$) can have randomization for some type $i \in \Theta$ with a support a subset of an interval (w, w') over which i's point ratio r_i is strictly increasing; and in any solution i's allocation is constant on any interval (w, w') over which r_i is strictly decreasing.[16] Further, the characterization of candidate solutions to (MRP) by Lemma 5 completely generalizes. If type $i = 1, \dots, I-1$ has a point ratio of surplus-to-slack function r_i that is single dipped, then there exist unique $a_i^* < b_i^*$ satisfying

$$r_i(a_i^*) \geq R_i(a_i^*, b_i^*) \geq r_i(b_i^*),$$

and $a_i^* \geq \underline{\omega}$ and $b_i^* \leq \overline{\omega}$, both with corresponding complementary slackness, such that, at any solution to (MRP$_\Theta$), the interval where randomization occurs is $[a_i^*, b_i^*]$ if x_i^* is stochastic, and the threshold k_i lies outside of $[a_i^*, b_i^*]$ if x_i^* is deterministic.

Although the general idea of using Lagrangian relaxation to construct a solution to (MRP$_\Theta$) is applicable in specific examples, extending Proposition 3 requires additional infor-

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[14] The argument is simply noting that we can treat the difference in multipliers $\lambda_{i-1,i} - \lambda_{i,i+1}$ as a single multiplier in the proof of part (i) of Lemma 4.

[15] The proof of part (ii) of Lemma 4 can be partially generalized: if at some solution $\left(\chi_i^{[a_i^*,b_i^*]}\right)_{i\in\Theta}$ there exist some $i_1, i_2 \in \Theta$ with $i_2 \geq i_1 + 1$ such that IC'_{i_1-1,i_1} and IC'_{i_2,i_2+1} are both slack, and $\lambda_{i,i+1} > 0$ for all $i = i_1, \dots, i_2 - 1$, then the value of (MRP$_\Theta$) does not depend on χ_i, $i = i_1, \dots, i_2$. However, in general we no longer have the freedom to change the values of χ_i to reduce the number of random allocations between i_1 and i_2, because changing χ_i for any $i = i_1, \dots, i_2$ can violate $\text{IC}'_{i,i+1}$ and/or $\text{IC}'_{i,i-1}$.

[16] Even though the allocation of type i affects two upward constraints $\text{IC}'_{i,i+1}$ and $\text{IC}'_{i-1,i}$ (if $i \geq 2$ and $i \leq I-1$), under condition (A) the effects are the same. This implies that whenever we switch type i's allocation $x_i(\omega)$ from a random one to a deterministic one, or vice versa, so long as we keep as fixed the weight average of $x_i(\omega)$, neither of the two upward constraints is unaffected.

mation about the shape of each point ratio of surplus-to-slack r_i and the structure of binding upward constraints. Here, we generalize the strong alignment assumption introduced in Section 4. Suppose that (i) condition (A) holds, and (ii) $f_I(\omega)/f_1(\omega)$ is strictly increasing for all $\omega \in [\underline{\omega}, \overline{\omega}]$. Then, $r_i(\omega) \geq r_i(\omega')$ for any $\omega > \omega' > c$ implies that $r_{i'}(\omega) > r_{i'}(\omega')$ for all $i, i' \in \Theta$ with $i' \geq i + 1$. Under the assumption that $r_i(\omega)$ is single dipped for each $i \in \Theta$, with (ω_i^p, ω_i^t) being the largest interval over which $r_i(\omega)$ is strictly decreasing, if $c = \underline{\omega}$, then the intervals are all ordered by type, so that $\omega_{i'}^p \geq \omega_i^p$ with strict inequality if $\omega_i^p > \underline{\omega}$, and $\omega_{i'}^t \leq \omega_i^t$ with strict inequality if $\omega_i^t < \overline{\omega}$. These results can help us make the correct guesses about the values of the multipliers in order to apply the argument of Proposition 3. This will be illustrated in Appendix A with the class of examples with explicit distribution functions.

Under strong alignment with $c = \underline{\omega}$, the argument in Proposition 4 can be extended to more than three types. We can show that randomization for any type $i = 2, \ldots, I$ at a solution to (P_Θ) implies we cannot have both a deterministic allocation for $i - 1$ and a binding $\mathrm{IC}'_{i-1,i}$. This suggests that randomization occurs in "clusters," where each cluster of adjacent types has binding upward constraints among them, and clusters are separated from each other. See Appendix A for an illustration of this idea.

6 Discussion

Our approach based on the modified relaxed problem is local. In (MRP) with three ex ante types, we keep IC'_{LM}, the equivalent of the local upward incentive compatibility constraint IC_{LM}, and drop IC'_{HL}, the equivalent of the global downward constraint IC_{HL}.[17] To establish the sufficient condition for randomization in solutions to the original problem (P), in the proof of Proposition 2 we rely on restrictions on the distribution functions $\{F_\theta\}_{\theta=H,M,L}$, because a solution to (P) may not solve (MRP). In particular, when the sufficient condition (DET_L) for deterministic solutions fails for type L, we construct a solution $(x_M, \hat{x}_L)$ to (MRP) with randomization for type L that binds IC'_{LM}, but we need F_M to be "increasingly aligned with" F_L (the weighting function $\tau(\omega)$ to be non-decreasing) in order to conclude that $(x_M, \hat{x}_L)$ satisfies IC'_{HL} and is thus feasible in (P). If instead F_M is decreasingly aligned with F_L,

[17]We also drop IC'_{MH} and IC'_{LH}, but they are never binding at any solution to (MRP).

$(x_M, \hat{x}_L)$ violates IC'_{HL}. Even though any solution to (MRP) has to be stochastic, we cannot conclude the same for (P).

An alternative to the local approach is to keep global as well as local IC constraints. In the main model with three ex ante types, this global approach leads to a constrained maximization problem (GP) that shares with (MRP) the same objective, the same choice variables x_M and x_L, which are non-decreasing with values in $[0, 1]$, and the same local upward constraint IC'_{LM}, but keeps all other IC constraints, including, in particular, the global downward constraint IC'_{HL}. We claim that if the solution $(x_\theta(\omega), u_\theta(\underline{\omega}))_{\theta=H,M,L}$ to (P) is deterministic, then (x_M, x_L) solves (GP).[18] This claim can be established from the following straightforward steps: first, IR_L binds at any solution to (P); second, either IC_{ML} or IC_{HL} binds, or both bind at any solution to (P); third, if IC_{ML} binds and IC_{HL} is slack at any solution to (P), then IC_{HM} binds; fourth, if IC_{ML} binds and IC_{HL} is slack at any solution $(x_\theta(\omega), u_\theta(\underline{\omega}))$ to (P), then (x_M, x_L) solves (GP); fifth, if the solution to the original problem is deterministic, then at the solution IC_{ML} binds and IC_{HL} is slack.

After we identify (GP) using the global approach, we can now extend Proposition 2. Suppose the solution $(x_\theta(\omega), u_\theta(\underline{\omega}))$ to (P) is deterministic. We have $x_L(\omega) = x_M(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$ for some $\hat{k}$. Consider part (i), with randomization for type L, fixing $x_M = \mathbb{1}_{\omega \geq \hat{k}}$ (part (ii) can be similarly extended). If there exists a random allocation $\hat{x}_L = \chi_L^{[a,b]}$ with $\chi_L \in (0, 1)$, $a < \hat{k}$ and $b > \hat{k}$, satisfying both IC'_{LM} and IC'_{HL}

$$
\begin{aligned}
&-\chi_L \int_a^{\hat{k}} (F_L(\omega) - F_M(\omega))d\omega + (1 - \chi_L) \int_{\hat{k}}^b (F_L(\omega) - F_M(\omega))d\omega \geq 0, \\
&-\chi_L \int_a^{\hat{k}} (F_M(\omega) - F_H(\omega))d\omega + (1 - \chi_L) \int_{\hat{k}}^b (F_M(\omega) - F_H(\omega))d\omega \geq 0,
\end{aligned}
$$

such that the value of the objective of (GP) is increased

$$
\chi_L \int_a^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega - (1 - \chi_L) \int_{\hat{k}}^b \phi_L \delta_L(\omega) f_L(\omega) d\omega > 0,
$$

then we have a contradiction to the assumption that (x_M, x_L) solves (GP).[19] This then

[18] This claim does not hold for (MRP), because we have dropped IC'_{HL} in (MRP). Even though IC'_{HL} is slack at the solution $(x_\theta(\omega), u_\theta(\underline{\omega}))_{\theta=H,M,L}$ to (P), (x_M, x_L) does not necessarily solve (MRP).

[19] It suffices to show that $(\hat{x}_L, x_M)$ satisfies IC'_{MH} and IC'_{LH}. Since $x_H(\omega) = \mathbb{1}_{\omega \geq c} \geq x_M(\omega) = \mathbb{1}_{\omega \geq \hat{k}}$ for all

establishes that the solution to (P) is stochastic.

The above conditions for the existence of $\hat{x}_L$ can be expressed in terms of surplus-to-slack ratios as in Proposition 2 as follows, but because both IC'_{LM} and IC'_{HL} are present in (GP), we have two different ratios for type L. Suppose that $\hat{k}_M > \hat{k}_L$ and $\hat{k}$ is interior. If there exist $a \le \hat{k}$ and $b \ge \hat{k}$ such that

$$\frac{\int_a^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_a^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega} > \frac{\int_{\hat{k}}^b \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_{\hat{k}}^b (F_L(\omega) - F_M(\omega)) d\omega} \tag{RAN$_L$}$$

$$\frac{\int_a^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_a^{\hat{k}} (F_M(\omega) - F_H(\omega)) d\omega} > \frac{\int_{\hat{k}}^b \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_{\hat{k}}^b (F_M(\omega) - F_H(\omega)) d\omega}, \tag{RAN$'_L$}$$

then changing the allocation for type L from $\mathbb{1}_{\omega \ge \hat{k}}$ to $\chi_L^{[a,b]}$ for any $\chi_L \in (0,1)$ such that

$$\max\left\{ \frac{\int_a^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega}{\int_{\hat{k}}^b (F_L(\omega) - F_M(\omega)) d\omega}, \frac{\int_a^{\hat{k}} (F_M(\omega) - F_H(\omega)) d\omega}{\int_{\hat{k}}^b (F_M(\omega) - F_H(\omega)) d\omega} \right\} < \frac{1 - \chi_L}{\chi_L} < \frac{\int_a^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_{\hat{k}}^b \phi_L \delta_L(\omega) f_L(\omega) d\omega}$$

improves the value of the objective of (GP) while satisfying both IC'_{LM} and IC'_{HL}. It follows that any solution to (P) is stochastic. With condition (RAN$_L$) just the reverse of (DET$_L$), and a new condition (RAN$'_L$), the above generalizes Proposition 2. When τ is non-decreasing, by Lemma 3 condition (RAN$_L$) implies (RAN$'_L$), and so we have Proposition 2. When τ is strictly increasing, Proposition 2 does not apply because the random allocation $\hat{x}_L$ constructed in the proof violates IC'_{HL}, but solutions to (P) can still be stochastic when condition (RAN$'_L$) holds because by Lemma 3 it implies (RAN$_L$). More generally, conditions (RAN$_L$) and (RAN$'_L$) are sufficient for solutions to (P) to be stochastic, without any restrictions on the distributions beyond first order stochastic dominance ranking.

Appendix A. Examples

In this section, we explicitly solve (P) for a class of sequential screening problems. We use them illustrate the results from both the main model with three ex ante types in Section 4

ω, IC'_{MH} is satisfied. For IC'_{LH}, we note that since by assumption $(x_\theta(\omega), u_\theta(\underline{\omega}))$ solves (P) with $x_M = \mathbb{1}_{\omega \ge \hat{k}}$, we have $\delta_L(\hat{k}) \ge 0$. As a result, we can assume that $a > c$, and thus $x_H(\omega) = \mathbb{1}_{\omega \ge c} \ge x_L(\omega) = \chi_L^{[a,b]}$ for all ω and IC'_{LH} is satisfied.

and the extension with more than three types in Section 5. Since the model in Section 4 is a special case of the model in Section 5, we use the latter, and specialize to three types when necessary.

For all $\omega \in [0, \infty)$, let

$$f_i(\omega) = \gamma_i e^{-\gamma_i \omega},$$

for $i = 1$ and $i = I \geq 3$, with $\gamma_1 > \gamma_I > 0$, and for each $i = 1, \ldots, I$ let

$$f_i(\omega) = (1 - \tau_i)\gamma_I e^{-\gamma_I \omega} + \tau_i \gamma_1 e^{-\gamma_1 \omega}$$

for some $\tau_i \in [0, 1]$, with $1 = \tau_1 > \tau_2 > \ldots > \tau_I = 0$. The resulting class of distributions $\{F_i(\omega)\}_{i=1,\ldots,I}$ satisfies conditions (i) and (ii) of strong alignment. We have $\delta_I(\omega) = \omega - c$, and for each $i = 1, \ldots, I - 1$,

$$\begin{aligned}
\delta_i(\omega) &= \omega - c - \frac{(\tau_i - \tau_{i+1})\left(e^{-\gamma_I \omega} - e^{-\gamma_1 \omega}\right) \sum_{i'=i+1}^{I} \phi_{i'}}{\left((1 - \tau_i)\gamma_I e^{-\gamma_I \omega} + \tau_i \gamma_1 e^{-\gamma_1 \omega}\right) \phi_i}, \\
r_i(\omega) &= \frac{\phi_i(\omega - c)\left((1 - \tau_i)\gamma_I e^{-\gamma_I \omega} + \tau_i \gamma_1 e^{-\gamma_1 \omega}\right)}{e^{-\gamma_I \omega} - e^{-\gamma_1 \omega}} - (\tau_i - \tau_{i+1}) \sum_{i'=i+1}^{I} \phi_{i'}.
\end{aligned}$$

It is straightforward to verify that $r_i(0) = -\infty$ and $dr_i(0)/d\omega = \infty$ if $c > 0$, and if $c = 0$,

$$\begin{aligned}
r_i(0) &= \frac{\phi_i((1 - \tau_i)\gamma_I + \tau_i \gamma_1)}{\gamma_1 - \gamma_I} - (\tau_i - \tau_{i+1}) \sum_{i'=i+1}^{I} \phi_{i'}, \\
\frac{dr_i(0)}{d\omega} &= \frac{\phi_i}{2}((1 - \tau_i)\gamma_I - \tau_i \gamma_1).
\end{aligned}$$

Also, $r_1(\infty) = -(1 - \tau_2)(1 - \phi_1)$, and $r_i(\infty) = \infty$ for $i = 2, \ldots, I - 1$. We first derive two claims we need for explicit characterizations of solutions to (P).

Claim 2 *For each $i = 1, \ldots, I - 1$, $r_i(\omega)$ is single dipped. Further, if $c = 0$, then $r_1(\omega)$ is strictly decreasing, and for any $i = 2, \ldots, I - 1$ there exists a strictly positive and finite ω_i^t such that $r_i(\omega)$ is strictly decreasing for any $\omega < \omega_i^t$ and strictly increasing for any $\omega > \omega_i^t$. If $c > 0$, then there exists a strictly positive and finite ω_1^p such that $r_1(\omega)$ is strictly increasing for any $\omega < \omega_1^p$ and strictly decreasing for any $\omega > \omega_1^p$, and $r_i(\omega)$ is strictly increasing in ω if $(1 - \tau_i)\gamma_I \geq \tau_i \gamma_1$.*

Proof. By taking derivatives, we can show that $dr_i(\omega)/d\omega$ has the same sign as

$$(1-\tau_i)\gamma_I\left(e^{(\gamma_1-\gamma_I)\omega}-1\right)+\tau_i\gamma_1\left(1-e^{-(\gamma_1-\gamma_I)\omega}\right)-(\gamma_1-\gamma_I)((1-\tau_i)\gamma_I+\tau_i\gamma_1)(\omega-c).$$

Thus, $dr_1(\omega)/d\omega > 0$ if and only if

$$1-e^{-(\gamma_1-\gamma_I)\omega} > (\gamma_1-\gamma_I)(\omega-c).$$

The left-hand side is strictly concave in ω, with a derivative equal to $\gamma_1-\gamma_I$ at $\omega=0$. It follows that if $c=0$, then $dr_1(\omega)/d\omega<0$ for all ω, and if $c>0$, there exists a strictly positive and finite ω_1^p which equates the two sides of the inequality above, such that $r_1(\omega)$ is strictly increasing for any $\omega<\omega_1^p$ and strictly decreasing for any $\omega>\omega_1^p$.

Next, fix any $i=2,\dots,I-1$. At any $\hat{\omega}$ such that $dr_i(\hat{\omega})/d\omega=0$, the sign $d^2r_i(\hat{\omega})/d\omega^2$ is the same as

$$(1-\tau_i)\gamma_I e^{(\gamma_1-\gamma_I)\hat{\omega}}+\tau_i\gamma_1 e^{-(\gamma_1-\gamma_I)\hat{\omega}}-((1-\tau_i)\gamma_I+\tau_i\gamma_1).$$

The sign of the above is the same as

$$(1-\tau_i)\gamma_I e^{(\gamma_1-\gamma_I)\hat{\omega}}-\tau_i\gamma_1.$$

Thus, the sign of $d^2r_i(\hat{\omega})/d\omega^2$ at any $\hat{\omega}$ such that $dr_i(\hat{\omega})/d\omega=0$ can only change from negative to positive. It follows that $r_i(\omega)$ is single dipped. If $c=0$, then since $dr_i(0)/d\omega<0$ and $r_i(\infty)=\infty$, and since $r_i(\omega)$ is single dipped, $r_i(\omega)$ has a unique interior trough. If $c>0$, then $(1-\tau_i)\gamma_I\geq\tau_i\gamma_1$ implies that $d^2r_i(\hat{\omega})/d\omega^2>0$ at any $\hat{\omega}$ such that $dr_i(\hat{\omega})/d\omega=0$. As a result, $\hat{\omega}$ is a local minimum of $r_i(\omega)$. Since $dr_i(0)/d\omega=\infty$, and since $r_i(\omega)$ is single dipped, it cannot have a local minimum without having a local maximum. This is a contradiction, and it follows there is no $\hat{\omega}$ such that $dr_i(\hat{\omega})/d\omega=0$ when $(1-\tau_i)\gamma_I\geq\tau_i\gamma_1$. Thus, $r_i(\omega)$ is strictly increasing in ω. ■

For each $i=1,\dots,I-1$, the total dynamic virtual surplus of type i under a threshold allocation rule $\mathbb{1}_{\omega \geq k}$ is given by

$$S_i(k) = \int_k^{\infty} r_i(\omega) \left(e^{-\gamma_I \omega} - e^{-\gamma_1 \omega} \right) d\omega.$$

The following claim provides a characterization of $S_i(k)$. Let $\hat{k}_i$ be the smallest maximizer of $S_i(k)$, $i = 1, \dots, I-1$. We have that $dS_i(k)/dk$ has the same sign as $-r_i(k)$. At any $\hat{\omega}$ such that $dS_i(\hat{\omega})/d\omega = 0$, the sign of $d^2 S_i(\hat{\omega})/dk^2$ is the same as $-dr_i(\hat{\omega})/dk$.

Claim 3 *If $c = 0$, then $\hat{k}_1 = 0$ when $S_1(0) \geq 0$ and $\hat{k}_1 = \infty$ otherwise, and for each $i = 2, \dots, I-1$, $\hat{k}_i$ is uniquely defined by $r_i(\hat{k}_i) = 0$ and $dr_i(\hat{k}_i)/d\omega > 0$ when $S_i(\hat{k}_i) \geq S_i(0)$, and $\hat{k}_i = 0$ otherwise. If $c > 0$, then $\hat{k}_1 = \infty$ when $r_1(\omega_1^p) \leq 0$, and is otherwise uniquely defined by $r_1(\hat{k}_1) = 0$ and $dr_1(\hat{k}_1)/d\omega > 0$, and $\hat{k}_i$ is uniquely defined by $r_i(\hat{k}_i) = 0$ for any i such that $(1 - \tau_i)\gamma_I \geq \tau_i \gamma_1$.*

Proof. Suppose that $c = 0$. By Claim 2, since $r_1(\omega)$ is strictly decreasing, $S_1(k)$ has no interior local maximum. It follows that $S_1(k)$ is maximized at either $\hat{k}_1 = 0$ or $\hat{k}_1 = \infty$. Since $S_1(\infty) = 0$, the maximum is either attained at $\hat{k}_1 = 0$ if $S_1(0) \geq 0$, or else at $\hat{k}_1 = \infty$. For any $i = 2, \dots, I-1$, by Claim 2, since $r_i(\omega)$ has a unique interior trough at ω_i^t, there are three cases. If $r_i(\omega_i^t) \geq 0$, then $S_i(k)$ is strictly decreasing for all k. The maximum of $S_i(k)$ is reached at $\hat{k}_i = 0$. If $r_i(0) < 0$, then since $dr_i(0)/d\omega < 0$ and $r_i(\infty) = \infty$, there exists a unique $\hat{w}$ strictly positive and finite, satisfying $r_i(\hat{w}) = 0$ with $dr_i(\hat{w})/d\omega > 0$, such that $S_i(k)$ is strictly increasing for all $k \in (0, \hat{w})$ and strictly decreasing for all $k > \hat{w}$. The maximum of $S_i(k)$ is reached at $\hat{k}_i = \hat{w}$. If $r_i(\omega_i^t) < 0 \leq r_i(0)$, then there is a unique $\hat{w} > \omega_i^t$ such that $r_i(\hat{w}) = 0$, with $dr_i(\hat{w})/d\omega > 0$. In this case $\hat{w}$ is a local maximizer of $S_i(k)$. The maximum of $S_i(k)$ is reached at $\hat{k}_i = \hat{w}$ if $S_i(\hat{w}) \geq S_i(0)$ and otherwise at $\hat{k}_i = 0$.

Suppose that $c > 0$. By Claim 2, $r_1(\omega)$ has a unique interior peak at some ω_1^p. If $r_1(\omega_1^p) \leq 0$, then $S_1(k)$ is increasing for all k, and is therefore maximized at $\hat{k}_1 = \infty$. Otherwise, by Claim 2 there exists a unique $\hat{w}$ such that $r_1(\hat{w}) = 0$ and $dr_1(\hat{w})/d\omega > 0$. It follows that $S_1(k)$ is maximized at $\hat{k}_1 = \hat{w}$. For any $i = 2, \dots, I-1$, by Claim 2, $r_1(\omega)$ is strictly increasing in ω when $(1 - \tau_i)\gamma_I \geq \tau_i \gamma_1$. Since $r_i(0) = -\infty$ and $r_i(\infty) = \infty$, there exists a unique $\hat{w}$ such that $r_i(\hat{w}) = 0$. It follows that $S_i(k)$ is maximized at $\hat{k}_i = \hat{w}$. ■

Now we are ready to illustrate explicitly constructed solutions to (P) through a series of examples. For the first two examples, we have $I = 3$. We revert back to the notation of H, M and L. So type I becomes type H, and type 1 becomes type L, with $\tau \in (0, 1)$ representing the weight on f_L in f_M. The first example provides a straightforward application of part (i) of Proposition 3.

Example 1: $I = 3$ and $c > 0$. We assume $(1 - \tau)\gamma_H \geq \tau\gamma_L$. By Claim 3, $r_M(\hat{k}_M) = 0$, and $r_L(\hat{k}_L) \leq 0$ and $\hat{k}_L \leq \infty$, with complementary slackness.

First, suppose that $\hat{k}_M \leq \hat{k}_L$. This corresponds to the regular case that the existing literature focuses on. The solution to (P) is deterministic, with threshold allocation for all three types: the threshold is c for type H, $\hat{k}_M$ for type M, and $\hat{k}_L$ for type L.

Second, suppose instead $\hat{k}_M > \hat{k}_L$. This requires $\hat{k}_L < \infty$ and thus $r_L(\omega_L^p) > 0$, where ω_L^p is the unique interior peak of r_L by Claim 2. The deterministic solution $\hat{k}$ to (MRP) is uniquely determined by $r_L(\hat{k}) + r_M(\hat{k}) = 0$, and is strictly between $\hat{k}_L$ and $\hat{k}_M$. By Claim 2, r_M is strictly increasing because $(1 - \tau)\gamma_H \geq \tau\gamma_L$. Lemma 5 then implies that, if there is a stochastic solution to (MRP) then randomization occurs only for type L. By Claim 2, $r_L(\omega)$ has a unique interior peak at ω_L^p with $r_L(0) = -\infty$ and $r_L(\infty) = 0$. By Lemma 5, in any stochastic solution (x_M^*, x_L^*) to (MRP), the support of type L's random allocation $x_L^*(\omega)$ is given by $[a_L^*, \infty)$, with a_L^* uniquely defined by

$$R_L(a_L^*, \infty) = r_L(a_L^*),$$

implying that $r_L(a_L^*) > 0$ and so $a_L^* \in (\hat{k}_L, \omega_L^p)$. Part (i) of Proposition 3 then establishes that if there exists $k_M > a_L^*$ such that

$$r_M(k_M) = -R_L(a_L^*, \infty) = -r_L(a_L^*),$$

then $x_L^*(\omega) = \chi_L^{[a_L^*, \infty)}$ and $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$ solve (MRP), with

$$\chi_L = \frac{\int_{k_M}^{\infty}(F_L(\omega) - F_M(\omega))d\omega}{\int_{a_L^*}^{\infty}(F_L(\omega) - F_M(\omega))d\omega},$$

and thus corresponds to an optimal stochastic mechanism. Since r_M is strictly increasing, such k_M exists if and only if

$$r_L(a_L^*) + r_M(a_L^*) < 0.$$

If the above condition is violated, there is no stochastic solution to (MRP). The solution is deterministic with a common threshold $\hat{k}$, and the solution to (P) is deterministic. We have an example where solutions to (P) are deterministic even though the unconstrained solution to (P) violates IC'_{LM}. ■

Our second example assumes $c = 0$ and uses Proposition 4 and Lemma 5 to pin down a unique candidate solution to (MRP) when the unconstrained solution violates IC'_{LM}. We then apply Corollary 1 to establish a sufficient condition to validate the candidate solution and thus correspond to an optimal stochastic mechanism.

Example 2: $I = 3$ and $c = 0$. By Claim 3, we have $\hat{k}_L = 0$ if $S_L(0) \geq 0$, and otherwise $\hat{k}_L = \infty$. For type M, by Claim 2, there is a unique minimizer ω_M^t of $r_M(\omega)$. By Claim 3, a sufficient condition for $\hat{k}_M$ to be interior is $r_M(0) < 0$.

First, suppose that $\hat{k}_L = \infty$, or $\hat{k}_L = \hat{k}_M = 0$. This is a regular case in the existing literature. The solution to (P) is deterministic, with threshold allocation for all three types: the threshold is 0 for type H, $\hat{k}_M$ for type M, and $\hat{k}_L$ for type L.

Second, suppose that $\hat{k}_L = 0$ and $\hat{k}_M > 0$. If $\hat{k} > 0$, then since $r_L(\omega)$ is strictly decreasing by Claim 2, Proposition 2 implies that any solution to (MRP) is stochastic. If $\hat{k} = 0$, Proposition 2 does not apply, and the solution to (MRP) may be stochastic, or deterministic given by $x_M^*(\omega) = x_L^*(\omega) = \mathbb{1}_{\omega \geq 0}$. By Proposition 4, if randomization occurs in any solution to (P), it occurs for type L and takes the form of $x_L^*(\omega) = \chi_L^{[a_L^*, b_L^*]}$ and $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$. By Lemma 5, since $r_L(\omega)$ is strictly decreasing, $a_L^* = 0$ and $b_L^* = \infty$. Further, since r_M has a unique interior trough and $r_M(\infty) = \infty$, we have $a_M^* = 0$ and b_M^* is uniquely defined by

$$r_M(b_M^*) = R_M(0, b_M^*).$$

Since $\hat{k}_L = 0$, we have $S_L(0) \geq S_L(\infty) = 0$, and thus $R_L(0, \infty) \geq 0$. By Corollary 1, if there exists $k_M \geq b_M^*$ such that

$$r_M(k_M) = -R_L(0, \infty),$$

then $x_L^*(\omega) = \chi_L^{[0,\infty)}$ and $x_M^*(\omega) = \mathbb{1}_{\omega \geq k_M}$ solve (MRP), with

$$\chi_L = \frac{\int_{k_M}^{\infty}(F_L(\omega) - F_M(\omega))d\omega}{\int_0^{\infty}(F_L(\omega) - F_M(\omega))d\omega},$$

and thus corresponds to an optimal stochastic mechanism. Since $r_M(\omega)$ is strictly increasing for $\omega > b_M^*$, the above condition is equivalent to

$$R_L(0, \infty) + r_M(b_M^*) \leq 0.$$

If $R_L(0, \infty) + r_M(b_M^*) > 0$, then there is no stochastic solution to (MRP), and $x_M^*(\omega) = x_L^*(\omega) = \mathbb{1}_{\omega \geq 0}$ correspond to a solution to (P). ■

The third example below illustrates what we call randomization clusters with $I = 4$ and $c = 0$. We construct a solution to (P) where types 1 and 2 have random allocations while types 3 and 4 have deterministic allocations. To do so, we first use Lemma 5 to propose the unique candidate solution to (MRP) that is consistent with this randomization cluster. We then apply the same Lagrangian relaxation method used in Proposition 3 and Corollary 1 to establish a sufficient condition for the candidate solution to solve (MRP) and thus correspond to optimal stochastic mechanisms.

Example 3: $I = 4$ and $c = 0$. We consider a solution $(x_i^*(\omega))_{i=1,2,3}$ to (MRP) of the form $x_i^*(\omega) = \chi_i^{[a_i,b_i]}$ for $i = 1, 2$, and $x_3^*(\omega) = \mathbb{1}_{\omega \geq k_3}$. By Claim 2, $r_1(\omega)$ is strictly decreasing, and both $r_2(\omega)$ and $r_3(\omega)$ have a unique interior trough. It follows from Lemma 5 that $a_1 = a_1^* = 0$ and $b_1 = b_1^* = \infty$, $a_2 = a_2^* = 0$ and $b_2 = b_2^*$, and $k_3 \geq b_3^*$ with $a_3^* = 0$, where b_i^* is uniquely defined by

$$r_i(b_i^*) = R_i(0, b_i^*)$$

for each $i = 2, 3$. As we have argued in Section 5, since $a_2^* = a_3^*$, we have $b_2^* > b_3^*$. We claim that if $R_1(0, \infty) \geq 0$, $R_1(0, \infty) + r_2(b_2^*) \geq 0$, and

$$-r_3(a_2^*) < R_1(0, \infty) + r_2(b_2^*) \leq -r_3(b_3^*),$$

then there is a unique value of k_3, together with some $\chi_1, \chi_2 \in (0, 1)$, such that $x_1^*(\omega) = \chi_1^{[0,\infty)}$,

$x_2^*(\omega) = \chi_2^{[0,b_2^*]}$, and $x_3^*(\omega) = \mathbb{1}_{\omega \geq k_3}$ form a solution to (MRP), and hence to (P).

The claim is established by a generalization of the Lagrangian relaxation argument in Proposition 3. Since $r_3(\omega)$ is strictly increasing for $\omega \geq b_3^*$ with $r_3(\infty) = \infty$, under the stated conditions there exists a unique $k_3 \in [b_3^*, b_2^*)$ such that

$$R_1(0,\infty) + r_2(b_2^*) + r_3(k_3) = 0.$$

We choose the multipliers as follows: $\lambda_{2,1} = R_1(0,\infty)$ and $\lambda_{3,2} = R_1(0,\infty) + r_2(b_2^*)$. By assumption, $\lambda_{2,1}, \lambda_{3,2} \geq 0$. With these values of the multipliers, we argue that for each type $i = 1,2,3$, the given allocation $x_i^*(\omega)$ maximizes the part of the Lagrangian function associated with type i among all weakly increasing functions $x_i(\omega)$ with the range of $[0,1]$. For type 1, with $\lambda_{2,1} = R_1(0,\infty)$, the argument is the same as for Corollary 1. For type 2, with $R_2(0,b_2^*) = r_2(b_2^*) = \lambda_{3,2} - \lambda_{2,1}$, the argument is the same as in Proposition 3. Finally, for type 3, with $\lambda_{3,2} = R_1(0,\infty) + r_2(b_2^*) = -r_3(k_3)$ and $k_3 \geq b_3^*$, the argument is the same for Corollary 1. The claim is then established by noting that since $k_3 < b_2^*$, we can find values of χ_1 and χ_2 to bind $\text{IC}'_{1,2}$ and $\text{IC}'_{2,3}$:

$$\chi_1 = \frac{\int_{k_3}^{\infty}(F_1(\omega) - F_4(\omega))d\omega}{\int_0^{\infty}(F_1(\omega) - F_4(\omega))d\omega}, \ \chi_2 = \frac{\int_{k_3}^{b_2^*}(F_1(\omega) - F_4(\omega))d\omega}{\int_0^{b_2^*}(F_1(\omega) - F_4(\omega))d\omega}.$$

By complementary slackness, the value of the Lagrangian function achieved by the proposed solution $x_1^*(\omega) = \chi_1^{[0,\infty)}$, $x_2^*(\omega) = \chi_2^{[0,b_2^*]}$, and $x_3^*(\omega) = \mathbb{1}_{\omega \geq k_3}$ is feasible in (MRP). It follows that the proposed solution solves (MRP), and thus corresponds to a solution to (P). ■

Appendix B. Proof of Lemma 4

(i) We establish the lemma for type L; the proof for type M is the same. We first show that there is a solution to (MRP) where $x_L^*(\omega)$ is piece-wise constant. Suppose instead that $x_L^*(\omega)$ is continuously strictly increasing for all $\omega \in (w', w'')$. We claim that the integrand of $\mathcal{L}(x_M, x_L; \lambda)$, given by

$$\phi_L \delta_L(\omega) f_L(\omega) - \lambda(F_L(\omega) - F_M(\omega))$$

is zero for all $\omega \in (w', w'')$. To see this, note that if it is strictly positive at some $\hat{w} \in (w', w'')$, we can find a neighborhood $(\hat{w} - \epsilon, \hat{w} + \epsilon)$ of $\hat{w}$ for some $\epsilon > 0$ such that the integrand is strictly positive for all $\omega \in (\hat{w} - \epsilon, \hat{w} + \epsilon)$. But then by changing $x_L^*(\omega)$ for all ω in the neighborhood to its highest value $x_L^*(\hat{w} + \epsilon)$ at $\hat{w} + \epsilon$ we can strictly increase the value of $\mathcal{L}(x_M, x_L; \lambda)$, which contradicts Luenberger's Theorem. A similar argument applies if the integrand is strictly negative at any $\omega \in (w', w'')$. By the Intermediate Value Theorem, there is a $\hat{w} \in (w', w'')$ such that

$$\int_{w'}^{w''} x_L^*(\hat{w})(F_L(\omega) - F_M(\omega))d\omega = \int_{w'}^{w''} x_L^*(\omega)(F_L(\omega) - F_M(\omega))d\omega.$$

Since the integrand is zero for all $\omega \in (w', w'')$, we have

$$\int_{w'}^{w''} x_L^*(\omega)\phi_L\delta_L(\omega)f_L(\omega)d\omega = \int_{w'}^{w''} x_L^*(\hat{w})\phi_L\delta_L(\omega)f_L(\omega)d\omega.$$

Thus, the value of the part of the objective function associated with $x_L^*(\omega)$ for $\omega \in (w', w'')$ is unchanged if we replace $x_L^*(\omega)$ with $x_L^*(\hat{w})$ for all $\omega \in (w', w'')$.

Next, we show that there is a solution where there is at most one intermediate value of $x_L^*(\omega)$ strictly between 0 and 1. Suppose instead that there exist w', $\hat{w}$ and w'', such that $x_L^*(\omega) = \chi$ for $\omega \in (w', \hat{w})$ and χ' for $\omega \in (\hat{w}, w'')$, with $x_L^*(w^-) < \chi < \chi' < x_L^*(w''^+)$. Consider changing $x_L^*(\omega)$ for $\omega \in (w', \hat{w})$ by some small amount ϵ, and simultaneously changing $x_L^*(\omega)$ for $\omega \in (\hat{w}, w'')$ by some other small amount ϵ', such that IC'_{LM} is unchanged:

$$\epsilon \int_{w'}^{\hat{w}} (F_L(\omega) - F_M(\omega))d\omega + \epsilon' \int_{\hat{w}}^{w''} (F_L(\omega) - F_M(\omega))d\omega = 0.$$

The change in the value of the objective function of (MRP) is given by

$$\epsilon \int_{w'}^{\hat{w}} \phi_L\delta_L(\omega)f_L(\omega)d\omega + \epsilon' \int_{\hat{w}}^{w''} \phi_L\delta_L(\omega)f_L(\omega)d\omega.$$

Since both $\epsilon > 0 > \epsilon'$ and $\epsilon < 0 < \epsilon'$ are feasible perturbations, and since $x_L^*(\omega)$ is optimal, we must have

$$R_L(w', \hat{w}) = R_L(\hat{w}, w'').$$

Then there is also a solution to (MRP) with one fewer intermediate value strictly between 0 and 1, by setting ϵ and ϵ' such that $\chi + \epsilon = \chi' + \epsilon'$.

(ii) By part (i) above, (MRP) always has a solution in the form of $(\chi_L^{[a,b]}, \chi_M^{[a',b']})$. Suppose that $a < b$, $a' < b'$, $\chi_L \in (0,1)$ and $\chi_M \in (0,1)$. Then, by Luenberger's Theorem, since $x_\theta^*(\omega)$ maximizes $\mathcal{L}(x_M, x_L; \lambda)$ among all non-decreasing $x_\theta(\cdot)$, for each $\theta = M, L$, we have the first order condition

$$\int_{a'}^{b'} (\phi_M \delta_M(\omega) f_M(\omega) + \lambda(F_L(\omega) - F_M(\omega)))\, d\omega = \int_a^b (\phi_L \delta_L(\omega) f_L(\omega) - \lambda(F_L(\omega) - F_M(\omega)))\, d\omega = 0.$$

The objective of (MRP) evaluated at the solution $(\chi_L^{[a,b]}, \chi_M^{[a',b']})$ is

$$\chi_M \int_{a'}^{b'} \phi_M \delta_M(\omega) f_M(\omega) d\omega + \int_{b'}^{\overline{\omega}} \phi_M \delta_M(\omega) f_M(\omega) d\omega + \chi_L \int_a^b \phi_L \delta_L(\omega) f_L(\omega) d\omega + \int_b^{\overline{\omega}} \phi_L \delta_L(\omega) f_L(\omega) d\omega.$$

If $\lambda = 0$ at the solution, then by the first order condition the objective function is independent of the values of χ_M and χ_L. We can change χ_M to 1, which keeps IC'_{LM} satisfied, because the allocation of type M is weakly increased for all ω. Thus, there is also a solution where the allocation for type M is deterministic.

If $\lambda > 0$, then by complementary slackness, IC'_{LM} is binding. By the first order condition, the objective function is again independent of the values of χ_M and χ_L. As a result, if we replace either χ_M or χ_L with 0 or 1, then so long as IC'_{LM} holds, the resulting allocations, which have randomization for at most one type, yield the same value for the objective function of (MRP). Since IC'_{LM} is binding, the set $[a, b] \cap [a', b']$ has a positive measure. Then, there are four cases we need to consider: i) $a \leq a' < b' \leq b$, ii) $a' \leq a < b' \leq b$, iii) $a' \leq a < b \leq b'$, and iv) $a \leq a' < b \leq b'$. For i), IC'_{LM} is satisfied with either $\tilde{\chi}_M = 1$ and

$$\tilde{\chi}_L = \frac{\int_{a'}^b (F_L(\omega) - F_M(\omega))\, d\omega}{\int_a^b (F_L(\omega) - F_M(\omega))\, d\omega} \in (0, 1]\,,$$

or $\tilde{\chi}_M = 0$ and

$$\tilde{\chi}_L = \frac{\int_{b'}^b (F_L(\omega) - F_M(\omega))\, d\omega}{\int_a^b (F_L(\omega) - F_M(\omega))\, d\omega} \in [0, 1)\,.$$

For case ii), IC'_{LM} is satisfied with $\tilde{\chi}_M = 0$ and

$$\tilde{\chi}_L = \frac{\int_{b'}^{b} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega}{\int_{a}^{b} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega} \in [0, 1)\,,$$

or $\tilde{\chi}_L = 1$ and

$$\tilde{\chi}_M = \frac{\int_{a}^{b'} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega}{\int_{a'}^{b'} \left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega} \in (0, 1]\,.$$

Case iii) is symmetric to case i), and case iv) is symmetric to case ii), both with roles of the types switched. The lemma follows immediately.

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