Unobserved Mechanism Design: Targeted Offers

Li, Hao and Michael Peters*

Vancouver School of Economics

University of British Columbia

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Abstract

In a standard auction problem where buyers have identically and independently distributed private valuations for an object, we consider a game of imperfect information in which buyers may not be informed of the commitments made by the seller of the object. There exist communicative equilibria in which uninformed buyers with low valuations signal their disinterest in an offer and allow the seller to target the take-it-or-leave-it offer to uninformed buyers who signal their interests. Informed buyers, who observe any deviation by the seller, make the seller's commitment to not making an offer to uninterested uninformed buyers credible. Under an assumption on the revenue function from a take-it-or-leave-it offer that is stronger than concavity, we rule out all other communicative equilibria.

*Li: hao.li@ubc.ca. Peters: peters@econ.ubc.ca. We are grateful to Social Sciences and Humanities Research Council of Canada for financial support.

1 Introduction

In our earlier paper Li and Peters (2021), we have introduced the idea of unobserved mechanism design in a standard environment where buyers all have identical independently distributed valuations but they have the same probability of being uninformed of the mechanism that the seller has committed to. In this game of imperfect informative, there exist uncommunicative equilibria where uninformed buyers babble. Under the assumption that the revenue function from a take-it-or-leave-it offer is concave, any equilibrium outcome corresponds to an optimal equal priority auction, in which informed buyers with all valuations on a pooling interval receive the good with the same probability as all uninformed buyers receive a take-it-or-leave-it offer.

In this follow-up paper, we consider communicative equilibria where uninformed buyers send messages to the seller's mechanism that are informative of their valuations. Since uninformed buyers would not be able to observe the seller's deviations, their communication strategy must be a best response to how the seller's equilibrium mechanism responds to the messages. In other words, the equilibrium communication strategy needs to be incentive compatible, which is the counterpart of incentive compatibility with respect to truthful reporting of valuations for informed buyers in a direct mechanism. We show that a strengthening of the concavity assumption on the revenue function is sufficient to imply that in any communicative equilibrium a message from uninformed buyers generates at most one offer, and all such offers are the same. The intuition is particularly simple, and does not rely on the strong concavity assumption, for the claim that if two messages generate only one offer, not only they have to be the same but they are also generate with the same probability. In equilibrium there must be uninformed buyers who send the message that generates each offer and who obtain a payoff of zero; otherwise the seller could increase the revenue by raising the offer in response to the message, without affecting the incentives or revenues from informed buyers or being detected by uninformed buyers. If the two offers are different, then uninformed buyers who send the message that generates the higher offer should deviate and send the message that generates the lower offer. If the two offers are the same, but one is generated with a greater probability than the other, then all uninformed buyers with valuations strictly above the offer should choose the message that generates it with a greater probability. In either case, the communication strategy would not be incentive compatible for uninformed buyers.

The only other possibility remaining is communicative equilibria in which one message generates an offer from the equilibrium mechanism while another generates no offers. This communication strategy by uninformed buyers can be informative of their valuations. We consider only the class of equilibria in which there is a threshold valuation such that uninformed buyers with valuations above the threshold send the message that generates an offer while those with valuations below the threshold send the message that generate no offers. We refer to the offer-generating message as "interested," and the other message as "uninterested." In equilibrium the offer to interested uninformed buyers exceeds the threshold valuation, and as a result, uninterested buyers are indifferent between the two messages. This means that a communication strategy with such two messages is incentive compatible for uninformed buyers. We show that for any sufficiently small cutoff, the seller's best response is a direct mechanism that combines an equal priority auction where informed buyers with all valuations on a pooling interval receive the good with the same probability as all interested uninformed buyers receive a take-it-or-leave-it offer, with no offers to uninterested uninformed buyers. Intuitively, any revenue from making an offer to uninterested buyers is too small relative to the incentive cost of inviting informed buyers with low valuations to pretend to be uninterested uninformed buyers. Of course, if all buyers are known to be uninformed, the seller would not be able to commit to not making an offer to uninterested buyers when there is no one interested. Thus, it is the presence of informed buyers who discipline the seller and give the seller necessary credibility to make targeted offers.

2 The Model

The model is the same as in Li and Peters (2021), and is presented here for the present paper to be self-containing. There are n potential buyers of a single object owned by a seller. Each buyer i has a privately known valuation v_i that is independently drawn from some distribution F with a continuously differentiable density function f on $[0, 1]$. The seller's reservation value for the object is zero. Both the buyers and the seller are risk-neutral.

The seller moves first and commits to a mechanism. The message space is $\mathcal{M} = [0,1]^2$, which embeds the support $[0,1]$ of valuations and the support $[0,1]$ of a randomization device, and is common knowledge among the buyers and the seller. Denote a typical message from buyer i as b_i. A mechanism maps a profile of n messages to a single take-it-or-leave-it offer. More precisely, for any profile of messages, the output of the mechanism is a profile of pairs (p_i, q_i), representing the probability of making buyer i an offer q_i and the offer itself p_i, with $\sum_{i=1}^{n} q_i \leq 1$. A pure strategy of the seller is a a mechanism $\gamma = \{\mathcal{M}, p_i, q_i\}_{i=1}^{n}$. Let Γ be the set of all mechanisms. We will allow the seller to randomize over mechanisms.

With probability $\alpha \in (0,1)$, independent their valuation, each buyer privately draws the information type "uninformed," representing that they do not observe the mechanism already committed by the seller. With the residual probability $1-\alpha$, the buyer is "informed," meaning they observe the mechanism. We denote uninformed type as $\tau = \mu$ and the informed type as $\tau = \epsilon$. A pure strategy σ_i for each buyer i is a function $\sigma_i : [0,1] \times \{\epsilon, \mu\} \times \Gamma \to \mathcal{M}$, satisfying the informational constraint

$$\sigma_i (v_i, \mu, \gamma) = \sigma_i (v_i, \mu, \gamma') = \sigma_i (v_i, \mu)$$

for all $\gamma, \gamma' \in \Gamma$. Throughout the paper we restrict buyers to use pure strategies.

3 Direct Mechanisms

We look for symmetric perfect Bayesian equilibria of the game. Li and Peters (2021) consider uncommunicative equilibria in which uninformed buyers babble, or equivalently, choose a single message from $\mathcal{M}$. To characterize equilibrium outcomes, they define direct mechanisms and show that a version of the revelation principle holds, so that any equilibrium outcome can be supported via an optimal direct mechanism, and conversely, an optimal mechanism characterizes an equilibrium outcome. In this section, we extend the revelation principle to communicative equilibria.

For now we consider only communicative equilibria in which uninformed buyers use a finite strategy, represented by a mapping $\sigma^\mu : [0,1] \to \mathcal{M}^\mu$, where $\mathcal{M}^\mu$ is a finite subset of $\mathcal{M}$. Without loss we assume that every message $b \in \mathcal{M}^\mu$ is used with a positive probability. Since we restrict buyers to use pure strategies, any such strategy induces a finite partition of the support of valuations $[0,1]$. All strategies with the same partition are equivalent.

Let v^ϵ be the profile of $n-m$ valuations of informed buyers, and let b^μ be the profile of m messages of uninformed buyers, where m is the number of uninformed buyers. Reorder n buyers such that the first $n-m$ of them are informed. For each $v = (v_1, \ldots, v_n) \in [0,1]^n$, and for each $i = 1, \ldots, n-m$, let

$$\rho_i^\epsilon(v^\epsilon) = (v_i, v_2, \ldots, v_{i-1}, v_1, v_{i+1}, \ldots, v_{n-m}, v_{n-m+1}, \ldots, v_n\},$$

and for each $j = n-m+1, \ldots, n$, let

$$\rho_j^\mu(b^\mu) = (b_n, b_{n-m+2}, \ldots, b_{n-1}, \ldots, b_j\}.$$

A feasible direct mechanism δ is a collection of functions

$$\delta = \left\{ \left(q_m^\epsilon(v^\epsilon; b^\mu), p_m^\epsilon(v^\epsilon; b^\mu)\right)_{m=0}^{n-1}, \left(q_m^\mu(v^\epsilon; b^\mu), p_m^\mu(v^\epsilon; b^\mu)\right)_{m=1}^{n} \right\}$$

where $q_m^\tau(v^\epsilon; b^\mu)$ and $p_m^\tau(v^\epsilon; b^\mu)$ for each $\tau = \epsilon, \mu$ map $[0,1]^{n-m} \times (\mathcal{M}^\mu)^m \to [0,1]$, and satisfy

- $(q_m^\tau(v^\epsilon; b^\mu), p_m^\tau(v^\epsilon; b^\mu))$ for each $\tau = \epsilon, \mu$ is variant to permutations of $(v_2, \ldots, v_{n-m})$, and to permutations of $(b_{n-m+1}, \ldots, b_{n-1})$;

- for all v^ϵ and b^μ,

$$\sum_{i=1}^{n-m} q_m^\epsilon\left(\rho_i^\epsilon\left(v^\epsilon\right); b^\mu\right) + \sum_{j=n-m+1}^{n} q_m^\mu(v^\epsilon; \rho_j^\mu(b^\mu)) \leq 1. \tag{1}$$

Fix a strategy of uninformed buyers σ^μ, and let $\sigma^\mu(v^\mu)$ be the profile of messages from uninformed buyers when the profile of their valuations is v^μ. Given any direct mechanism δ, for each $m = 0, \dots, n-1$ define

$$Q^{\epsilon}_{m}(w;\sigma^{\mu}) = \mathbb{E}_{v^{\epsilon}_{-1};v^{\mu}}\left[q^{\epsilon}_{m}(w, v^{\epsilon}_{-1};\sigma^{\mu}(v^{\mu}))\right],$$

where (w, v^{ϵ}_{-1}) is the profile of valuations of informed buyers, with $v_1 = w$. Similarly,

$$P^{\epsilon}_{m}(w;\sigma^{\mu}) = \mathbb{E}_{v^{\epsilon}_{-1};v^{\mu}}\left[q^{\epsilon}_{m}(w, v^{\epsilon}_{-1};\sigma^{\mu}(v^{\mu}))p^{\epsilon}_{m}(w, v^{\epsilon}_{-1};\sigma^{\mu}(v^{\mu}))\right].$$

Taking expectations over m, we define

$$Q^{\epsilon}(w;\sigma^{\mu}) = \sum_{m=0}^{n-1} B(m;n-1,\alpha)Q^{\epsilon}_{m}(w;\sigma^{\mu})$$
$$P^{\epsilon}(w;\sigma^{\mu}) = \sum_{m=0}^{n-1} B(m;n-1,\alpha)P^{\epsilon}_{m}(w;\sigma^{\mu}),$$

where

$$B(m;n-1,\alpha) = \binom{n-1}{m}(1-\alpha)^{n-1-m}\alpha^{m}$$

is the probability that m out of $n-1$ buyers are uninformed. Let

$$U^{\epsilon}(w;\sigma^{\mu}) = Q^{\epsilon}(w;\sigma^{\mu})\,w - P^{\epsilon}(w;\sigma^{\mu}).$$

Then, for fixed σ^{μ}, we say that the mechanism δ is σ^{μ}-incentive compatible for informed buyers with respect to valuations, if the payoff to an informed buyer with valuation w can be written as

$$U^{\epsilon}(w;\sigma^{\mu}) = \int_{0}^{w} Q^{\epsilon}(x;\sigma^{\mu})\,dx, \tag{2}$$

with $Q^{\epsilon}(w;\sigma^{\mu})$ non-decreasing in w.

The payoff $U^{\mu}(w, b;\sigma^{\mu})$ to an uninformed buyer with valuation w from using any message $b \in \mathcal{M}^{\mu}$, when all other uninformed buyers use the strategy σ^{μ}, is

$$\sum_{m=0}^{n-1} B(m;n-1,\alpha)\mathbb{E}_{v^{\epsilon};v^{\mu}_{-n}}\left[q^{\mu}_{m+1}(v^{\epsilon};\sigma^{\mu}_{-n}(v^{\mu}_{-n}), b)\max\left\{w - p^{\mu}_{m+1}(v^{\epsilon};\sigma^{\mu}_{-n}(v^{\mu}_{-n}), b), 0\right\}\right], \tag{3}$$

where v^{μ}_{-n} is the profile of valuations of other m uninformed buyers, and $(\sigma^{\mu}_{-n}(v^{\mu}_{-n}), b)$ is the profile of messages from all other uninformed buyers and the present uninformed buyer with valuation w, with $b_n = b$. This is the same payoff $U^{\epsilon}(w, b; \sigma^{\mu})$ received by an informed buyer with valuation w from using the same message b. We say that δ is σ^{μ}-incentive compatible for uninformed buyers if for all w,

$$U^{\mu}\left(w, \sigma^{\mu}(w); \sigma^{\mu}\right) \geq U^{\mu}\left(w, b; \sigma^{\mu}\right)$$

for all $b \in \mathcal{M}^{\mu}$.

We say that δ is σ^{μ}-incentive compatible for informed buyers if for all w, if δ is incentive compatible for informed buyers with respect to valuations and if

$$U^{\epsilon}(w; \sigma^{\mu}) \geq U^{\mu}\left(w, b; \sigma^{\mu}\right)$$

for all $b \in \mathcal{M}^{\mu}$. Finally, δ is σ^{μ}-incentive compatible if it is σ^{μ}-incentive compatible for both informed and uninformed buyers.

Define

$$\phi(w) = w - \frac{1 - F(w)}{f(w)}$$

as the virtual valuation of w. The seller's revenue from informed buyers under any incentive compatible mechanism is given by

$$\begin{aligned}
&\sum_{m=0}^{n-1} B(m; n, \alpha) \mathbb{E}_{v^{\epsilon}; v^{\mu}} \left[\sum_{i=1}^{n-m} p^{\epsilon}_{m}(\rho^{\epsilon}_{i}(v^{\epsilon}); \sigma^{\mu}(v^{\mu})) \right] \\
=&\sum_{m=0}^{n-1} B(m; n, \alpha)(n - m) \mathbb{E}_{w} \left[P^{\epsilon}_{m}(w; \sigma^{\mu}) \right] \\
=&n(1 - \alpha) \mathbb{E}_{w} \left[Q^{\epsilon}\left(w; \sigma^{\mu}\right) w - U^{\epsilon}\left(w; \sigma^{\mu}\right) \right] \\
=&n(1 - \alpha) \int_{0}^{1} Q^{\epsilon}\left(w; \sigma^{\mu}\right) \phi(w) f(w) dw, \qquad (4)
\end{aligned}$$

where the last step uses the incentive compatibility with respect to valuations for informed buyers and integration by parts, and then imposes $U^{\epsilon}(0; \sigma^{\mu}) = 0$.

The seller's revenue from uninformed buyers is given by

$$\sum_{m=1}^{n} B(m;n,\alpha)\mathbb{E}_{v^\epsilon;v^\mu}\left[\sum_{j=n-m+1}^{n} q_m^\mu(v^\epsilon;\rho_j^\mu(\sigma^\mu(v^\mu)))p_m^\mu(v^\epsilon;\rho_j^\mu(\sigma^\mu(v^\mu)))\mathbb{1}_{v_j\geq p_m^\mu(v^\epsilon;\rho_j^\mu(\sigma^\mu(v^\mu)))}\right]$$

$$=\sum_{m=1}^{n} B(m;n,\alpha)m\mathbb{E}_w\Big[\mathbb{E}_{v^\epsilon;v^\mu_{-n}}\big[q_m^\mu(v^\epsilon;\sigma^\mu_{-n}(v^\mu_{-n}),\sigma^\mu(w))\cdot$$

$$p_m^\mu(v^\epsilon;\sigma^\mu_{-n}(v^\mu_{-n}),\sigma^\mu(w))\mathbb{1}_{w\geq p_m^\mu(v^\epsilon;\sigma^\mu_{-n}(v^\mu_{-n}),\sigma^\mu(w))}\big]\Big]$$

$$=n\alpha\mathbb{E}_w\left[\sum_{m=0}^{n-1} B(m;n-1,\alpha)\mathbb{E}_{v^\epsilon;v^\mu_{-n}}\Big[q_{m+1}^\mu(v^\epsilon;\sigma^\mu_{-n}(v^\mu_{-n}),\sigma^\mu(w))\cdot\right.$$

$$\left.p_{m+1}^\mu(v^\epsilon;\sigma^\mu_{-n}(v^\mu_{-n}),\sigma^\mu(w))\mathbb{1}_{w\geq p_{m+1}^\mu(v^\epsilon;\sigma^\mu_{-n}(v^\mu_{-n}),\sigma^\mu(w))}\Big]\right]. \quad (5)$$

The seller's total revenue $R(\delta)$ from a direct mechanism δ when uninformed buyers use strategy σ^μ is the sum of (4) and (5).

We can now state a revelation principle as follows:

Theorem 1 *Fix a strategy σ^μ of uninformed buyers. For any symmetric equilibrium in which uninformed buyers use σ^μ, there is a feasible, σ^μ-incentive compatible direct mechanism δ^* that achieves the equilibrium expected revenue for the seller and $R(\delta^*) \geq R(\delta)$ for every σ^μ-incentive compatible direct mechanism δ. Conversely, any feasible, σ^μ-incentive compatible direct mechanism δ^* that maximizes $R(\delta)$ can be used to construct an equilibrium in which uninformed buyers use σ^μ.*

For a fixed communication strategy σ^μ of uninformed buyers, Theorem 1 reduces the problem of finding symmetric equilibrium outcomes to an optimal mechanism design problem. There are at least three differences between this optimization problem from a standard mechanism design problem. First, in addition to the standard truthful reporting of valuations, incentive compatibility constraints of informed buyers require them not to deviate to any message used by an uninformed buyer. Second, incentive compatibility constraints of uninformed buyers are based on their inferences about the equilibrium mechanism, because they do not any deviation by the seller. Third, the seller also faces the incentive compatibility constraint that there are no profitable deviations that take advantage of uninformed buyers.

4 Multiple Offers

In this section, we consider equilibria where uninformed buyers use some strategy σ^μ with at least one message that generates at least two offers. We refer to them as communicative equilibria with multiple offers. We will show that such equilibria do not exist under an assumption on the monopoly revenue function $\pi(\cdot)$, defined as

$$\pi(w) = w(1 - F(w)).$$

Fix some μ. By Theorem 1, any equilibrium outcome is associated with an optimal direct mechanism

$$\left\{ (q_m^\epsilon(v^\epsilon; b^\mu), p_m^\epsilon(v^\epsilon; b^\mu))_{m=0}^{n-1}, (q_m^\mu(v^\epsilon; b^\mu), p_m^\mu(v^\epsilon; b^\mu))_{m=1}^{n} \right\}.$$

From (3), we say that a message b generates an offer t if the probability $\theta(t)$, defined below, is strictly positive:

$$\theta(t) \equiv \sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_{v^\epsilon; v_{-n}^\mu} \left[q_{m+1}^\mu(v^\epsilon; \sigma_{-n}^\mu(v_{-n}^\mu), b) \,\middle|\, p_{m+1}^\mu(v^\epsilon; \sigma_{-n}^\mu(v_{-n}^\mu), b) = t \right]. \quad (6)$$

Given the above definition of $\theta(t)$, by the seller's revenue from uninformed buyers (5), the total probability that the seller's equilibrium mechanism responds to an equilibrium message b by uninformed buyers with an offer t is

$$n\alpha\theta(t) \int_{\{w | \sigma^\mu(w) = b\}} dF(w). \quad (7)$$

For now we make the simplifying assumption that the support $\mathcal{M}^\mu$ of σ^μ is finite. Then, given δ and σ^μ, the set of offers t generated by all messages in $\mathcal{M}^\mu$ is also finite.

Suppose that there are two messages, b and $\tilde{b}$, that are in the support of σ^μ and generate at least one offer. For notational simplicity, denote the l distinct offers generated by b as $t^1 < \ldots < t^l$, with corresponding positive probabilities $\theta^1, \ldots, \theta^l$. Similarly, let the $\tilde{l}$ offers generated by $\tilde{b}$ be $\tilde{t}^1 < \ldots < \tilde{t}^{\tilde{l}}$, with corresponding positive probabilities $\tilde{\theta}^1, \ldots, \tilde{\theta}^{\tilde{l}}$. We first show that in any equilibrium $t^j = \tilde{t}^j$ and $\theta^j = \tilde{\theta}^j$ for each $j = 1, \ldots, \min\{l, \tilde{l}\}$, so the two offer distributions coincide for the lowest offers.

Lemma 1 *For any $b, \tilde{b} \in \mathcal{M}^\mu$ that each generate at least one offer, there exists an offer τ generated by both b and $\tilde{b}$ such that the two offer distributions are the same for all offers $\tau' \leq \tau$.*

Proof. Consider t^j for any $j = 1, \ldots, l$. By construction, t^j is a serious offer, meaning that there exists valuation w of an uninformed buyer i such that for whom $\sigma^\mu(w) = b$ and $w > t^j$. Let

$$\beta(t^j) = \inf \left\{ w : \sigma^\mu(w) = b \text{ and } w > t^j \right\}.$$

By definition, we have $\beta(t^j) \geq t^j$. We claim that $\beta(t^j) = t^j$. If this were false, the seller could raise the revenue by increasing his price offer t^j on a set of positive measure, contradicting the equilibrium condition imposed by Theorem 1 that the seller's mechanism is a best response to σ^μ. The same argument establishes that for any $j = 1, \ldots, \tilde{l}$,

$$\tilde{\beta}(\tilde{t}^j) \equiv \inf \left\{ w : \sigma^\mu(w) = \tilde{b} \text{ and } w \geq \tilde{t}^j \right\} = \tilde{t}^j.$$

Now, compare t^j and $\tilde{t}^j$, and θ^j and $\tilde{\theta}^j$. If $t^j > \tilde{t}^j$, since t^j is the lowest offer generated by b, an uninformed buyer i with valuation $\beta(t^j) = t^j$ receives an expected payoff of zero by sending message b, but by deviating and sending message $\tilde{b}$, he would receive a strictly positive payoff of $\tilde{\theta}^j(t^j - \tilde{t}^j)$. This contradicts the equilibrium condition the equilibrium condition imposed by Theorem 1 that the seller's equilibrium mechanism is σ^μ-incentive compatible for uninformed buyers. Thus, $t^j = \tilde{t}^j$. If $\theta^j < \tilde{\theta}^j$, then given $t^j = \tilde{t}^j$, an uninformed buyer i with any valuation strictly between t^j and $\min\{t^{j+1}, \tilde{t}^{j+1}\}$ strictly prefers sending message b to sending message $\tilde{b}$, contradicting the result that $\beta(t^j) = t^j$. Thus, $\theta^j = \tilde{\theta}^j$. By induction, $t^j = \tilde{t}^j$ and $\theta^j = \tilde{\theta}^j$ for each $j = 1, \ldots, \min\{l, \tilde{l}\}$. ■

The above lemma implies that if all messages lead to the same number of offers, they will all have the same offer distribution. The seller does not discriminate among uninformed buyers by their messages, even though the valuation distribution of an uninformed buyer conditional on a message are different across messages. In other words, messages by uninformed buyers can be informative about their valuations, but the seller may be prevented from exploiting the information content because uninformed buyers have correct equilibrium beliefs about the offer distributions without observing the seller's mechanism.

Now we rule out all communicative equilibria where messages in the support of the communication strategy σ^μ generate two or more offers. The following result assume that $wf(w)$ is non-decreasing in w. This is stronger than the assumption that the monopoly revenue function $\pi(w)$ is strictly concave.

Theorem 2 *Suppose $wf(w)$ is non-decreasing in w. There is no equilibrium in which uninformed buyers use a finite number of messages and any one of them generates multiple offers.*

Proof. We already know that if two messages in the support of σ^μ lead to the same number of offers, they have the same offer distribution. Without loss we can assume that the equilibrium messages of uninformed buyers are strictly ranked in increasing order by the number of offers they generate.

We first show that there is no equilibrium in which the numbers of offers generated by two messages ranked next to each other differ by 2 or greater. Suppose b and $\tilde{b}$ are two such messages, and t^j and t^{j+1} are two offers generated by b but not by $\tilde{b}$. Since b and $\tilde{b}$ are ranked next to each other, t^j and t^{j+1} are generated by all messages ranked higher than b and not by any message ranked lower than $\tilde{b}$.

Let θ^j and θ^{j+1} be the probability of b generating t^j and t^{j+1} respectively. Consider another direct mechanism by increasing t^j marginally and decreasing t^{j+1} marginally so that the expected payoff $U^\epsilon(w, b; \sigma^\mu)$ for an informed buyer with any valuation $w > t^{j+1}$ from pretending to be uninformed by using b is unchanged. We have

$$dU^\epsilon(w, b; \sigma^\mu) = -\theta^j dt^j - \theta^{j+1} dt^{j+1} = 0.$$

The above condition ensures the changes have no effect on the incentive compatibility of informed buyers with respect to b.

By (7), the seller's total probability of making an offer t^j to an uninformed buyer who sends the message b is

$$n\alpha\theta^j \int_{\{w|\sigma^\mu(w)=b\}} dF(w).$$

The corresponding expressions for t^{j+1} are similarly given. Define

$$\pi(t|b) \equiv t \int_{\{w \geq t | \sigma^{\mu}(w)=b\}} dF(w)$$

as the seller's "conditional revenue function" from making offer t^j in response to message b. Then, given by how we have constructed dt^j and dt^{j+1} to satisfy the incentive compatibility constraint of informed buyers, from (5), (6) and (7) we have that the total effects on the seller's expected revenue of marginally increasing t^j and marginally decreasing t^{j+1} with respect to b can be written as

$$n\alpha\theta^j \int_{\{w|\sigma^{\mu}(w)=b\}} dF(w) \left(\frac{d\pi(t^j|b)}{dt} - \frac{d\pi(t^{j+1}|b)}{dt} \right) dt^j.$$

By construction, no message in the support of σ^{μ} ranked lower than b generates t^j and t^{j+1}, while all messages ranked higher than b generate t^j and t^{j+1} with the same probabilities θ^j and θ^{j+1}. For each message ranked higher than b, consider marginally increasing t^j and decreasing t^{j+1} such that the deviation payoff of an informed buyer with valuation $w > t^{j+1}$ who uses this message is unchanged, and derive a similar expression for the impact on the seller's expected revenue of these changes. Then, summing over all messages in the support of σ^{μ}, we have the total effects as

$$n\alpha\theta^j \left(\frac{d\pi(t^j)}{dt} - \frac{d\pi(t^{j+1})}{dt} \right) dt^j.$$

Since $\pi(\cdot)$ is strictly concave when $vf(v)$ is non-decreasing, $t^j < t^{j+1}$ and $dt^j > 0$, the above is strictly positive. This contradicts the equilibrium condition from Theorem 1 that the seller's equilibrium mechanism is optimal given uninformed buyers' strategy σ^{μ}.

Thus, in any equilibrium with a finite number l of messages in the support of σ^{μ}, there are exactly l offers generated for uninformed buyers and we can order the messages such that, for each $j = 1, \ldots, l$, message b^j generates offers $t^1, \ldots, t^j$. Now we show that $l = 1$ in equilibrium. Suppose not and consider messages b^{l-1} and b^l. By construction, offer t^l is only generated by b^l, with probability denoted as θ^l, and offer t^{l-1} is only generated by message b^{l-1} and b^l, with the same probability, denoted as θ^{l-1}. Consider marginally increasing t^{l-1}

and marginally decreasing t^l such that the expected payoff $U^\epsilon(w, b; \sigma^\mu)$ for an informed buyer with any valuation $w > t^l$ from pretending to be uninformed by using b^l is unchanged. This requires:

$$dU^\epsilon(w, b; \sigma^\mu) = -\theta^{l-1}dt^{l-1} - \theta^l dt^l = 0.$$

The effect on the seller's revenue from offers t^l and t^{l-1} to an uninformed buyer sending message b^l is given by

$$n\alpha\theta^{l-1} \int_{\{w|\sigma^\mu(w)=b^l\}} dF(w) \left(\frac{d\pi(t^{l-1}|b^l)}{dt} - \frac{d\pi(t^l|b^l)}{dt} \right) dt^{l-1}.$$

Using the definition of conditional revenue function $\pi(t|b^l)$, we have, for each $t = t^{l-1}, t^l$,

$$\frac{d\pi(t|b^l)}{dt} = \int_{\{w\geq t|\sigma^\mu(w)=b^l\}} dF(w) + t\frac{d}{dt} \int_{\{w\geq t|\sigma^\mu(w)=b^l\}} dF(w).$$

Since offer t^{l-1} is generated by both message b^l and b^{l-1}, there is a positive measure of uninformed buyers with valuations above t^{l-1} that use each of the two messages, and so

$$\frac{d}{dt} \int_{\{w\geq t^{l-1}|\sigma^\mu(w)=b^l\}} dF(w) > -\frac{f(t^{l-1})}{\int_{\{w|\sigma^\mu(w)=b^l\}} dF(w)}.$$

On the other hand, since offer t^l is generated only by message b^l, uninformed buyers with valuations above t^l in equilibrium use b^l exclusively, and so

$$\frac{d}{dt} \int_{\{w\geq t^l|\sigma^\mu(w)=b^l\}} dF(w) = -\frac{f(t^l)}{\int_{\{w|\sigma^\mu(w)=b^l\}} dF(w)}.$$

Since $t^{l-1} < t^l$ and $wf(w)$ is non-decreasing in w, we have

$$\frac{d\pi(t^{l-1}|b^l)}{dt} - \frac{d\pi(t^l|b^l)}{dt} > \int_{\{t^{l-1}\leq w\leq t^l|\sigma^\mu(w)=b^l\}} dF(w) > 0.$$

Thus, the total effect on the seller's expected revenue of the marginal changes in t^{l-1} and t^l with respect to b^l is strictly positive. This contradicts the equilibrium condition from Theorem 1 that the seller's equilibrium mechanism is optimal given uninformed buyers' strategy σ^μ. ■

The proof of Theorem 2 is divided into two parts. In the first part, we only need the weaker assumption that $\pi(w)$ is strictly concave. Thus, with the weaker assumption, we already have that there is no equilibrium in which all equilibrium messages of uninformed buyers generate the same two or more offers. The second part uses the stronger assumption of that $wf(w)$ is non-decreasing to rule out all equilibria in which some messages in the support of σ^μ generate two or more offers. Putting the two parts together, we conclude that there is no equilibrium in which every message from uninformed buyers generates at least one offer.

5 Single Targeted Offers

By Theorem 2, under the stronger concavity assumption, the only possibility of communicative equilibria is that there are only two messages used by uninformed buyers, one generating a single offer while the other generating no offers. We refer to these two messages as "interested," denoted as b^1, and "uninterested," denoted as b^0. Such equilibrium is necessarily informative, because uninformed buyers with valuations above the offer generated by b^1 strictly prefer b^1 to b^0. Since an interested uninformed weakly prefers b^1 to b^0, and an uninterested buyer is indifferent between the two messages, the incentive compatibility constraints for uninformed buyers are always satisfied.

In this section we consider a particular class of such equilibria, where the strategy σ^μ of uninformed buyers takes a threshold form, with $\sigma^\mu(w) = b^1$ for $w \geq \xi$ and $\sigma^\mu(w) = b^0$ for $w < \xi$ for some threshold $\xi \in (0, 1)$. We will refer to them as communicative equilibria with targeted offers, because in equilibrium the seller commits to mechanism that targets take-it-or-leave-it offers to only interested uninformed buyers, i.e., those with valuations above ξ who in equilibrium signal their interests in receiving an offer with message b^1. When $\xi = 0$, the equilibrium degenerates into the uncommunicative equilibrium constructed in Li and Peters (2021). For any $\xi > 0$, however, since the commitment to not making an offer to uninformed buyers sending b^0 is unobservable to uninformed buyers, the seller faces the temptation to deviate and modify the mechanism by making an offer to uninformed buyers when this is the only profitable option. We show by construction that it is informed buyers who discipline the seller and ensure that the seller does not deviate.

We construct a communicative equilibrium for each ξ positive and sufficiently small. To save notation, we will not index the variables with ξ, and we drop σ^{μ} from all relevant expressions. By Theorem 1, it is necessary and sufficient to find an optimal direct mechanism given the threshold strategy σ^{μ}. Based on the insights from Li and Peters (2021), we will construct an equal priority auction, in which informed buyer with valuations on some pooling interval have the same allocation priority as interested uninformed, with no offers made to uninterested uninformed buyers. The construction takes two steps. First, we characterize the optimal equal priority auction, conditional on not making an offer to uninterested uninformed buyers. Second, we show that it is an optimal direct mechanism given σ^{μ}; in particular, it is optimal not to make an offer to uninterested uninformed buyers, even when it is the only option.

5.1 A conditionally optimal equal priority auction

As in Li and Peters (2021), an equal priority auction consists of a pooling interval for informed buyers $[v_{-}, v_{+}]$, a take-it-or-leave-it offer t to interested uninformed buyers, and a reserve price r for informed buyers when there are no interested uninformed buyers, satisfying $r \leq t \leq v_{-} \leq v_{+}$. Conditional on not making an offer to uninterested buyers, the allocation rules and offers are the same as in Li and Peters (2021). It suffices here to specify the trading probabilities for informed buyers and interested uninformed buyers, with the transfer rule for informed buyers determined through standard arguments of incentive compatibility with respect to valuations.

Let m be the number of uninformed buyers, including both interested and uninterested uninformed buyers. Among them, let m^1 be the number of interested uninformed buyers. The probability of trade function $Q^{\epsilon}(\cdot)$ for an informed buyer is calculated as follows. For $w < r$,

$$Q^{\epsilon}(w) = 0.$$

For $w \in [r, v_-)$, we have

$$Q^\epsilon(w) = \begin{pmatrix} n-1 \\ m \end{pmatrix} ((1-\alpha)F(w))^{n-1-m}(\alpha F(\xi))^m = ((1-\alpha)F(w) + \alpha F(\xi))^{n-1}.$$

For $w > v_+$,

$$Q^\epsilon(w) = \begin{pmatrix} n-1 \\ m \end{pmatrix} ((1-\alpha)F(w))^{n-1-m}(\alpha F(\xi))^m = ((1-\alpha)F(w) + \alpha F(\xi))^{n-1}.$$

Finally, for $w \in [v_-, v_+]$,

$$Q^\epsilon(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+) \sum_{m^1=0}^{m} B_{m_1}^m \frac{1}{m^1 + k + 1},$$

where

$$B_k^{n-1-m}(v_-, v_+) = \begin{pmatrix} n-1-m \\ k \end{pmatrix} (F(v_+) - F(v_-))^k F^{n-1-m-k}(v_-)$$

is the probability that k out of $n-1-m$ informed buyers have valuations in the interval $[v_-, v_+]$, and

$$B_{m^1}^m = \begin{pmatrix} m \\ m^1 \end{pmatrix} (1 - F(\xi))^{m^1} F^{m-m^1}(\xi)$$

is the probability that m^1 out of m uninformed buyers send message b^1. We denote $Q^\epsilon(w)$ in this case as $\chi(v_-, v_+)$, and provide a convenient formula for it in the following lemma. The proof is in the appendix.

Lemma 2 *For $w \in [v_-, v_+]$, the allocation probability $Q^\epsilon(w)$ under an equal priority auction (r, t, v_-, v_+) is*

$$\chi(v_-, v_+) = \frac{((1-\alpha)F(v_+) + \alpha)^n - ((1-\alpha)F(v_-) + \alpha F(\xi))^n}{n((1-\alpha)(F(v_+) - F(v_-)) + \alpha(1 - F(\xi)))}.$$

Given the above allocation function $Q^\epsilon(w)$, the payoff for an informed buyer is

$$U^\epsilon(w) = \int_0^w Q^\epsilon(x)dx,$$

and the revenue from informed buyers is

$$n(1-\alpha)\int_r^1 Q^\epsilon(w)f(w)dw.$$

By construction, the probability of trade for interested uninformed buyers is $\chi(v_-, v_+)$. The payoff for an interested uninformed buyer is

$$U^\mu(w, b^1) = \chi(v_-, v_+)\max\{w-t, 0\},$$

and the seller's revenue from interested buyers is

$$n\alpha\chi(v_-, v_+)\pi(t).$$

The incentive condition for an informed buyer not to pretend to be an interested uninformed buyer is

$$\int_0^w Q^\epsilon(x)dx \geq \chi(v_-, v_+)\max\{w-t, 0\}$$

for all $w \in [0,1]$. Since by construction $U^\epsilon(w)$ is strictly convex for $w \in [r, v_-]$ and for $w \geq v_+$ and $U^\mu(w, b^1)$ is piece-wise linear, the above is satisfied for all $w \in [0,1]$ if and only if it is satisfied at $w = v_-$:

$$\int_r^{v-} Q^\epsilon(x)dx \geq \chi(v_-, v_+)\max\{v_- - t, 0\} \tag{8}$$

The following lemma characterizes the optimal equal-priority auction $\{r, t, v_-, v_+\}$ with $0 \leq r \leq t \leq v_- \leq v_+ \leq 1$ that maximizes the sum of the seller's revenue from informed and uninformed buyers subject to the incentive compatibility constraint (8), conditional on not making an offer to uninterested uninformed buyers. We refer to it as ξ-conditionally optimal equal-priority auction. The proof is in the appendix. We assume that the revenue function

π is strictly concave, with $r^* \in (0,1)$ the unique maximizer.

Lemma 3 *Suppose that $\pi(\cdot)$ is strictly convex, and σ^μ is a threshold strategy for uninformed buyers with threshold ξ. If an equal priority auction $\{r, t, v_-, v_+\}$ is ξ-conditionally optimal, then (8) binds, and $r < r^* < t < v_- < v_+$. Further, for sufficiently small ξ, we have $r > \xi$ and $v_+ < 1$, and*

$$\alpha(\pi(t) - (1 - F(\xi))\phi(v_+)) = (1 - \alpha)\Big((v_- - t)(\phi(v_+) - \phi(v_-))f(v_-) + \int_{v_-}^{v_+} f(w)(\phi(v_+) - \phi(w))dw\Big); \tag{9}$$

$$\phi(r)f(r) + (\phi(v_+) - \phi(v_-))f(v_-) = 0; \tag{10}$$

$$\alpha\pi'(t) + (1 - \alpha)(\phi(v_+) - \phi(v_-))f(v_-) = 0. \tag{11}$$

5.2 Optimal mechanism

The main result in this section is that for ξ sufficiently small, the optimal priority auction characterized by Lemma 3 conditional on not making an offer to uninterested uninformed buyers, is in fact an optimal direct mechanism. In particular, for ξ sufficiently small, it is optimal for the seller not to make an offer to uninterested buyers, even when it is the only profitable option. The reason for this is that the incentive cost of preventing informed buyers with low valuations from claiming to be uninterested buyers outweighs any revenue gain from making such an offer.

We establish the main result using the Lagrangian relaxation approach that captures the above intuition. The rough idea is as follows. We construct two non-negatively valued multiplier functions, one for the continuum of incentive compatibility constraints that for each valuation $w \in [0,1]$ an informed buyer's expected payoff $U^\epsilon(w)$ from truthfully reporting w is greater than or equal to the deviation payoff $U^\mu(w, b^1)$ from pretending to be an interested uninformed buyer, and another for the continuum of incentive compatibility constraints that $U^\epsilon(w)$ is greater than or equal to the deviation payoff $U^\mu(w, b^0)$ from pretending to be an uninterested uninformed buyer. These two multiplier functions, denoted as $\lambda^1(w)$ and $\lambda^0(w)$

respectively, are chosen to ensure that the direct mechanism constructed from the optimal equal-priority auction $\{r, t, v_-, v_+\}$ characterized by Lemma 3 is a point-wise maximizer of the Lagrangian. In particular, as in Li and Peters (2021), for each $w \in [v_-, v_+]$, a properly weighted sum of the incentive benefit $\lambda^1(w)$ and the revenue $\phi(w)$ from increasing the allocation for an informed buyer with valuation w is equal to a weighted sum of the total incentive cost $\int_0^1 \lambda^1(x)dx$ and the revenue $\pi(t)$ from increasing the allocation for interested uninformed buyers. As a result, it maximizes the Lagrangian to give an equal priority to informed buyers with valuations on $[v_-, v_+]$ and interested uninformed buyers. At the same time, for each $w \in [\xi, r]$, both a weighted sum of the incentive benefit $\lambda^0(w)$ and the revenue $\phi(w)$ from increasing the allocation for an informed buyer with valuation w, and a weighted sum of the incentive cost $\int_0^1 \lambda^0(x)dx$ and the revenue from increasing the allocation for uninterested uninformed buyers, are both non-positive. This implies that it maximizes the Lagrangian not to give an offer to informed buyers with valuations below r or uninterested uninformed buyers. Further, $\lambda^1(w) = 0$ for all w outside $[v_-, v_+]$ and $\lambda^0(w) = 0$ for all w outside $[\xi, r]$. Under the optimal equal priority auction $\{r, t, v_-, v_+\}$ characterized by Lemma 3, informed buyers with valuations on $[v_-, v_+]$ are indifferent between participating in the auction and pretending to be interested uninformed buyers, while those with valuations on $[\xi, r]$ are also indifferent between participating in the auction and the pretending to be uninterested uninformed buyers as they receive a payoff of zero either way. By complementary slackness, the direct mechanism constructed from the optimal equal-priority auction $\{r, t, v_-, v_+\}$ characterized by Lemma 3 is a point-wise maximizer of the Lagrangian. Since the value of the Lagrangian is an upper-bound of the value of the objective function, which is the sum of (4) and (5), and since the former is achieved by the equal-priority auction $\{r, t, v_-, v_+\}$ characterized by Lemma 3, the latter is indeed an optimal direct mechanism given the threshold strategy μ. It then follows from Theorem 1 that the outcome of $\{r, t, v_-, v_+\}$ corresponds to a symmetric perfect Bayesian equilibrium.

Before we define the Lagrangian, we use the concavity assumption on the monopoly revenue function $\pi(w)$ to simplify the analysis for optimal direct mechanism. Recall that a direct mechanism is a collection of functions

$$\left\{(q_m^{\epsilon}(v^{\epsilon};b^{\mu}),p_m^{\epsilon}(v^{\epsilon};b^{\mu}))_{m=0}^{n-1},(q_m^{\mu}(v^{\epsilon};b^{\mu}),p_m^{\mu}(v^{\epsilon};b^{\mu}))_{m=1}^{n}\right\}.$$

We show that, given the threshold form of σ^{μ}, concavity of $\pi(w)$ implies that $p_m^{\mu}(v^{\epsilon};b^{\mu})$ is independent of the profile of valuations v^{ϵ} of informed buyers, and depends on the profile of messages b^{μ} only through the message b_n to whom the offer $p_m^{\mu}(v^{\epsilon};b^{\mu})$ is made. The proof of the lemma below is in the appendix.

Lemma 4 *If $\pi(\cdot)$ is strictly concave, and uninformed buyers use threshold strategy, then in any optimal direct mechanism, $p_m^{\mu}(v^{\epsilon};b^{\mu})$ depends only on b_n.*

Given the threshold strategy σ^{μ}, the only relevant information about the profile of messages from uninformed is the number m^1 of buyers that send b^1. For notational brevity, we drop the dependence on σ^{μ}. By Lemma 4, we can further rewrite direct mechanisms under consideration as

$$\left\{\left(\left(q_{m,m^1}^{\epsilon}(v^{\epsilon}),p_{m,m^1}^{\epsilon}(v^{\epsilon})\right)_{m^1=0}^{m}\right)_{m=0}^{n-1},\left(\left(\left(q_{m,m^1}^{\mu}(v^{\epsilon};b^j)\right)_{m^1=0}^{m}\right)_{m=1}^{n},p^{\mu}(b^j)\right)_{j=0,1}\right\},$$

where $p_{m,m^1}^{\epsilon}(v^{\epsilon})$ and $q_{m,m^1}^{\epsilon}(v^{\epsilon})$ are, respectively, offer and offer probability for an informed buyer with valuation v_1 in the profile v^{ϵ} when m^1 out of m uninformed buyers are interested, and $q_{m,m^1}^{\mu}(v^{\epsilon};b^j)$ are offer probability for the uninformed buyer n who sends message $b_n=b^j$, $j=0,1$, when the profile of valuations of informed buyers is v^{ϵ} and, including bidder n, m^1 out of m uninformed buyers are interested, and $p^{\mu}(b^j)$ is the offer for uninformed buyer n who sends message $b_n=b^j$. The feasibility constraint (1) becomes

$$\sum_{i=1}^{n-m}q_{m,m^1}^{\epsilon}\left(\rho_i^{\epsilon}\left(v^{\epsilon}\right)\right)+m^1q_{m,m^1}^{\mu}(v^{\epsilon};b^1)+(m-m^1)q_{m,m^1}^{\mu}(v^{\epsilon};b^0)\leq 1.$$

For each $m = 0, \ldots, n-1$, define

$$Q^{\epsilon}_m(w) = \mathbb{E}_{v^{\epsilon}_{-1}}\left[\sum_{m^1=0}^{m} B^m_{m^1} q^{\epsilon}_{m,m^1}(w, v^{\epsilon}_{-1})\right],$$

$$Q^{\mu}_{m+1}(b^j) = \mathbb{E}_{v^{\epsilon}}\left[\sum_{m^1=0}^{m} B^m_{m^1} q^{\mu}_{m+1,m^1+j}(v^{\epsilon}; b^j)\right]$$

for each message b^j, $j = 0, 1$. We can now write the expected payoff (2) for an informed buyer as

$$U^{\epsilon}(w) = \int_0^w \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_m(x)\, dx,$$

and the seller's revenue (4) from all informed buyers as

$$n(1-\alpha)\int_0^1 \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_m(w)\phi(w) f(w) dw.$$

From (3), the expected payoff for an interested uninformed buyer is

$$U^{\mu}(w, b^1) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\mu}_{m+1}(b^1) \max\left\{w - p^{\mu}(b^1), 0\right\},$$

and the expected payoff for an uninterested uninformed buyer is

$$U^{\mu}(w, b^0) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\mu}_{m+1}(b^0) \max\left\{w - p^{\mu}(b^0), 0\right\}.$$

From (5), the seller's revenue from all for interested uninformed buyers is

$$n\alpha \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_{m+1}(b^1) p^{\mu}(b^1) \frac{1 - F(p^{\mu}(b^1)}{1 - F(\xi)},$$

while the seller's revenue from all for uninterested uninformed buyers is

$$n\alpha \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_{m+1}(b^0) p^{\mu}(b^0) \frac{F(\xi) - F(p^{\mu}(b^1)}{F(\xi)}.$$

Using $\lambda(w;b^j)$ for the incentive compatibility constraints

$$U^\epsilon(w) \geq U^\mu(w,b^j)$$

for each message b^j, $j=0,1$, we can now write the Lagrangian as

$$\begin{aligned} & n(1-\alpha)\sum_{m=0}^{n-1} B(m;n-1,\alpha)\int_0^1 Q_m^\epsilon(w)\phi(w)f(w)dw \\ & + n\alpha \sum_{m=0}^{n-1} B(m;n-1,\alpha)\left(Q_{m+1}^\mu(b^1)\pi(p^\mu(b^1)) + Q_{m+1}^\mu(b^0)p^\mu(b^0)(F(\xi)-F(p^\mu(b^0)))\right) \\ & + \sum_{j=0,1}\int_0^1 \lambda(w;b^j)\sum_{m=0}^{n-1} B(m;n-1,\alpha)\left(\int_0^w Q_m^\epsilon(x)dx - Q_{m+1}^\mu(b^j)\max\left\{w-p^\mu(b^j),0\right\}\right)dw. \end{aligned}$$

Using integration by parts, we can further rewrite the above Lagrangian as

$$\sum_{m=0}^{n-1} B(m;n-1,\alpha)\left(\int_0^1 K^\epsilon(w)Q_m^\epsilon(w)f(w)dw + K^\mu(b^1)Q_{m+1}^\mu(b^1) + K^\mu(b^0)Q_{m+1}^\mu(b^0)\right), \quad (12)$$

where

$$K^\epsilon(w)f(w) = n(1-\alpha)\phi(w)f(w) + \int_w^1 (\lambda(x;b^1)+\lambda(x;b^0))dx,$$

for each $w \in [0,1]$, and

$$\begin{aligned} K^\mu(b^1) &= n\alpha\pi(p^\mu(b^1)) - \int_0^1 \lambda(w;b^1)\max\{w-p^\mu(b^1),0\}dw, \\ K^\mu(b^0) &= n\alpha p^\mu(b^0)(F(\xi)-F(p^\mu(b^0))) - \int_0^1 \lambda(w;b^0)\max\{w-p^\mu(b^0),0\}dw. \end{aligned}$$

The expressions of $K^\epsilon(w)$, $K^\mu(b^1)$ and $K^\mu(b^0)$ have straightforward interpretations. For informed buyers, $K^\epsilon(w)$ represents the total contribution to the Lagrangian whenl allocation to an informed buyer with valuation w is increased: the first term in $K^\epsilon(w)$ is the virtual surplus gained by the seller, while the second term is the incentive benefit, because increasing the allocation to informed buyer with valuation w raises the expected payoff to informed buyers with all greater valuations and relaxes the incentive constraints that informed buyers weakly prefer participating in the auction to pretending to be interested or uninterested uninformed buyers. For interested uninformed buyers, $K^{\mu}(b^1)$ represents the total contribution to the Lagrangian when allocation to an interested uninformed buyer is increased: the first term is the revenue gained by the seller, while the second term is the incentive cost, because increasing the allocation to interested buyers increases the deviation payoff informed buyers obtain when they pretend to be interested uninformed buyer. The interpretation of $K^{\mu}(b^0)$ for uninterested buyers is similar to the interpretation of $K^{\mu}(b^1)$ for interested buyers.

Theorem 3 *Suppose that $\pi(\cdot)$ is strictly concave, σ^{μ} is a threshold strategy for uninformed buyers with threshold ξ, and $\{r, t, v_-, v_+\}$ is an ξ-conditionally optimal equal priority auction. If ξ is sufficiently small, $\{r, t, v_-, v_+\}$ together with no offer to uninterested uninformed buyers is an optimal direct mechanism.*

Proof. Suppose that $\{r, t, v_-, v_+\}$ is an ξ-conditionally optimal equal priority auction. By Lemma 3, the critical incentive compatibility constraint (8) binds and the first order conditions (9), (10), and (11) are satisfied. We will show below, in seven steps, that for two appropriately chosen multiplier functions, the allocations given by $\{r, t, v_-, v_+\}$, together with t to interested uninformed buyers and with no offers to uninterested uninformed buyers, are a point-wise maximizer of the Lagrangian (12), among all feasible, weakly increasing allocations for informed buyers and all offers to uninformed buyers. Since transfers to informed buyers can be constructed to make truthful reporting of valuations incentive compatible given that informed buyers have weakly increasing allocations, we have a direct mechanism from $\{r, t, v_-, v_+\}$, with no offers to uninterested uninformed buyers, that maximizes the Lagrangian (12). Further, by construction the incentive compatibility constraints for informed buyers with respect to both messages b^0 and b^1 are satisfied, with complementary slackness. As a result, the maximum value of the Lagrangian (12), which is an upper bound of the seller's revenue among all feasible, incentive compatible mechanisms, is achievable through the direct mechanism constructed from $\{r, t, v_-, v_+\}$. It follows that the direct mechanism is optimal.

(i) Construction of the two non-negatively valued multiplier functions.

Let $\lambda(w; b^1) = 0$ for all $w \notin [v_-, v_+]$, and let

$$\begin{aligned}\lambda(w; b^1) =& n(1-\alpha)\frac{d}{dw}((\phi(w) - \phi(v_+))f(w)) \\ =& n(1-\alpha)(2f(w) + (w - \phi(v_+))f'(w))\end{aligned}$$

for all $w \in [v_-, v_+]$. Since by assumption $\pi(\cdot)$ is strictly concave, $\phi(w)f(w)$ is strictly increasing in w, and thus $\lambda(w; b^1) > 0$ at any $w \in [v_-, v_+]$ such that $f'(w) \leq 0$. By (9) we have $(1 - F(\xi))\phi(v_+) < \pi(t) < \pi(r^*)$. Since $w \geq v_- > t > r^*$, for any $\xi \leq r^*$ we have $\lambda(w; b^1) > 0$ at any $w \in [v_-, v_+]$ such that $f'(w) > 0$. Thus, $\lambda(w; b^1)$ as constructed is non-negative for any w.

Let $\lambda(w; b^0) = 0$ for all $w \notin [\xi, r]$, and let

$$\lambda(w; b^0) = n(1-\alpha)\frac{d}{dw}(\phi(w)f(w))$$

for all $w \in [\xi, r]$. Since $\pi(\cdot)$ is strictly concave, $\lambda(w; b^0)$ is non-negative for all w.

(ii) Computing $K^\epsilon(w)$.

By construction

$$\int_w^1 \lambda(x; b^1)dx = n(1-\alpha)(\phi(v_+) - \phi(w))f(w)$$

for any $w \in [v_-, v_+]$. Thus

$$K^\epsilon(w) = \begin{cases} n(1-\alpha)\phi(w) & \text{if} \quad w > v_+ \\ n(1-\alpha)\phi(v_+) & \text{if} \quad w \in [v_-, v_+] \\ n(1-\alpha)\left(\phi(w) + (\phi(v_+) - \phi(v_-))f(v_-)/f(w)\right) & \text{if} \quad w \in (r, v_-). \end{cases}$$

By construction,

$$\int_w^1 \lambda(x; b^0)dx = n(1-\alpha)(\phi(r)f(r) - \phi(w)f(w))$$

for any $w \in [\xi, r]$. Thus,

$$\begin{aligned} K^\epsilon(w)f(w) &= n(1-\alpha)\left(\phi(w)f(w) + \phi(v_+) - \phi(v_-))f(v_-) + \phi(r)f(r) - \phi(w)f(w)\right) \\ &= n(1-\alpha)\left(\phi(v_+) - \phi(v_-))f(v_-) + \phi(r)f(r)\right) \\ &= 0, \end{aligned}$$

for all $w \in [\xi, r]$, where the last equality follows from (10). For $w < \xi$,

$$K^\epsilon(w)f(w) = n(1-\alpha)\left(\phi(w)f(w) - \phi(\xi)f(\xi)\right) < 0.$$

(iii) We claim that for ξ sufficiently small, $K^\epsilon(w)$ is strictly increasing for $w > v_+$ and for $w \in (r, v_-)$.

By step (ii), for $w > v_+$, the claim follows if we show that $\phi(w)$ is strictly increasing for all $w > v_+$. By definition,

$$\phi(w) = w - \frac{1 - F(w)}{f(w)},$$

and so $\phi'(w) > 0$ for all w such that $f'(w) \geq 0$. By strict concavity of $\pi(w)$, we have

$$(\phi(w)f(w))' = \phi'(w)f(w) + \phi(w)f'(w) > 0.$$

Since $v_+ > r^*$, we have $\phi(w) > 0$ for all $w > v_+$. The above inequality implies that $\phi'(w)f(w) > -\phi(w)f'(w)$, and thus $\phi'(w) > 0$ for all $w > v_+$ such that $f'(w) < 0$.

By step (ii), for $w \in (r, v_-)$, the claim that $K^\epsilon(w)$ is strictly increasing follows if we show that

$$\tilde{\phi}(w) \equiv w - \frac{1 - F(w) - (\phi(v_+) - \phi(v_-))f(v_-)}{f(w)}$$

is strictly increasing. Since

$$(1 - F(\xi))\phi(v_+) < \pi(t) < \pi(r^*) = r^*(1 - F(r^*)),$$

and since $v_- > r^*$, for any $\xi \le r^*$,

$$(\phi(v_+) - \phi(v_-))f(v_-) = (\phi(v_+) - v_-)f(v_-) + 1 - F(v_-) < 1 - F(v_-) < 1 - F(w)$$

for all $w < v_-$. It follows that $\tilde{\phi}'(w) > 0$ for any $w < v_-$ such that $f'(w) \ge 0$. By strict concavity of $\pi(w)$,

$$(\tilde{\phi}(w)f(w))' = \tilde{\phi}'(w)f(w) + \tilde{\phi}(w)f'(w) > 0.$$

By (10), $\tilde{\phi}(w) > 0$ for all $w \in (r, v_-)$. It follows that $\tilde{\phi}'(w) > 0$ for all $w \in (r, v_-)$ such that $f'(w) < 0$.

(iv) We claim that $p^\mu(b^1) = t$ maximizes $K^\mu(b^1)$, with the maximum value given by

$$K^\mu(b^1) = n\alpha(1 - F(\xi))\phi(v_+) > 0.$$

For any $p^\mu(b^1) > v_-$, using integration by parts and step (ii), we have

$$\begin{aligned}
&\int_0^1 \lambda(w; b^1) \max\{w - p^\mu(b^1), 0\} dw \\
=&-\int_{v_-}^{v_+} (w - p^\mu(b^1))\; d\left(\int_w^1 \lambda(x; b^1)dx\right) \\
=&n(1-\alpha)\left((v_- - p^\mu(b^1))(\phi(v_+) - \phi(v_-))f(v_-) + \int_{v_-}^{v_+} (\phi(v_+) - \phi(w))f(w)dw\right).
\end{aligned}$$

By (9), we have

$$K^\mu(b^1) = n\alpha(1-F(\xi))\phi(v_+) + n\alpha(\pi(p^\mu(b^1)) - \pi(t)) + (p^\mu(b^1) - t)n(1-\alpha)(\phi(v_+) - \phi(v_-))f(v_-).$$

The above is strictly concave in $p^\mu(b^1)$. By (11), it is maximized at $p^\mu(b^1) = t$, which is strictly greater than v_-. Any $p^\mu(b^1) \ge v_-$ can only decrease $K^\mu(b^1)$, and so $p^\mu(b^1) = t$ is a global maximizer of $K^\mu(b^1)$. Substituting $p^\mu(b^1) = t$ in the above expression for $K^\mu(b^1)$ we have expression for the maximum value.

(v) We claim that

$$\frac{K^\epsilon(w)}{1-\alpha} \ge \frac{K^\mu(b^1)}{\alpha(1 - F(\xi))}. \tag{13}$$

if and only if $w \geq v_-$, with equality for all $w \in [v_-, v_+]$, where for simplicity we continue to use the notation $K^\mu(b^1)$ for the maximum value given in step (iv).

For all $w \in [v_-, v_+]$, from step (ii) and step (iv), (13) holds as equality.

For all $w > v_+$, by step (iii),

$$K^\epsilon(w) = n(1-\alpha)\phi(w) > n(1-\alpha)\phi(v_+) = K^\epsilon(v_+).$$

Thus, (13) holds as a strict inequality for all $w > v_+$.

For all $w \in (r, v_-)$, by step (iii),

$$K^\epsilon(w) = n(1-\alpha)\left(w - \frac{1 - F(w) - (\phi(v_+) - \phi(v_-))f(v_-)}{f(w)}\right) < n(1-\alpha)\phi(v_+) = K^\epsilon(v_-).$$

Thus, (13) holds strictly in reverse for all $w \in (r, v_-)$.

For all $w \leq r$, using step (ii) we have

$$K^\epsilon(w) \leq 0 < n(1-\alpha)\phi(v_+) = K^\epsilon(v_-).$$

Thus, (13) holds strictly in reverse for all $w \leq r$.

(vi) We claim that if ξ is sufficiently small, then $p^\mu(b^0) = \xi$ maximizes $K^\mu(b^0)$, with the maximum satisfying

$$K^\mu(b^0) = -n(1-\alpha)\int_\xi^r (\phi(r)f(r) - \phi(w)f(w))dw < 0.$$

Without loss we can assume $p^\mu(b^0) \leq \xi$. Then, using integration by parts and step (ii) we have

$$\begin{aligned}
&\int_0^1 \lambda(w; b^0)\max\{w - p^\mu(b^0), 0\}dw \\
=&-\int_\xi^r (w - p^\mu(b^0))\; d\left(\int_w^1 \lambda(x; b^0)dx\right) \\
=&n(1-\alpha)\left((\xi - p^\mu(b^0))(\phi(r)f(r) - \phi(\xi)f(\xi)) + \int_\xi^r (\phi(r)f(r) - \phi(w)f(w))dw\right).
\end{aligned}$$

Maximize $K^{\mu}(b^0)$ is the same as maximizing

$$
\begin{aligned}
&n\alpha p^{\mu}(b^0)(F(\xi) - F(p^{\mu}(b^0))) + n(1-\alpha)p^{\mu}(b^0))(\phi(r)f(r) - \phi(\xi)f(\xi)) \\
=&n\alpha\pi(p^{\mu}(b^0)) + np^{\mu}(b^0)\left((1-\alpha)(\phi(r)f(r) - \phi(\xi)f(\xi)) - \alpha(1-F(\xi))\right).
\end{aligned}
$$

By strict concavity of $\pi(\cdot)$, the above is maximized at $p^{\mu}(b^0) = \xi$ if and only if

$$n\alpha\pi'(\xi) + n\left((1-\alpha)(\phi(r)f(r) - \phi(\xi)f(\xi)) - \alpha(1-F(\xi))\right) \geq 0,$$

which is equivalent to

$$(1-\alpha)\phi(r)f(r) \geq \xi f(\xi) - (1-\alpha)(1-F(\xi)). \tag{14}$$

In step (iii), we have shown that

$$(\phi(v_+) - \phi(v_-))f(v_-) < 1 - F(v_-).$$

Since $v_- > t > r^*$, from (10) we have

$$\phi(r)f(r) > -(1-F(r^*)).$$

Thus, (14) holds when $\xi = 0$. By continuity, it continues to hold for ξ sufficiently small. It follows that $K^{\mu}(b^0)$ is maximized at $p^{\mu}(b^0) = \xi$ when ξ sufficiently is small. Substituting $p^{\mu}(b^0) = \xi$ in the expression for $K^{\mu}(b^0)$, we obtain the maximized value as stated.

(vii) Verifying the direct mechanism given by $\{r, t, v_-, v_+\}$ point-wise maximizes the Lagrangian (12).

By step (v), to maximize the Lagrangian, we set $q^{\mu}_{m,m^1}(v^{\epsilon}; b^0) = 0$ for all $m = 1, \ldots, n$, all $m^1 = 0, \ldots, m$, and all v^{ϵ}, together with $p^{\mu}(b^0) = \xi$. This means no offers to uninterested uninformed buyers. We can then rewrite (12) as

$$(1-\alpha)^{n-1}\int_0^1 K^\epsilon(w)Q_0^\epsilon(w)f(w)dw+\alpha^{n-1}K^\mu(b^1)Q_n^\mu(b^1)$$
$$+\sum_{m=1}^{n-1}\left(\int_0^1 B(m;n-1,\alpha)K^\epsilon(w)Q_m^\epsilon(w)f(w)dw+B(m-1;n-1,\alpha)K^\mu(b^1)Q_m^\mu(b^1)\right).$$

The first term in the above Lagrangian can be further disaggregated

$$(1-\alpha)^{n-1}\int_0^1 K^\epsilon(w)\mathbb{E}_{v_{-1}^\epsilon}\left[q_{0,0}^\epsilon(w,v_{-1}^\epsilon)\right]f(w)dw=(1-\alpha)^{n-1}\mathbb{E}_{v^\epsilon}\left[\sum_{i=1}^n\frac{K^\epsilon(v_i)}{n}q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon))\right].$$

By step (ii), $K^\epsilon(w)$ is a positive constant for $w\in[v_-,v_+]$, strictly increasing for $w>v_+$ and for $w\in(r,v_-)$, and 0 for $w\in[\xi,r]$, and strictly negative for $w<\xi$. Thus, it is feasible and point-wise maximizing to set $q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon))=1$ if $v_i=\max\{v_1,\ldots,v_n\}$ and $v_i>v_+$, or if $v_i=\max\{v_1,\ldots,v_n\}$ and $v_i\in[r,v_-)$; set $q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon))=1/k$ if $v_i\in[v_-,v_+]$, $\max\{v_1,\ldots,v_n\}\in[v_-,v_+]$ and $\#\{j:v_j\in[v_-,v_+]\}=k$; and set $q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon))=0$ otherwise. This coincides with the allocations under $\{r,t,v_-,v_+\}$ for the case of $m=0$.

For the second term in the Lagrangian, recall from step (iii) that $K^\mu(b^1)>0$. It is feasible and point-wise maximizing to set $q_{m^1,n}^\mu(b^1)=1/m^1$ for all $m^1=1,\ldots,n$. This coincides with the allocations under $\{r,t,v_-,v_+\}$ for the case of $m=n$.

For each $m=1,\ldots,n-1$ in the third term in the Lagrangian, we disaggregate it as follows:

$$\int_0^1 B(m;n-1,\alpha)K^\epsilon(w)\mathbb{E}_{v_{-1}^\epsilon}\left[\sum_{m^1=0}^m B_{m^1}^m q_{m,m^1}^\epsilon(w,v_{-1}^\epsilon)\right]f(w)dw$$
$$+B(m-1;n-1,\alpha)K^\mu(b^1)\mathbb{E}_{v^\epsilon}\left[\sum_{m^1=0}^{m-1}B_{m^1}^{m-1}q_{m,m^1+1}^\mu(v^\epsilon;b^1)\right]$$
$$=B(m;n-1,\alpha)\mathbb{E}_{v^\epsilon}\left[\sum_{i=1}^{n-m}\frac{K^\epsilon(v_i)}{n-m}\sum_{m^1=0}^m B_{m^1}^m q_{m,m^1}^\epsilon(\rho_i^\epsilon(v^\epsilon))\right]$$
$$+B(m-1;n-1,\alpha)\mathbb{E}_{v^\epsilon}\left[K^\mu(b^1)\sum_{m^1=1}^m B_{m^1-1}^{m-1}q_{m,m^1}^\mu(v^\epsilon;b^1)\right]$$

When $m^1=0$, it is feasible and point-wise maximizing to set $q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon))=1$ if $v_i=$

$\max\{v_1, \ldots, v_{n-m}\}$ and $v_i > v_+$, or if $v_i = \max\{v_1, \ldots, v_{n-m}\}$ and $v_i \in [r, v_-)$; set $q^\epsilon_{0,0}(\rho^\epsilon_i(v^\epsilon)) = 1/k$ if $v_i \in [v_-, v_+]$, $\max\{v_1, \ldots, v_{n-m}\} \in [v_-, v_+]$ and $\#\{j : v_j \in [v_-, v_+]\} = k$; and set $q^\epsilon_{0,0}(\rho^\epsilon_i(v^\epsilon)) = 0$ otherwise. For each $m^1 = 1, \ldots, n-1$, from step (iv) we have

$$B(m; n-1, \alpha)\frac{K^\epsilon(v_i)}{n-m}B^m_{m^1} \geq B(m-1; n-1, \alpha)\frac{K^\mu(b^1)}{m^1}B^{m-1}_{m^1-1}$$

if and only if $v_i \geq v_-$, with equality for all $w \in [v_-, v_+]$. Thus, it is feasible and point-wise maximizing to set $q^\epsilon_{m,m^1}(\rho^\epsilon_i(v^\epsilon)) = 1$ for all m^1, if $v_i = \max\{v_1, \ldots, v_{n-m}\}$ and $v_i > v_+$; set $q^\epsilon_{m,m^1}(\rho^\epsilon_i(v^\epsilon)) = q^\mu_{m,m^1}(b^1) = 1/(m^1 + k)$ if $v_i \in [v_-, v_+]$, $\max\{v_1, \ldots, v_{n-m}\} \in [v_-, v_+]$ and $\#\{j : v_j \in [v_-, v_+]\} = k$; and otherwise set $q^\epsilon_{m,m^1}(\rho^\epsilon_i(v^\epsilon)) = 0$ and $q^\mu_{m,m^1}(b^1) = 1/(m^1 + k)$, where $k = \#\{j : v_j \in [v_-, v_+]\}$. This coincides with the allocations under $\{r, t, v_-, v_+\}$ for the case of $m = 1, \ldots, n-1$. ■

For ξ sufficiently small, Theorem 3 specifies an optimal direct mechanism. By Theorem 1, it corresponds to a communicative equilibrium. Therefore, we have a continuum of communicative equilibria by varying ξ.

Appendix

Proof of Lemma 2

We have

$$\begin{aligned}
\chi(v_-,v_+) =& \sum_{l=0}^{n-1} \binom{n-1}{l} ((1-\alpha)F(v_-))^{n-1-l} \sum_{l^1=0}^{l} \binom{l}{l^1} (\alpha F(\xi))^{l-l^1} \\
& \frac{1}{l^1+1} \sum_{k=0}^{l^1} \binom{l^1}{k} ((1-\alpha)(F(v_+)-F(v_-)))^k (\alpha(1-F(\xi))^{l^1-k} \\
=& \sum_{l=0}^{n-1} \binom{n-1}{l} ((1-\alpha)F(v_-))^{n-1-l} \sum_{l^1=0}^{l} \binom{l}{l^1} (\alpha F(\xi))^{l-l^1} \\
& \frac{1}{l^1+1} ((1-\alpha)(F(v_+)-F(v_-)) + \alpha(1-F(\xi))^{l^1} \\
=& \sum_{l=0}^{n-1} \binom{n-1}{l} ((1-\alpha)F(v_-))^{n-1-l} \frac{1}{l+1} \sum_{l'=1}^{l+1} \binom{l+1}{l'} (\alpha F(\xi))^{l+1-l'} \\
& ((1-\alpha)(F(v_+)-F(v_-)) + \alpha(1-F(\xi))^{l'-1} \\
=& \sum_{l=0}^{n-1} \binom{n-1}{l} ((1-\alpha)F(v_-))^{n-1-l} \frac{((1-\alpha)(F(v_+)-F(v_-)) + \alpha)^{l+1} - (\alpha F(\xi))^{l+1}}{(l+1)((1-\alpha)(F(v_+)-F(v_-)) + \alpha(1-F(\xi)))} \\
=& \frac{((1-\alpha)F(v_+)+\alpha)^n - ((1-\alpha)F(v_-)+\alpha F(\xi))^n}{n((1-\alpha)(F(v_+)-F(v_-)) + \alpha(1-F(\xi)))}.
\end{aligned}$$

Proof of Lemma 3

Define

$$D(r,t,v_-,v_+) = \int_r^{v_-} ((1-\alpha)F(w)+\alpha F(\xi))^{n-1} dw - \chi(v_-,v_+)(v_- - t),$$

and let R be the revenue from the equal-priority auction. We have

$$\begin{aligned}
\frac{\partial D}{\partial r} &= -((1-\alpha)F(r)+\alpha F(\xi))^{n-1}; \\
\frac{\partial R}{\partial r} &= -n(1-\alpha)((1-\alpha)F(r)+\alpha F(\xi))^{n-1}\phi(r)f(r)
\end{aligned}$$

and

$$\frac{\partial D}{\partial t} = \chi(v_-, v_+);$$
$$\frac{\partial R}{\partial t} = n\alpha\chi(v_-, v_+)\pi'(t).$$

If $v_- < v_+$, or if $v_- = v_+$ and $dv_- < 0$, we have

$$\frac{\partial \chi(v_-, v_+)}{\partial v_-} = (1-\alpha)f(v_-)\frac{\chi(v_-, v_+) - ((1-\alpha)F(v_-) + \alpha F(\xi))^{n-1}}{(1-\alpha)(F(v_+) - F(v_-)) + \alpha(1 - F(\xi))};$$
$$\frac{\partial D}{\partial v_-} = ((1-\alpha)F(v_-) + \alpha F(\xi))^{n-1} - \chi(v_-, v_+) - \frac{\partial \chi(v_-, v_+)}{\partial v_-}(v_- - t);$$
$$\frac{\partial R}{\partial v_-} = n(1-\alpha)((1-\alpha)F(v_-) + \alpha F(\xi))^{n-1} - \chi(v_-, v_+))\phi(v_-)f(v_-)$$
$$+ n((1-\alpha)(\pi(v_-) - \pi(v_+)) + \alpha\pi(t))\frac{\partial \chi(v_-, v_+)}{\partial v_-}.$$

If $v_- < v_+$, or if $v_- = v_+$ and $dv_+ > 0$, we have

$$\frac{\partial \chi(v_-, v_+)}{\partial v_+} = (1-\alpha)f(v_+)\frac{((1-\alpha)F(v_+) + \alpha)^{n-1} - \chi(v_-, v_+)}{(1-\alpha)(F(v_+) - F(v_-)) + \alpha(1 - F(\xi))};$$
$$\frac{\partial D}{\partial v_+} = -\frac{\partial \chi(v_-, v_+)}{\partial v_+}(v_- - t);$$
$$\frac{\partial R}{\partial v_+} = n(1-\alpha)\left(\chi(v_-, v_+) - ((1-\alpha)F(v_+) + \alpha)^{n-1}\right)\phi(v_+)f(v_+)$$
$$+ n((1-\alpha)(\pi(v_-) - \pi(v_+)) + \alpha\pi(t))\frac{\partial \chi(v_-, v_+)}{\partial v_+}.$$

Let (r, t, v_-, v_+) be an optimal equal-priority auction. We first show that it is interior.

Suppose that $r = t < v_- \le v_+$. Recall that $U^\epsilon(w)$ is strictly convex for $w \in (r, v_-)$ while $U^\mu(w, b^1)$ is linear for $w \in (t, v_-)$. Since $Q^\epsilon(w)$ has an upward jump at $w = v_-$, we have $U^\epsilon(v_-) < U^\mu(v_-, b^1)$, violating (8).

Suppose that $r < t = v_- \le v_+$. We have $U^\epsilon(v_-) > U^\mu(v_-, b) = 0$, and so the critical incentive compatibility constraint (8) is slack. Since $r < t$, we have $r < r^*$ or $t > r^*$, or both. If $r < r^*$, then by raising r marginally, the seller could increase the revenue because $\phi(r) < 0$ implies $\partial R/\partial r > 0$. If $t > r^*$, then by lowering t marginally, the seller could increase the revenue because $\pi'(t) < 0$ implies $\partial R/\partial t < 0$. With the critical incentive compatibility constraint (8) slack, we have a contradiction to the assumption of optimality.

Suppose that $r = t = v_- \leq v_+$. If $r = t < r^*$, then by raising t marginally, the seller relaxes the critical incentive compatibility constraint (8) because $\partial D/\partial t > 0$, and increases the revenue because $\pi'(t) > 0$ implies $\partial R/\partial t > 0$. If $r = t > r^*$, then by lowering r marginally, the seller relaxes the critical incentive compatibility constraint (8) because $\partial D/\partial r < 0$, and increases the revenue because $\phi(r) > 0$ implies $\partial R/\partial r < 0$. If $r = t = r^* = v_-$, then by lowering r marginally, the seller relaxes the critical bidding condition (8) because $\partial D/\partial r < 0$, without changing the revenue because $\partial R/\partial r = 0$. With (8) slack, the seller could then increase the revenue by either further raising v_- marginally if $v_- = r^* < v_+$, because $\phi(v_-) = 0$ implies $\partial R/\partial v_- > 0$, or by raising both v_- and v_+ by the same infinitesimal amount if $v_- = v_+ = r^*$, because $\partial R/\partial \hat{v} > 0$ when $\hat{v} = r^*$. In each case, we have a contradiction to the assumption of optimality.

Suppose that $r < t < v_- = v_+ = \hat{v}$. We have $\partial D/\partial \hat{v} < 0$ and $\partial R/\partial \hat{v} < 0$ has the same sign as $\pi(t) - \phi(\hat{v})$. Thus, $\pi(t) > \phi(\hat{v})$: otherwise, by decreasing v_- and v_+ by the same marginal amount, the seller relaxes the critical incentive compatibility constraint (8) without decreasing the revenue, which would then allow the seller to increase the revenue by either raising r or lowering t, as $r < t$ implies $r < r^*$ or $t > r^*$, or both. Since $\phi(1) = 1$, this implies that $\hat{v} < 1$. Now, consider perturbing the equal priority auction by reducing v_- from $\hat{v}$ and raising v_+ from $\hat{v}$ such that

$$-(\chi(\hat{v}, \hat{v}) - ((1-\alpha)F(\hat{v}) + \alpha F(\xi))^{n-1} dv_- = (((1-\alpha)F(\hat{v}) + \alpha)^{n-1} - \chi(\hat{v}, \hat{v}))dv_+.$$

By construction,

$$-\frac{\partial \chi(\hat{v}, \hat{v})}{\partial v_-} dv_- = \frac{\partial \chi(\hat{v}, \hat{v})}{\partial v_+} dv_+.$$

This implies that the critical incentive compatibility constraint (8) is relaxed, because

$$\frac{\partial D}{\partial v_-} dv_- + \frac{\partial D(\hat{v})}{\partial v_+} dv_+ = (((1-\alpha)F(\hat{v}) + \alpha F(\xi))^{n-1} - \chi(\hat{v}, \hat{v}))dv_-,$$

which is strictly positive. The seller's revenue is unchanged, because

$$
\begin{aligned}
&\frac{\partial R}{\partial v_-}dv_- + \frac{\partial R}{\partial v_+}dv_+\\
=&n(1-\alpha)(((1-\alpha)F(\hat{v})+\alpha F(\xi))^{n-1} - \chi(\hat{v},\hat{v}))\phi(\hat{v})f(\hat{v})dv_-\\
&\quad + n(1-\alpha)\left(\chi(\hat{v},\hat{v}) - ((1-\alpha)F(\hat{v})+\alpha)^{n-1}\right)\phi(\hat{v})f(\hat{v})dv_+,
\end{aligned}
$$

which is equal to 0 by construction. The seller could now increase the revenue by either raising r or lowering t, as $r < t$ implies $r < r^*$ or $t > r^*$, or both. This contradicts the assumption of optimality.

Now, we establish the first-order conditions stated in the lemma. To begin, the critical incentive compatibility constraint (8) binds at any optimal equal-priority auction. Otherwise, since $r < t$ implies that $r < r^*$ or $t > r^*$, or both, the seller could increase the revenue by either raising r or lowering t, a contradiction to the assumed optimality. Further, $r < r^* < t$. Otherwise, if $r^* \leq r < t$, the seller could relax (8) by lowering r marginally without decreasing the revenue, which then would allow the seller to increase the revenue by lowering t. Similarly, if $r < t \leq r^*$, the seller could relax (8) by raising t marginally without decreasing the revenue, which then would allow the seller to increase the revenue by raising r. Finally, $\pi(t) > (1-F(\xi))\phi(v_+)$. Otherwise, by lowering v_+ marginally, the seller relaxes (8) because $\partial D/\partial v_+ < 0$, and increases the revenue, as $\partial R/\partial v_+$ has the same sign as

$$
\begin{aligned}
&\alpha(\pi(t) - (1-F(\xi))\phi(v_+)) + (1-\alpha)(\pi(v_-) - \pi(v_+)) - \phi(v_+)(F(v_+) - F(v_-))\\
=&\alpha(\pi(t) - (1-F(\xi))\phi(v_+)) - (1-\alpha)\int_{v_-}^{v_+}(\phi(v_+) - \phi(w))f(w)dw\\
<&\alpha(\pi(t) - (1-F(\xi))\phi(v_+)),
\end{aligned}
$$

contradicting the assumed optimality. Given that $\pi(t) > (1-F(\xi))\phi(v_+)$, for ξ satisfying

$$
(1-F(\xi))\phi(1) = (1-F(\xi)) \geq \pi(r^*),
$$

we have $v_+ < 1$.

To obtain (9), consider perturbations dv_- and dv_+, while keeping r and t unchanged. An optimality condition is that

$$\frac{\partial R}{\partial v_-}dv_- + \frac{\partial R}{\partial v_+}dv_+ = 0,$$

for all perturbations dv_- and dv_+ satisfying

$$\frac{\partial D}{\partial v_-}dv_- + \frac{\partial D}{\partial v_+}dv_+ = 0.$$

Thus we have

$$\frac{\partial R/\partial v_-}{\partial D/\partial v_-} = \frac{\partial R/\partial v_+}{\partial D/\partial v_+}.$$

Using the expressions for $\chi(v_-, v_+)$, $\partial\chi(v_-, v_+)/\partial v_-$ and $\partial\chi(v_-, v_+)/\partial v_+$, straightforward algebra lead us to the first-order condition (9) for an optimal equal-priority auction with respect to v_- and v_+. Note that (9) implies that

$$\frac{\partial R/\partial v_+}{\partial D/\partial v_+} = -n(1-\alpha)(\phi(v_+) - \phi(v_-))f(v_-).$$

Next, to obtain (11), consider perturbations dt and dv_+. The resulting optimality condition is

$$\frac{\partial R/\partial t}{\partial D/\partial t} = \frac{\partial R/\partial v_+}{\partial D/\partial v_+}.$$

This gives the first order condition (11) with respect to t and v_+.

Lastly, to obtain (10), consider perturbations dr and dv_+, while keeping t and v_- unchanged. The resulting optimality condition is

$$\frac{\partial R/\partial r}{\partial D/\partial r} \geq \frac{\partial R/\partial v_+}{\partial D/\partial v_+},$$

and $r \geq 0$, with complementary slackness. This gives the first-order condition

$$-\phi(r)f(r) \leq (\phi(v_+) - \phi(v_-))f(v_-),$$

and $r \geq 0$, with complementary slackness. Since $(1 - F(\xi))\phi(v_+) < \pi(t)$ and $v_- > t > r^*$, for $\xi < r^*$ satisfying,

$$\phi(\xi)f(\xi) + 1 - F(r^*) \leq 0,$$

we have

$$(\phi(v_+) - \phi(v_-))f(v_-) = (\phi(v_+) - v_-)f(v_-) + 1 - F(v_-) < 1 - F(r^*) \leq -\phi(\xi)f(\xi),$$

Thus, the first-order condition implies that $r > \xi$.

Proof of Lemma 4

Fix a direct mechanism $\{q^\epsilon_m, p^\epsilon_m\}_{m=0}^{n-1}$ and $\{q^\mu_m, p^\mu_m\}_{m=1}^{n}$. For notational brevity, we drop the dependence on σ^μ. The expected payoff for an uninformed buyer who sends message $b \in \mathcal{M}^\mu$ is given by (3). From (5), the seller's revenue from such buyers is

$$n\alpha\mathbb{E}_w\Big[\sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_{v^\epsilon; v^\mu_{-n}}\big[q^\mu_{m+1}(v^\epsilon; \sigma^\mu_{-n}(v^\mu_{-n}), b)\cdot \\ p^\mu_{m+1}(v^\epsilon; \sigma^\mu_{-n}(v^\mu_{-n}), b)\mathbb{1}_{w \geq p^\mu_{m+1}(v^\epsilon; \sigma^\mu_{-n}(v^\mu_{-n}), b)}\big]\Big| \sigma^\mu(w) = b\Big].$$

In any optimal mechanism, without loss we can assume all offers to uninformed buyers who send b are "serious," i.e.,

$$p^\mu_{m+1}(v^\epsilon; \sigma^\mu_{-n}(v^\mu_{-n}), b) \leq \sup\{w : \sigma^\mu(w) = b\}$$

for all v^ϵ and all v^μ_{-n}. This is because, if the opposite holds for some v^ϵ and v^μ_{-n}, the seller can always reduce the corresponding offer probability $q^\mu_{m+1}(v^\epsilon; \sigma^\mu_{-n}(v^\mu_{-n}), b)$ to 0, without affecting the seller's revenue from buyers who send message b while maintaining the incentive compatibility of informed buyers with respect to b.

For each message $b \in \mathcal{M}^\mu$, define $p^\mu(b) \in [0, 1]$ to be the expected offer to uninformed buyers who send message b, given by

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_{v^\epsilon; v^\mu_{-n}}\left[q^\mu_{m+1}(v^\epsilon; \sigma^\mu_{-n}(v^\mu_{-n}), b)(p^\mu(b) - p^\mu_{m+1}(v^\epsilon; \sigma^\mu_{-n}(v^\mu_{-n}), b))\right] = 0,$$

Consider replacing $p^\mu_{m+1}(v^\epsilon; \sigma^\mu_{-n}(v^\mu_{-n}), b))$ for all v^ϵ and v^μ_{-n} with $p^\mu(b)$. By (3), since $U^\mu(w, b)$ is piece-wise linear and convex, such replacements weakly reduce $U^\mu(w, b)$. This implies that after the replacements the incentive compatibility constraint of informed buyers with respect to message b is maintained. From (5), if $b = b^1$, after the replacements the seller's revenue from interested uninformed buyers becomes

$$n\alpha p^{\mu}(b^1)(1 - F(p^{\mu}(b^1)).$$

Strict concavity of $\pi(\cdot)$ implies that the seller's revenue from interested uninformed buyers increases. If instead $b = b^0$, then since all offers to uninformed buyers who send b^0 are serious, after the replacements the seller's revenue from uninterested uninformed buyers becomes

$$n\alpha p^{\mu}(b^0)(F(\xi) - F(p^{\mu}(b^0)).$$

Strict concavity of $\pi(\cdot)$ implies that the above is strictly concave in $p^{\mu}(b^0)$. It follows that replacements increases the seller's revenue from uninterested uninformed buyers. The lemma follows immediately.

References

[1] Li, Hao and Michael Peters (2021): "Unobservable Mechanism Design: Equal Priority Auctions." Working paper, University of British Columbia.