# Equal-Priority Auctions

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#### Abstract

We study mechanism design under stochastic participation. A seller of a single indivisible good faces buyers with symmetric independent private values. Each buyer has an independent probability of not participating in the seller's mechanism, and to sell to "non-participants" the seller must randomly select one of them to make a takeit-or-leave-it offer. A buyer who is able to participate ("participant") can however behave like a non-participant. We show that in any optimal mechanism there is an interval of valuations such that all participants with valuations in the interval are given an equal allocation priority as non-participants. Optimal mechanisms can be implemented as an auction where bids in the interval are not distinguished and pooled with non-participants, together a fixed-price offer to a buyer in the pool when there are no distinguishable bids higher than the interval. Participants in the pool always accept their offers, while non-participants may not. This offers an explanation of why fixed prices instead of auctions are used more often than needed to sell to non-participants.

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# 1 Introduction

Advances of technologies have made online auctions increasingly accessible to buyers. Proxy bidding, by asking buyers to submit a maximum bid and letting a computer run an automatic ascending auction, allows a buyer to participate without paying constant attention. However, some buyers continue to find auctions "inconvenient" and do not participate in them. Indeed, the fraction of such non-participants is not only significant, but has also increased in recent years (Einav, Farronato, Levin and Sundaresan, 2018). For sellers, how to run auctions without giving up on non-participants is a practical problem. In this paper we argue that solution to this problem has implications to observed online auctions.

We study a problem for a seller of a single indivisible good facing a fixed number of buyers with symmetric independent private values, where each buyer has an independent probability of not participating in the seller's mechanism. We call this "mechanism design with stochastic participation." If the seller cannot reach non-participants, stochastic participation would not have any impact and standard auctions are optimal. To highlight implications of nonparticipants to auction design, we assume that to sell to "non-participants" the seller must randomly select one of them to make a take-it-or-leave-it offer. As in standard mechanism design problems, the seller commits to a mechanism at the interim stage after valuations are privately realized and the participation status of a given buyer is privately revealed independently of their valuations. A buyer who is able to participate in bidding ("participant") can however behave like a non-participant. This creates a "one-sided" incentive compatibility constraint for buyers with any given valuation for the good, but for participants only. It is the one novel feature in mechanism design with stochastic participation.

In an auction for participants, the prospect of making a fixed-price offer to non-participants gives the seller a stochastic outside option. The incentive problem this outside option creates is that participants can exploit this option by pretending to be a non-participant. By the revelation principle, we can assume that the seller's fixed-price offer is made only to "truthful" non-participants. Then, the choice of the fixed-price offer determines not only the expected value of this outside option to the seller, but also how high the reserve price can be to incentivize participants with low valuations to bid in the auction. In particular, under the assumption that the revenue from a fixed-price offer is concave in the offer, which implies the monopoly price is equal to the optimal reserve price (Myerson, 1981), setting a fixed-price offer to the monopoly price or higher requires a reserve price lower than the optimal reserve price. The seller faces a trade-off between the auction revenue from participants and the fixed-price revenue from non-participants. In particular, to provide incentives for participants with low valuations not to pretend to be non-participants, the seller sets a reserve price lower than Myerson's optimal reserve price and a fixed-price offer higher than the monopoly price.

In an incentive compatible mechanism with a reserve price for an auction among participants and a fixed price offer to non-participants, participants with sufficiently high valuations have higher allocation priorities than non-participants, and those with low valuations above the reserve price get the good only when there are no non-participants, or participants with higher allocation priorities. A necessary condition for optimality of such mechanism is that there exists a valuation at which a participant has the same allocation priority as non-participants, being indifferent between in bidding in the auction and pretending to be a non-participant. We show that any optimal mechanism has an interval of valuations for participants, rather than a single one, that have the same allocation priority as non-participants.

The optimality of an equal-priority interval of valuations results from the trade-off between the auction revenue from participants and the fixed-price revenue from non-participants. By the envelope theorem, the expected payoff of participants is increasing and convex in their valuation. If there is a single equal-priority valuation for participant, the expected payoff function has a kink at the valuation, because the probability of getting the good for a participant at just above the valuation is greater than that for a non-participant, and is smaller at just below the valuation. By revenue equivalence, we can keep the revenue from the auction unchanged at the margin by creating an arbitrarily small interval around the equal-priority valuation, as the probability of getting the good for participants is increased below the equal-priority valuation and decreased above it. However, replacing the single equal-priority valuation with an interval lifts up the expected payoff of participants for the entire interval. This allows the seller to reduce the fixed-price offer, increasing the revenue from non-participants. To establish that an equal-priority mechanism is optimal, we still have to show that an optimal mechanism takes the form of a reserve price, a fixed price and an equal-priority interval, all independent of the realized number of non-participants. This is challenging because the one-sided incentive constraint that ensures no participant wishes to pretend to be a non-participant has to hold for each valuation. By first optimizing over all equal-priority mechanisms and establishing necessary conditions for an optimal equal-priority mechanism, we construct a multiplier function for the one-sided incentive constraint and use a Lagrangian relaxation approach to show that any optimal equal-priority mechanism is indeed revenue-maximizing over all incentive compatible mechanisms.

One appealing feature of the independent private value auction problem for mechanism design is that finding the revenue maximizing mechanism can be reduced to a problem of solving a maximization problem with a single parameter - the reserve price. In our problem of mechanism design with stochastic participation, an optimal equal-priority mechanism can also be implemented as an "equal-priority auction," where bids outside the equal-priority interval are registered as distinguished bids and those in the interval are scrambled and pooled together with non-participants, together with a fixed-price offer made randomly to a buyer in the pool. The winner of the auction is the one with the highest distinguished bid if it is above the equal-priority interval, or if it is below the interval but the pool is empty. The price paid by the winner is the second highest distinguished bid if it is above the equalpriority interval, or the maximum of the second highest distinguished bid and the reserve price if the second highest distinguished bid is below the interval but the pool is empty, and is otherwise adjusted by the size of the pool. The optimal equal-priority auction can be found by solving a problem with four parameters - a reserve price, two cutoff valuations that define the equal-priority interval, and a fixed-price offer to the equal-priority pool. This is a harder problem than Myerson's (1981) optimal auction problem, but still computationally tractable. The numerical solutions we have found in simple environments suggest that the optimal equal-priority pool is significant.

Participants in the optimal equal-priority pool are indifferent between bidding in the auction and waiting for a fixed-price offer. For any degree of inconvenience that causes some buyers to become non-participants in the first place, some of these participants may decide to opt out. A consequence is that fixed prices are used more frequently to determine the trade than needed by sellers to reach non-participants. Conversely, auctions are used less often. Moreover, an econometrician who tries to recover distributions of valuations based on bids placed in an auction would get biased estimates, some valuations in the equal-priority pool might be missing from the bidding data. Generally speaking, our results suggest that non-participation should be taken seriously in understanding online auctions.

### 1.1 Related literature

The premise in our paper is that buyers may not participate in a seller's auction. This is similar to the idea in the marketing literature that consumers may not be responsive to advertised prices. Dickson and Sawyer (1990) asked buyers in supermarkets about their price knowledge as they were shopping. Even when the item being placed in their basket had been specially marked down and heavily advertised, 25% of consumers did not even realize the good was on special. The approach used earlier in economics, as in, say Butters (1977), was that buyers randomly observe price offers in a competitive environment. In that literature, firms advertise prices which some buyers see, while others do not.<sup>1</sup> These papers considered the same problem that we do, which is how non-responsive buyers would affect the prices that sellers offer. The difference is that we are interested in mechanisms, not prices.

Sellers' attempt to sell to non-participants provides type-dependent outside options to participants as they are free to pretend to be non-participants. This is the basic problem in the literature on competing mechanisms. One example is the paper by McAfee (1993). His model had buyers whose outside option involved waiting until next period to purchase in a competing auction market just like the one in the current period. He imposed large market assumptions to ensure that the value of these outside options was independent of the reserve price that any seller in the existing market chose. In our paper, the value of this outside option depends on the nature of the mechanism the seller chooses for participants. This makes it resemble later papers on competing mechanisms in terms of outside options, like Virag (2010) who studies finite competing auction models where a seller who raises her

<sup>&</sup>lt;sup>1</sup>See also Varian (1980), or Stahl (1994). In Varian (1980), some buyers are loyal to a specific seller, while others are just interested in the lowest price.

reserve price increases congestion in other auctions, or Hendricks and Wiseman (2020) who study the same problem in a sequential auction environment.

Our participants can "prove" their identity in the same sense as Porath, Dekel and Lipman (2014). The main difference is that they assume that all players have this option at some cost under a social choice function, while in our model the driving force is the stochastic presence of buyers who cannot participate in the seller's mechanism. They also assume players have complete information about the state, but in our model only buyers know their own valuations.

Finally, our participants can pretend they are non-participants but not the other way around. The one-sidedness of this incentive condition is similar to Denekere and Severinov (2006), who study an optimal non linear pricing problem with a fraction of consumers constrained to reporting their valuations truthfully.<sup>2</sup> As in our paper, their mechanism separates "honest" consumers from "strategic" consumers who can misrepresent their valuations costlessly. The main difference is that we start with a standard independent private value auction problem rather than a non linear pricing problem.

# 2 Mechanisms with Stochastic Participation

There are *n* potential buyers of a single homogeneous good for sale by a seller. Each buyer has a privately known valuation w that is independently drawn from the interval [0, 1]. We assume that all valuations are distributed according to some distribution F with a continuously differentiable density f. Buyers' payoff when they buy at price p is given by w - p. The seller's cost is zero, so the profit from selling at price p is just p.

Define

$$\pi(w) = (1 - F(w))w$$

as the revenue function from a take-it-or-leave-it offer w to a buyer. Throughout the paper we maintain the following assumption.

#### Assumption 1 $\pi(\cdot)$ is strictly concave.

<sup>&</sup>lt;sup>2</sup>See also Sher and Vohra (2015). They use graph theory to study a more general non linear pricing problem with voluntary provision of hard evidence.

Following the standard auction literature, we also define

$$\phi(w) = w - \frac{1 - F(w)}{f(w)}$$

as the virtual valuation function. We have  $\phi(0) < 0$  and  $\phi(1) = 1$ , and so  $\phi(w)$  crosses 0 at least once. Since  $\pi'(w) = -\phi(w)f(w)$ , concavity of  $\pi(\cdot)$  implies that  $\phi(w)$  crosses 0 only once. Let the crossing point be  $r^*$ ; this is also the unique maximizer of  $\pi(w)$ . Furthermore,  $\phi(w)$  is strictly increasing in v for  $w \ge r^*$ .<sup>3</sup> The valuation  $r^*$  represents the optimal reserve price in a standard auction, regardless of the number of buyers.<sup>4</sup>

Buyers are either *participants* or a *non-participants* in the seller's mechanism, each buyer with probability  $\alpha \in (0, 1)$  being a non-participant independent of their valuation. Since the revelation principle applies to participants, we restrict the seller to direct mechanisms and Bayesian Nash equilibria in which all realized participants report their valuations truthfully. By assumption, non-participants do not report their valuations; instead, we assume that the seller's mechanism may randomly select one buyer from those who do not report their valuations and make a take-it-or-leave-it offer.

We need to add some notation to formally describe a symmetric mechanism with clarity. In what follows the notation m always means the number of non-participants. We reorder n buyers such that the first n - m of them are participants; the orders among participants and among non-participants are arbitrary. For each profile of reported valuations of n - m participants,  $v = (v_1, \ldots, v_{n-m}) \in [0, 1]^{n-m}$ , and for each  $i = 1, \ldots, n - m$ , let

$$\rho_m^i(v) = (v_i, v_2, \dots, v_{i-1}, v_1, v_{i+1}, \dots, v_{n-m});$$

that is,  $\rho_m^i(v)$  switches the positions of  $v_1$  and  $v_i$ . Now we have

<sup>&</sup>lt;sup>3</sup>At any  $w \in (0,1)$ , if f(w) is non-decreasing, then by definition  $\phi(w)$  is strictly increasing; if f(w) is strictly decreasing at w and if  $\phi(w) \ge 0$ , then  $\phi(w)$  is strictly increasing in w, because concavity of  $\pi(w)$  implies that  $\phi(w)f(w)$  is strictly increasing in w.

<sup>&</sup>lt;sup>4</sup>In much of the auction literature, the seller has the fixed outside option of keeping the good. The virtual valuation function  $\phi(w)$  is assumed to be strictly increasing to simplify the analysis (the "regular case" in Myerson (1981)). In our model, the seller's outside option in an auction with participants is to give it to an unobservant buyer with a take-it-or-leave-it offer, and is endogenous. We do not need to assume that  $\phi(w)$  is strictly increasing for valuations below  $r^*$ .

**Definition 1** A symmetric mechanism  $\delta$  is a collection of functions

$$\left\{ (q_m^{\sigma}, p_m^{\sigma})_{m=0}^{n-1}, (q_m^{\mu}, p_m^{\mu})_{m=1}^n \right\}$$

where  $q_m^{\sigma}, p_m^{\sigma} : [0, 1]^{n-m} \to [0, 1]$  for each m = 0, ..., n-1, and  $q_m^{\mu}, p_m^{\mu} : [0, 1]^{n-m} \to [0, 1]$  for each m = 1, ..., n, satisfying

- (q<sup>\sigma</sup><sub>m</sub>(v), p<sup>\sigma</sup><sub>m</sub>(v)) are invariant to permutations of (v<sub>2</sub>,..., v<sub>n-m</sub>), and (q<sup>\mu</sup><sub>m</sub>(v), p<sup>\mu</sup><sub>m</sub>(v)) are invariant to permutations of (v<sub>1</sub>,..., v<sub>n-m</sub>);
- for all v and for all m,

$$\sum_{i=1}^{n-m} q_m^{\sigma} \left( \rho_m^i(v) \right) + m q_m^{\mu}(v) \le 1.$$
 (1)

The function  $q_m^{\mu}(v)$  gives the probability with which a take-it-or-leave-it offer  $p_m^{\mu}(v)$  is made to a non-participant, given that there are m non-participants and the profile of reported valuations is  $v = \{v_1, \ldots, v_{n-m}\}$ . The function  $q_m^{\sigma}(v)$  gives the probability that buyer 1 gets the good, and conditional on getting the good, the payment  $p_m^{\sigma}(v)$  made to the seller, given that there are m non-participants and the reported valuation profile of buyers  $i = 2, \ldots, n-m$ is  $v_{-1} = \{v_2, \ldots, v_{n-m}\}$ . Symmetry requires the allocation and the offer functions of nonparticipants to be invariant to permutations of the valuation profile of participants, and the allocation and the payment functions of each participant. Since  $\rho_m^i(v)$  switches the positions of the reported valuation profile of the other participants. Since  $\rho_m^i(v)$  gives the probability that one of the first element of v and its *i*-th element, the sum  $\sum_{i=1}^{n-m} q_m^{\sigma}(\rho_m^i(v))$  gives the probability that one of the first n - m elements of v gets the good. Then (1) ensures that when participants have valuations given by the first n - m valuations in v, the probability that one of them gets the good plus the probability that the good is offered to one of non-participants is less than or equal to 1.

### 2.1 Incentive compatibility

In our mechanism design problem, non-participants are reduced to waiting for a take-or-leaveit offer. Participants must be induced to report their valuation truthfully by the seller's mechanism, and to participate in the mechanism in the first place instead of pretending to be a non-participant. We now use the standard methodology to characterize incentive compatibility, and to derive a revenue equivalence result.

Under a mechanism  $\delta = \{(q_m^{\sigma}, p_m^{\sigma})_{m=0}^{n-1}, (q_m^{\mu}, p_m^{\mu})_{m=1}^n\}$ , by truthfully reporting their valuation, the probability that a participant with valuation w gets the good when there are  $m \leq n-1$  non-participants is

$$Q_m^{\sigma}(w) = \mathbb{E}_{v_{-1}}\left[q_m^{\sigma}(w, v_{-1})\right],$$

where  $v_{-1} = (v_2, ..., v_{n-m})$ . Similarly

$$P_m^{\sigma}(w) = \mathbb{E}_{v_{-1}} \left[ q_m^{\sigma}(w, v_{-1}) p_m^{\sigma}(w, v_{-1}) \right]$$

is the expected payment the participant with valuation w makes. By taking expectations of  $Q_m^{\sigma}$  and  $P_m^{\sigma}$  over m, we have the ex ante probability of getting the good and the expected payment for a participant with valuation w:

$$Q^{\sigma}(w) = \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) Q_m^{\sigma}(w)$$
$$P^{\sigma}(w) = \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) P_m^{\sigma}(w) ,$$

where  $B_{\tilde{m}}^{\tilde{n}}(\alpha)$  is the probability that  $\tilde{m}$  randomly drawn buyers out of  $\tilde{n} \geq \tilde{m}$  buyers are non-participants, given by

$$B_{\tilde{m}}^{\tilde{n}}(\alpha) = \begin{pmatrix} \tilde{n} \\ \tilde{m} \end{pmatrix} (1-\alpha)^{\tilde{n}-\tilde{m}} \alpha^{\tilde{m}}.$$

For each  $m = 0, \ldots, n - 1$ , define

$$U_m^{\sigma}(w) = wQ_m^{\sigma}(w) - P_m^{\sigma}(w)$$

as the indirect payoff of a participant with valuation w from truthful reporting when there are m non-participants, under a mechanism  $\delta = \{(q_m^{\sigma}, p_m^{\sigma})_{m=0}^{n-1}, (q_m^{\mu}, p_m^{\mu})_{m=1}^n\}$ . Define

$$U^{\sigma}(w) = \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) U_m^{\sigma}(w) = w Q^{\sigma}(w) - P^{\sigma}(w) \,.$$

This is the expected payoff of a truthful participant with valuation w under  $\delta$ . If the mechanism is incentive compatible *with respect to valuations*, the expected payoff to a participant with valuation w can be written as

$$U^{\sigma}(w) = \int_{0}^{w} Q^{\sigma}(x) \, dx, \qquad (2)$$

with  $Q^{\sigma}(\cdot)$  non-decreasing.<sup>5</sup> The (interim) payoff to a non-participant with valuation w is

$$U^{\mu}(w) = \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) \mathbb{E}_v \left[ q_{m+1}^{\mu}(v) \max\left\{ w - p_{m+1}^{\mu}(v), 0 \right\} \right].$$

**Definition 2** The mechanism  $\delta = \{(q_m^{\sigma}, p_m^{\sigma})_{m=0}^{n-1}, (q_m^{\mu}, p_m^{\mu})_{m=1}^n\}$  is incentive compatible if  $Q^{\sigma}(\cdot)$  is non-decreasing, and

$$U^{\sigma}\left(w\right) \ge U^{\mu}\left(w\right) \tag{3}$$

for every  $w \in [0, 1]$ .

Condition (3) can be viewed as a type-independent participation condition. It ensures that participants do not have incentives to pretend to be non-participants.

The seller's expected revenue from participants is given by

$$\sum_{m=0}^{n-1} B^n_m(\alpha) \mathbb{E}_v \left[ \sum_{i=1}^{n-m} q^\sigma_m(\rho^i_m(v)) p^\sigma_m(\rho^i_m(v)) \right].$$

<sup>&</sup>lt;sup>5</sup>See, for example, Myerson (1981). We have assumed  $U^{\sigma}(0) = 0$  for simplicity. This is usually not part of requirement for incentive compatibility, but clearly necessary for any revenue maximizing direct mechanism.

By the symmetry of the mechanism, we can write the above as

$$\sum_{m=0}^{n-1} B_m^n(\alpha)(n-m) \int_0^1 (wQ_m^{\sigma}(w) - U_m^{\sigma}(w))f(w)dw.$$

Using the definitions of  $Q^{\sigma}$  and  $U^{\sigma}$ , and the envelope condition (2), integration by parts implies that the revenue from participants is

$$n(1-\alpha)\int_0^1 Q^{\sigma}(w)\,\phi(w)f(w)dw,\tag{4}$$

in any mechanism that is incentive compatible with respect to evaluations. The above is the familiar revenue equivalence result regarding participants, adapted to stochastic participation in our model.

The revenue from non-participants is

$$\sum_{m=1}^{n} B_{m}^{n}(\alpha) \mathbb{E}_{v} \left[ m q_{m}^{\mu}(v) \pi \left( p_{m}^{\mu}(v) \right) \right].$$
(5)

Since non-participants do not accept the fixed-price offers with probability one, revenue equivalence does not apply. The revenue cannot be written as a function of allocations  $\{q_m^{\mu}\}$  alone. Further, condition (3) has to be dealt with separately when the seller chooses a mechanism  $\delta$  to maximize  $R(\delta)$  given by the sum of (4) and (5).

## 3 Equal-Priority Mechanisms

Instead of directly solving the problem of optimal mechanism with stochastic participation directly, we first consider a special class of mechanisms which we call "equal-priority" mechanisms. We describe the set of all incentive compatible equal-priority mechanisms, and then characterizes the one that gives the seller the highest expected revenue. In the next section we will verify that the seller cannot do strictly better among all mechanisms.

An equal-priority mechanism is fully characterized by four numbers, a reserve price r, a take-it-or-leave-it offer t, and the upper and lower bound  $\overline{w}$  and  $\underline{w}$  of an interval of buyer valuations, satisfying  $r \leq \underline{w} \leq \overline{w}$ . Let m be the number of non-participants, and k be the

number of reported valuations in the "equal-priority interval"  $[\underline{w}, \overline{w}]$ . The mechanism treats the *m* non-participants and the *k* participants with the same allocation priority; we refer to them as "equal-priority pool." Participants with reported valuations above  $\overline{w}$  have higher priorities than the m + k buyers in the equal-priority pool, and those with valuations below  $\underline{w}$  but above *r* have lower priorities. When  $\underline{w} = \overline{w}$ , with probability one the equal-priority pool contains only the *m* non-participants.

In words, the allocations, payments and offers in an equal-priority mechanism are determined in the following way:

- When the highest reported valuation is less than r: no participant gets the good; for each  $m \ge 1$ , with probability 1/m an offer t is made to a non-participant.
- When the highest reported valuation by a participant is between r and  $\underline{w}$ : if m = 0, the participant gets the good with probability one and pays the maximum of the second highest reported valuation and r; if  $m \ge 1$ , with probability 1/m, an offer t is made to a non-participant.
- When the highest reported valuation by a participant is between  $\underline{w}$  and  $\overline{w}$ : if m+k = 1, the participant gets the good with probability one and pays the maximum of the second highest reported valuation and r; if  $m+k \geq 2$ , each participant with reported valuation between  $\underline{w}$  and  $\overline{w}$  gets the good with probability 1/(m+k), and conditional on getting the good, pays  $\underline{w}$ , while each non-participant gets t with probability 1/(m+k).
- When the highest reported valuation by a participant is above  $\overline{w}$ : the participant gets the good with probability one, pays the second highest reported valuation if it is above  $\overline{w}$ , pays  $(\underline{w} + (m + k)\overline{w})/(m + k + 1)$  if the second highest reported valuation is in  $[\underline{w}, \overline{w}]$  or if  $m \ge 1$ , and otherwise pays the maximum of the second highest reported valuation and r.

An equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  is a modified second-price auction with a reserve price r for participants, combined with a take-it-or-leave-it offer t to non-participants. However, we have an equal-priority pool consisting of participants with valuations between  $\underline{w}$  and  $\overline{w}$  and non-participants. As a result, the payment made by the participant with the highest reported valuation, is the maximum of r and the second highest reported valuation, only if the second highest reported valuation is outside  $[\underline{w}, \overline{w}]$ , and only if there are no non-participants when the second highest reported valuation is lower than  $\underline{w}$ .

Formally, using the notation of direct mechanisms introduced in section 2, we can represent an equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  as follows. Suppose that  $v_1$  is the highest reported valuation, and  $v_2$  be the second highest reported valuation. The collection of functions  $\{(q_m^{\sigma}(v), p_m^{\sigma}(v))_{m=0}^{n-1}, (q_m^{\mu}(v), p_m^{\mu}(v))_{m=1}^n\}$  given by an equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  is

$$\begin{aligned} q_m^{\sigma}(v) &= 0 & \text{if } v_1 < r, \text{ or } v_1 \in [r, \underline{w}) \text{ and } m \ge 1 \\ q_m^{\sigma}(v) &= 1/(m+k), \ p_m^{\sigma}(v) = \underline{w} & \text{if } v_1 \in [\underline{w}, \overline{w}] \text{ and } m+k \ge 2 \\ q_m^{\sigma}(v) &= 1, \ p_m^{\sigma}(v) = (\underline{w} + (m+k)\overline{w})/(m+k+1) & \text{if } v_1 > \overline{w}, \text{ and } v_2 \in [\underline{w}, \overline{w}] \text{ or } m \ge 1 \\ q_m^{\sigma}(v) &= 1, \ p_m^{\sigma}(v) = \max\{v_2, r\} & \text{if otherwise,} \end{aligned}$$

and

$$\begin{cases} q_m^{\mu}(v) = 0 & \text{if } v_1 > \overline{w} \\ q_m^{\mu}(v) = 1/(m+k), \ p_m^{\mu}(v) = t & \text{if otherwise.} \end{cases}$$

Suppose that participants buyers truthfully report their valuations in an equal-priority mechanism. Then using the allocation rule, we can calculate the probability  $Q^{\sigma}(w)$  with which each valuation of participants trades as follows.

For w < r, we have  $Q^{\sigma}(w) = 0$  as participants with valuation w below the reserve price r are excluded from allocation. For  $w \in [r, \underline{w})$ , we have

$$Q^{\sigma}(w) = (1 - \alpha)^{n-1} F^{n-1}(w),$$

as participants with valuation w below the pooling interval  $[\underline{w}, \overline{w}]$  have lower allocation priorities than non-participants. They get the good only when all other n-1 buyers are participants, and only when w is the highest among them above the reserve price r. For  $w > \overline{w}$ , we have

$$Q^{\sigma}(w) = \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) F^{n-1-m}(w) = \left( (1-\alpha) F(w) + \alpha \right)^{n-1}$$

as participants with valuation w above the pooling interval  $[\underline{w}, \overline{w}]$  have higher allocation priorities than non-participants. They get the good when any of the n-1 buyers is either a participant with a valuation lower than w, or a non-participant.

Finally, for  $w \in [\underline{w}, \overline{w}]$ , we have

$$Q^{\sigma}(w) = \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(\underline{w}, \overline{w}) \frac{1}{m+k+1},$$

where

$$B_{\tilde{k}}^{\tilde{n}}(\underline{w},\overline{w}) = \begin{pmatrix} \tilde{n} \\ \tilde{k} \end{pmatrix} (F(\overline{w}) - F(\underline{w}))^{\tilde{k}} F^{\tilde{n}-\tilde{k}}(\underline{w})$$

is the probability that  $\tilde{k}$  participants, who are randomly drawn out of  $\tilde{n} \geq \tilde{k}$  participants with valuations  $w \leq \overline{w}$ , have valuations in the pooling interval  $[\underline{w}, \overline{w}]$ . The allocation probability  $Q^{\sigma}(w)$  is independent of  $w \in [\underline{w}, \overline{w}]$ , because all participants with valuations on the pooling interval have the same allocation priority. As it plays a critical role in the analysis below, for convenience we denote the trading probability  $Q^{\sigma}(w)$  for  $w \in [\underline{w}, \overline{w}]$  as  $\chi(\underline{w}, \overline{w})$ . We re-do the double summations over m and k by first summing over k for fixed l = m + k then summing over l:

$$\begin{split} \chi(\underline{w},\overline{w}) &= \sum_{l=0}^{n-1} \binom{n-1}{l} \left( (1-\alpha)F(\underline{w}) \right)^{n-1-l} \frac{1}{l+1} \sum_{k=0}^{l} \binom{l}{k} \left( (1-\alpha)(F(\overline{w}) - F(\underline{w})) \right)^{k} \alpha^{l-k} \\ &= \sum_{l=0}^{n-1} \binom{n-1}{l} \left( (1-\alpha)F(\underline{w}) \right)^{n-1-l} \frac{1}{l+1} \left( (1-\alpha)(F(\overline{w}) - F(\underline{w})) + \alpha \right)^{l}. \end{split}$$

It follows that

$$\chi(\underline{w},\overline{w}) = \frac{((1-\alpha)F(\overline{w}) + \alpha)^n - ((1-\alpha)F(\underline{w}))^n}{n((1-\alpha)(F(\overline{w}) - F(\underline{w})) + \alpha)}.$$
(6)

The logic of (6) is that, the probability that a participant whose valuation is in the equalpriority interval  $[\underline{w}, \overline{w}]$  gets the good,  $\chi(\underline{w}, \overline{w})$ , is the same for all participants with valuation in the pool, and the same as any non-participant, as long as there are no participants reporting a valuation above  $\overline{w}$ . This explains why in the formula (6) the denominator is the expected number of buyers in the equal-priority pool, and the numerator is the ex ante probability that there is at least one buyer, participant or non-participant, in that equalpriority pool.

To summarize, we have the following expected allocation for participants with any valuation w:

$$Q^{\sigma}(w) = \begin{cases} 0 & \text{if } w < r \\ (1-\alpha)^{n-1}F^{n-1}(w) & \text{if } w \in [r,\underline{w}) \\ \chi(\underline{w},\overline{w}) & \text{if } w \in [\underline{w},\overline{w}] \\ ((1-\alpha)F(w)+\alpha)^{n-1} & \text{if } w > \overline{w}. \end{cases}$$
(7)

The following result gives the necessary and sufficient condition for an equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  to be incentive compatible.

**Lemma 1** An equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  is incentive compatible if and only if

$$\int_{r}^{\underline{w}} (1-\alpha)^{n-1} F^{n-1}(w) dw \ge \chi(\underline{w}, \overline{w})(\underline{w}-t)$$
(8)

Two arguments are needed to establish Lemma 1. The first is standard: we need to show that the rules of allocation and payment are the ones that make truthful reporting by participants incentive compatible. Since the allocation rule is monotone, we just need to show that the payoff of participants  $U^{\sigma}(w)$  from truthful reporting satisfies (2) for each w. Indeed, the payment rule is constructed from the allocation rule of the equal-priority mechanism to ensure that it is incentive compatible with respect to valuations. The second is to show that when t satisfies condition (8) no participant can improve their payoff by pretending to be a non-participant. Since  $Q^{\sigma}(w) = \chi(\underline{w}, \overline{w})$  for all  $w \in [\underline{w}, \overline{w}]$ , it follows from (2) that  $U^{\sigma}(w)$  is linear with slope  $\chi(\underline{w}, \overline{w})$ . By construction, this is the same slope as the increasing part of the payoff function  $U^{\mu}(w)$  for non-participants. This is

$$U^{\mu}(w) = \chi(\underline{w}, \overline{w}) \max\{w - t, 0\},\tag{9}$$

because non-participants have the same allocation priority as participants whose valuations are in  $[\underline{w}, \overline{w}]$ . Moreover, since by construction  $Q^{\sigma}(w)$  is strictly increasing for  $w \in [r, \underline{w})$ and  $w > \overline{w}$ , it follows from (2) that the payoff function  $U^{\sigma}(w)$  is strictly convex for  $w \ge r$  outside  $[\underline{w}, \overline{w}]$ . The equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  is therefore incentive compatible if and only if

$$U^{\sigma}(\underline{w}) \ge U^{\mu}(\underline{w})$$

This is precisely (8).

Figure 1 shows the payoffs to participants and non-participants in an equal-priority mechanism with a binding incentive compatible constraint (8). The green line represents the payoff function  $U^{\mu}(\cdot)$  of a non-participant or a participant that acts as one. It is zero for valuations below t, and has a slope equal to  $\chi(\underline{w}, \overline{w})$  above t. The red curve represents the payoff function  $U^{\sigma}(w)$  to a participant. It coincides with the green line for valuations in the equal-priority interval  $[\underline{w}, \overline{w}]$  because the incentive condition (8) is binding, and is strictly convex for valuations between r and  $\underline{w}$ , and above  $\overline{w}$ .



Figure 1. An equal-priority mechanism with a binding incentive constraint.

In any equal-priority mechanism, participants with low valuations, between r and  $\underline{w}$ , and those with high valuations, above  $\overline{w}$ , are strictly worse off by pretending to a non-participant. If the incentive compatibility constraint (8) is binding, it is a matter of indifference for participants with valuations in  $[\underline{w}, \overline{w}]$  whether they truthfully report their valuations or wait for fixed-price offer t just like a non-participant. Indeed, the same truth telling equilibrium among participants is implemented if we change the payment rule, so that a participant with valuations in the equal-priority interval  $[\underline{w}, \overline{w}]$  gets a fixed-price offer always equal to t, instead of being asked to pay the maximum of the second highest bid and reserve price r when there are no other buyers in the equal-priority pool, and  $\underline{w}$  when there is at least one other buyer in the pool. Furthermore, by revenue equivalence, the seller's revenue from participants is the same if all participants with valuations in the equal-priority interval  $[\underline{w}, \overline{w}]$ behave in the same way as non-participants. Since the allocation probability  $q^{\mu}(v)$  and the offer  $p^{\mu}(v)$  for non-participants depend only on the size of the equal-priority pool, i.e., m+k, and not on its composition, the seller's revenue from non-participants is also unaffected by whether or not participants with valuations in  $[\underline{w}, \overline{w}]$  pretend to be non-participants.

Any equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  with a binding incentive condition (8) is therefore payoff-equivalent for all buyers and the seller to the following auction, where a bid is registered as a distinguishable bid it is outside the equal-priority interval  $[\underline{w}, \overline{w}]$ , and is otherwise scrambled and pooled with non-participants, together with a fixed-price offer t made to a randomly selected buyer in the pool (l is the size of the pool below).

- When the highest distinguishable bid is less than r: the seller keeps the object if l = 0; otherwise, with probability 1/l the seller makes an offer t to each buyer in the equal-priority pool.
- When the highest distinguishable bid is between r and  $\underline{w}$ : if l = 0, the bidder that makes the bid wins, and pays the maximum of the second highest distinguishable bid and r; if  $l \ge 1$ , with probability 1/l, the seller makes an offer t to each buyer in the equal-priority pool.
- When the highest distinguishable bid is above  $\overline{w}$ : the bidder that makes bid wins; the winner pays the second highest distinguishable bid if it is above  $\overline{w}$ , pays the maximum of r and the second highest distinguishable bid if it is below  $\underline{w}$  and l = 0, and otherwise pays  $(\underline{w} + l\overline{w})/(l+1)$ .

The above indirect mechanism is what we refer to as the "equal-priority auction" in the introduction. It is a modified second-price auction. The winner is the one with the highest

distinguishable bid above the reserve price r, except when it is lower than  $\underline{w}$  and the equalpriority pool is non-empty, in which case the good is sold through the fixed-price offer of tto a random buyer in the pool. The price paid by the winner is the maximum of the reserve price r and the second highest distinguishable bid, except when the latter is lower than  $\underline{w}$ and the equal-priority pool is non-empty, in which case the price is adjusted by the size of the equal-priority pool, given by  $(\underline{w} + l\overline{w})/(l+1)$ .

### 3.1 Optimal equal-priority mechanism

Under an equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$ , the seller's expected revenue from participants is given by (4) with  $Q^{\sigma}(w)$  specified in (7). The revenue from non-participants, given by (5), becomes

$$\sum_{m=1}^{n} B_m^n(\alpha) \sum_{k=0}^{n-m} B_k^{n-m}(\underline{w}, \overline{w}) \frac{m}{m+k} \pi(t).$$

Using a similar method as in computing  $\chi(\underline{w}, \overline{w})$ , we redo the double summations above by first summing over l = m + k and then over l:

$$\begin{split} &\sum_{l=1}^{n} \binom{n}{l} \left( (1-\alpha)F(\underline{w}) \right)^{n-l} \sum_{k=0}^{l} \binom{l}{k} \left( (1-\alpha)(F(\overline{w}) - F(\underline{w})) \right)^{k} \alpha^{l-k} \frac{l-k}{l} \pi(t) \\ &= \sum_{l=1}^{n} \binom{n}{l} \left( (1-\alpha)F(\underline{w}) \right)^{n-l} \left( (1-\alpha)(F(\overline{w}) - F(\underline{w})) + \alpha \right)^{l-1} \alpha \pi(t) \\ &= \frac{\left( (1-\alpha)F(\overline{w}) + \alpha \right)^{n} - \left( (1-\alpha)F(\underline{w}) \right)^{n}}{(1-\alpha)(F(\overline{w}) - F(\underline{w})) + \alpha} \alpha \pi(t). \end{split}$$

Thus, we can write the objective of the problem of optimal equal-priority mechanism as

$$n(1-\alpha)\int_0^1 Q^{\sigma}(w)\,\phi(w)f(w)dw + n\alpha\chi(\underline{w},\overline{w})\pi(t).$$
(10)

The choice variables are  $\{r, \underline{w}, \overline{w}; t\}$ , and the constraints are  $r \leq \underline{w} \leq \overline{w}$  and (8). The following lemma provides a characterization of the solution in terms of necessary first-order conditions.

**Lemma 2** If  $\{r, \underline{w}, \overline{w}; t\}$  is an optimal equal-priority mechanism, then

$$0 < r < r^* < t < \underline{w} < \overline{w} < 1.$$

Further, (8) holds with equality, and

$$\alpha(\pi(t) - \phi(\overline{w})) = (1 - \alpha) \left( (\underline{w} - t)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) + \int_{\underline{w}}^{\overline{w}} (\phi(\overline{w}) - \phi(w))f(w)dw \right); (11)$$

$$-\alpha \pi'(t) = (1 - \alpha)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w});$$
(12)

$$-\phi(r)f(r) = (\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}).$$
(13)

Incentive compatibility of an equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$ , condition (8), requires that r < t, so that participants with valuations just above r do not have incentives to pretend to be a non-participant.<sup>6</sup> Our proof of Lemma 2 (in the appendix) first shows that  $r < r^* < t$ . This means that the seller provides incentives for participants with low valuations by selling to participants with negative virtual valuations, and simultaneously raising the fixed-price offer to non-participants to above the monopoly price  $r^*$ . That the optimal reserve price r is lower than  $r^*$ , the optimal reserve price of Myerson (1981), is one impact of non-participants on auctions for participants.

A more drastic impact is the existence of an equal-priority interval for participants. In the appendix, we use a variational argument to show that  $\underline{w} < \overline{w}$ , so that there is a nondegenerate equal-priority interval  $[\underline{w}, \overline{w}]$  for participants as long as participation is stochastic, i.e.,  $\alpha > 0$ . If the interval is degenerate, with  $\underline{w} = \overline{w} = \hat{w}$ , the equal-priority pool contains only non-participants. From the expressions of  $Q^{\sigma}(\cdot)$  in (7), we can easily verify that  $U^{\sigma}(w)$ has an upward kink at  $w = \hat{w}$ , with the slope just below  $\hat{w}$  and the slope just above  $\hat{w}$ bracketing  $\chi(\hat{w}, \hat{w})$ . By marginally decreasing  $\underline{w}$  and increasing  $\overline{w}$  to create an arbitrarily small equal-priority interval around  $\hat{w}$ , the seller can make the loss in the revenue from participants negligible. However, these variations in  $Q^{\sigma}(\cdot)$  raise the payoff function  $U^{\sigma}(w)$ strictly above  $U^{\mu}(w)$  over the newly created interval. As a result, the incentive condition (8) is relaxed. Since  $t > r^*$ , this allows the seller to reduce t marginally, with a first-order

<sup>&</sup>lt;sup>6</sup>The slope of  $U^{\sigma}(w)$  at w just above r is 0 while the slope of  $U^{\mu}(w)$  at w just above t is strictly positive.

increase in the revenue from non-participants.

In any optimal equal-priority mechanism, the incentive condition (8) for participants with valuations in the equal-priority interval  $[\underline{w}, \overline{w}]$  is binding. Otherwise, in Figure 1 we would have a line segment in the payoff function  $U^{\sigma}(\cdot)$  for participants parallel to, and above, the linear part of the payoff function  $U^{\mu}(\cdot)$  for non-participants. The seller would then want to either shrink the equal-priority interval, by increasing  $\underline{w}$  and decreasing  $\overline{w}$ , or raise the take-it-or-leave-it offer t to none-participants.

#### 3.2 Properties of optimal equal-priority mechanism

The conditions in Lemma 2 can be used derive properties of an optimal equal-priority mechanism.<sup>7</sup> First, if the seller does not give the object to a participant, the seller can always make a take-it-or-leave-it offer to a non-participant if there is one. Absent incentives, the seller would set the reserve price  $\overline{r}(t)$  for participants so that the virtual valuation is equal to the expected profit  $\pi(t)$  of making the offer t to a non-participant:

$$\phi(\overline{r}(t)) = \pi(t).$$

By condition (11), the optimal equal-priority mechanism has  $\phi(\overline{w}) < \pi(t)$ . This means that the seller gives the object to participants even though their virtual valuations are lower than the value of the seller's "outside option"  $\pi(t)$ . The reason for doing this is to provide incentives for participants with valuations just above  $\overline{w}$  to truthfully report their valuation rather than wait for the fixed-price offer by pretending to be a non-participant.

Second, when all buyers are surely participants, the revenue from the optimal equalpriority mechanism converges to the revenue from the standard auction with reserve price  $r^*$ . From equation (13), it becomes optimal for the seller not to distort the reserve price r

<sup>&</sup>lt;sup>7</sup>They are all derived with variational arguments without explicitly using a multiplier for (8). From the proof in the appendix, it can be seen that the value of the multiplier associated with (8) is the right-hand side of (12) multiplied by n. This turns out to be the integral of the multiplier function  $\lambda(\cdot)$  in the proof of Theorem 1 over the valuation support [0, 1]. Under a similar Lagrangian relaxation approach as in the proof of Theorem 1, we can use this value of the multiplier to show that (11), (12) and (13), together with binding (8), are also sufficient for an optimal equal-priority mechanism. The sufficiency of the conditions in Lemma 2 is also implied by Theorem 1, because they are shown to imply allocations and prices that are optimal among all mechanisms.

at all to provide incentives. The equal-priority interval shrinks to a single valuation  $w_0$  as  $\alpha$  goes to 0,<sup>8</sup> satisfying the binding constraint (8) that a participant with valuation  $w_0$  is indifferent between truthfully reporting it and receiving a fixed-price offer  $t_0$  when all other participants have valuations below  $w_0$ ,

$$\int_{r^*}^{w_0} F^{n-1}(w) dw = F^{n-1}(w_0)(w_0 - t).$$

The limit values of  $w_0$  and  $t_0$  satisfy the above indifference condition and the limit version of first order conditions (11) and (12), given by

$$\pi'(t_0)(w_0 - t_0) + \pi(t_0) - \phi(w_0) = 0.$$

We have  $t_0 > r^*$  and  $\pi(t_0) > \phi(w_0)$ . When  $\alpha$  is arbitrarily close to 0, the incentives for participants not to pretend to be a non-participant are provided by raising the fixed-price offer to an unlikely non-participant above  $r^*$ , and not selling to non-participants even when the profit from doing so exceeds virtual valuations of participants.

Third, in the opposite limit of  $\alpha = 1$ , buyers are surely non-participants, and the revenue from the optimal equal-priority mechanism converges to the revenue from a fixed-price offer  $r^*$ . By (12), the seller no longer distorts t to provide incentives for participants. From (11), the upper-bound of the equal-priority interval  $\overline{w}$  converges to  $\overline{r}(r^*)$ , satisfying

$$\phi(\overline{r}(r^*)) = \pi(r^*),$$

as the need for the seller to provide incentives for participants with valuations just above the upper-bound becomes second order. From the binding constraint (8), the lower-bound of the equal-priority interval becomes  $r^{*,9}$  This prevents an unlikely participant with a valuation equal to the lower bound from pretending to be a non-participant, as the buyer has almost

<sup>&</sup>lt;sup>8</sup>The limit of  $\chi(\underline{w}, \overline{w})$  as  $\alpha$  goes to 0 and  $\underline{w}$  and  $\overline{w}$  shrink to the same point of  $w_0$  is  $F^{n-1}(w_0)$ . That is, when all other buyers are almost surely participants, a deviating participant will be the only buyer in the equal-priority pool and will win the object with probability one if all other buyers (who are participants) have valuation below  $w_0$ .

<sup>&</sup>lt;sup>9</sup>The limit of  $\chi(\underline{w}, \overline{w})$  as  $\alpha$  goes to 1 is 1/n, as an unlikely participant will surely face n-1 non-participants in the equal-priority pool after pretending to be a non-participant.

no chance of making the winning bid with the limit reserve price  $r_1$  satisfying (13)

$$-\phi(r_1)f(r_1) = \pi(r^*)f(r^*).$$

As long as  $\alpha$  is strictly less than 1, however, we still need  $\underline{w} > r_1$  to in order to provide incentives for participants with valuations just below  $\underline{w}$  not to deviate and pretend to be non-participants.

## 4 Optimal Mechanisms

The main result of this paper is that an optimal equal-priority mechanism provides the seller the highest expected revenue among all incentive compatible mechanisms given in Definitions 1 and 2. A mechanism  $\delta$  given in Definition 1 consists of a series of functions  $(q_m^{\sigma}(v), p_m^{\sigma}(v))_{m=0}^{n-1}$  and  $(q_m^{\mu}(v), p_m^{\mu}(v))_{m=1}^n$ . We first use Assumption 1 to simplify the optimal design problem. Replacing all these offers with the expected offer reduces the deviation payoff to participants from pretending to be a non-participant. Concavity then implies a greater revenue from non-participants.

**Lemma 3** In any optimal incentive-compatible direct mechanism,  $p_m^{\mu}(v)$  is independent of m and v.

Using Lemma 3, we denote the constant price offered to non-participants as  $p^{\mu}$ . Define

$$Q^{\mu} = \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) \mathbb{E}_v \left[ q_{m+1}^{\mu}(v) \right]$$

to be the total probability of an offer expected by a non-participant. The expected revenue from non-participants, given by (5), can be written as

$$\sum_{m=1}^{n} B_m^n(\alpha) \mathbb{E}_v \left[ m q_m^\mu(v) \pi \left( p_m^\mu(v) \right) \right] = n \alpha Q^\mu \pi \left( p^\mu \right).$$

Next, we drop the transfers  $(p_m^{\sigma}(v))_{m=0}^{n-1}$  to participants, and consider a maximization problem using only allocations  $(q_m^{\sigma}(v))_{m=0}^{n-1}$ . Once we show that an optimal equal-priority

mechanism  $\{r, \underline{w}, \overline{w}; t\}$  solves the problem, we can then use the payment rule in section 3 to construct the transfers  $(p_m^{\sigma}(v))_{m=0}^{n-1}$  and the resulting payoff function  $U^{\sigma}(\cdot)$ , and apply Lemma 1 to conclude that the solution is incentive compatible.

The optimal mechanism problem can be stated as choosing  $(q_m^{\sigma}(v))_{m=0}^{n-1}$ ,  $(q_m^{\mu}(v))_{m=1}^n$ , and  $p^{\mu}$  to maximize

$$n(1-\alpha)\int_0^1 Q^{\sigma}(w)\,\phi(w)f(w)dw + n\alpha Q^{\mu}\pi\left(p^{\mu}\right),$$

subject to the feasibility constraint (1),  $Q^{\sigma}(\cdot)$  is non-decreasing, and for every w

$$\int_{0}^{w} Q^{\sigma}(x) \, dx \ge Q^{\mu} \max\left\{w - p^{\mu}, 0\right\}.$$
(14)

**Theorem 1** There is no incentive compatible mechanism that yields a strictly greater revenue for the seller than an optimal equal-priority mechanism.

To establish Theorem 1, we need to show that an optimal equal-priority mechanism solves the maximization set up for the theorem. Optimizing over all incentive compatible direct mechanisms is difficult, due to the continuum of incentive constraints (14) for participants with any valuation w not to pretend to be a non-participant. Instead we adopt an indirect approach of Lagrangian relaxation method, by incorporating the continuum of constraints through a multiplier function. The construction of the multiplier function uses the necessary conditions established in Lemma 3 for an optimal equal-priority mechanism. This approach may be used in more general problems of mechanism with stochastic participation discussed in section 5. We outline it in the next subsection.

#### 4.1 Lagrangian relaxation approach

Let  $\lambda(\cdot)$  be an arbitrary non-negative valued Lagrangian function from [0, 1] into  $\mathbb{R}$ . The relaxed problem is to maximize

$$n(1-\alpha) \int_{0}^{1} Q^{\sigma}(w) \phi(w) f(w) dw + n\alpha Q^{\mu} \pi(p^{\mu}) + \int_{0}^{1} \lambda(w) \left( \int_{0}^{w} Q^{\sigma}(x) dx - Q^{\mu} \max\{w - p^{\mu}, 0\} \right) dw,$$

with the same choice variables and constraints except (14). That is, by introducing the Lagrangian function, we incorporate a continuum of constraints (14) into the objective function of the relaxed problem as an extra term.

The above relaxed problem has different solutions depending on the choice of  $\lambda(\cdot)$ . Regardless of the choice of  $\lambda(\cdot)$ , however, the value of the relaxed problem is an upper bound on the value of the full problem, because the solution to the full problem is feasible for the relaxed problem and because the extra term in the objective function of the relaxed problem is non-negative by construction. We will try to construct a function  $\lambda(\cdot)$  such that the solution to the relaxed problem is an optimal equal-priority mechanism. Since the equal-priority mechanism yields an upper bound on the seller's revenue in the full problem, and since it satisfies all the constraints in the full problem, it solves the full problem.

The multiplier function  $\lambda(\cdot)$  is the shadow cost (benefit) of violating (relaxing) the constraints (14). The second term in the relaxed Lagrangian is the total shadow value. The relaxed problem is then choosing feasible allocations  $(q_m^{\sigma}(v))_{m=0}^{n-1}$  and  $(q_m^{\mu}(v))_{m=1}^n$ , together with  $p^{\mu}$ , subject to the feasibility constraint (1), to maximize the sum of the resulting revenues from participants and non-participants, and the shadow values. The key to our construction of the desired  $\lambda(\cdot)$  is that, first, it satisfies complementary slackness so that the extra term in the relaxed Lagrangian is zero; and second, the allocations of an optimal equal-priority mechanism characterized by Lemma 2 maximize the sum of the revenues and the shadow values.

More precisely, we use integration by parts and rewrite the Lagrangian as

$$\int_0^1 K^{\sigma}(w)Q^{\sigma}(w)f(w)dw + K^{\mu}Q^{\mu},$$

where, for each  $w \in [0, 1]$ ,

$$\begin{split} K^{\sigma}(w)f(w) &= n(1-\alpha)\phi(w)f(w) + \int_{w}^{1}\lambda(x)dx;\\ K^{\mu} &= n\alpha\pi(p^{\mu}) - \int_{0}^{1}\lambda(x)\max\{x-p^{\mu},0\}dx. \end{split}$$

The two terms of  $K^{\sigma}(w)$  and  $K^{\mu}$  precisely capture how the shadow values are incorporated

into the objective of the relaxed Lagrangian separately through the revenue from participants and the revenue from non-participants, with the former dependent on the valuation but the latter independent. Disaggregating  $Q^{\sigma}(w)$  and  $Q^{\mu}$  over the realized number m of nonparticipants and over the realized valuation profile v of n - m participants, and using the symmetry of the mechanism, we have the final expression for the relaxed Lagrangian:

$$\frac{(1-\alpha)^{n-1}}{n} \mathbb{E}_{v} \Big[ \sum_{i=1}^{n} K^{\sigma}(v_{i}) q_{0}^{\sigma}(\rho_{0}^{i}(v)) \Big] + \sum_{m=1}^{n-1} \mathbb{E}_{v} \Big[ \frac{B_{m}^{n-1}(\alpha)}{n-m} \sum_{i=1}^{n-m} K^{\sigma}(v_{i}) q_{m}^{\sigma}(\rho_{m}^{i}(v)) + B_{m-1}^{n-1}(\alpha) K^{\mu} q_{m}^{\mu}(v) \Big] + \alpha^{n-1} K^{\mu} q_{n}^{\mu}.$$
(15)

The first term in (15) represents the objective of the relaxed Lagrangian when m = 0, where  $K^{\mu}$  does not appear because all n buyers are participants. Similarly, the last term represents the objective when m = n, where  $K^{\sigma}$  does not appear and  $q_n^{\mu}$  is constant because there are no participants. The middle term in (15) represents the objective of the relaxed Lagrangian when there are both participants and non-participants, that is, when m is between 1 and n-1, with the first part for participants and the second for non-participants.

Fix an equal-priority mechanism  $\{r, t, \underline{w}, \overline{w}\}$  that binds the incentive condition (8) and satisfies the necessary conditions for optimality, equations (11)-(13) in Lemma 2. We construct a multiplier function  $\lambda$ , with  $\lambda(w) = 0$  for all  $w \notin [\underline{w}, \overline{w}]$ , such that  $K^{\sigma}(w)$  is equal to a constant  $K^{\sigma}(\overline{w})$  for all  $w \in [\underline{w}, \overline{w}]$ .<sup>10</sup> Under Assumption 1, using the necessary conditions in Lemma 2 for  $\{r, t, \underline{w}, \overline{w}\}$  to be an optimal equal-priority mechanism we can show that  $\lambda(w) \geq 0$  for all  $w \in [\underline{w}, \overline{w}]$ , with  $K^{\sigma}(w)$  strictly increasing for all  $w < \underline{w}$  and  $w > \overline{w}$  such that  $K^{\sigma}(w) > 0$ . Further, we can show that  $p^{\mu} = t$  maximizes  $K^{\mu}$ , and hence the relaxed Lagrangian (15), and the maximized value  $K_t^{\mu}$  of  $K^{\mu}$  satisfies

$$\frac{B_m^{n-1}(\alpha)}{n-m}K^{\sigma}(\overline{w}) = \frac{B_{m-1}^{n-1}(\alpha)}{m}K_t^{\mu}.$$

Theorem 1 follows by showing that the allocations given by the equal-priority mechanism maximizes (15). In particular, when m is between 1 and n - 1, it is point wise maximizing

<sup>&</sup>lt;sup>10</sup>For any  $\underline{w} < \overline{w}$ , there is a unique such function  $\lambda$ . To see this, note that  $K^{\sigma}(\overline{w}) = n(1-\alpha)\phi(\overline{w})$  because  $\lambda(w) = 0$  for all  $w > \overline{w}$ . We can then solve for  $\lambda(w)$  for all  $w \in [\underline{w}, \overline{w}]$  from  $K^{\sigma}(w) = K^{\sigma}(\overline{w})$ .

to give the good to the participant with the highest valuation if it is greater than  $\overline{w}$ , that is, choose  $q_m^{\sigma}(\rho_m^i(v)) = 1$  if  $v_i$  is the highest among  $(v_1, \ldots, v_{n-m})$  and if  $v_i > \overline{w}$ . Otherwise, by (1), the seller should give the good with an equal probability to all k participants with valuations in  $[\underline{w}, \overline{w}]$  and m non-participants. That is, under the feasibility constraint (1), it is point wise maximizing to choose  $q_m^{\sigma}(\rho_m^i(v)) = q_m^{\mu}(v) = 1/(m+k)$  if  $v_i$  is the highest valuation among  $(v_1, \ldots, v_{n-m})$  and if  $v_i$  is on the interval  $[\underline{w}, \overline{w}]$ .

#### 4.2 Observable stochastic participation

To understand revenue and welfare effects of stochastic participation, we compare the optimal equal-priority mechanism established in Theorem 1 to the benchmark of "observable stochastic participation" where the seller can separate participants from non-participants. In this benchmark, the seller chooses a direct mechanism  $\left\{ \left(q_m^{\sigma}, p_m^{\sigma}\right)_{m=0}^{n-1}, \left(q_m^{\mu}, p_m^{\mu}\right)_{m=1}^n \right\}$  as defined in Definition 1. Only incentive compatibility with respect to valuations is required of participants. The optimal mechanism in the benchmark maximizes the sum of the expected revenue from participants (4) and from non-participants (5), subject only to the feasibility constraint (1). Lemma 3 applies, and the optimal  $p^{\mu}(v) = r^*, m = 1, \ldots, n$ , for all profile of valuations v of participants. For any m between 1 and n-1, the optimal allocation is to choose  $q_m^{\sigma}(\rho_m^i(v)) = 1$  if  $v_i$  is the highest among the profile  $v = (v_1, \ldots, v_{n-m})$  and if  $v_i > \overline{r}(r^*)$ , and otherwise  $q_m^{\mu}(v) = 1/m$ ; for m = 0, the optimal allocation is  $q_0^{\sigma}(\rho_m^i(v)) = 1$  if  $v_i$  is the highest among the profile v and if  $v_i > r^*$ , and  $q_0^{\sigma}(\rho_m^i(v)) = 0$  for all i. The optimal mechanism under observed stochastic participation has no equal-priority interval for participations, because the seller can separate participants from non-participants without paying any information rent to the former. For participants, in addition to the optimal reserve price of  $r^*$  of Myerson's (1981) that applies when there are no non-participants, there is a reserve price of  $\overline{r}(r^*)$  as the good is sold to a non-participant at the monopoly price of  $r^*$  when the highest virtual valuation among participants is below the monopoly revenue of  $\pi(r^*)$ . We can also interpret the optimal mechanism in the benchmark as the "unconstrained solution" to the mechanism design problem under stochastic participation, as the key incentive compatibility constraint (3) in Definition 2 is missing.

Under the unconstrained solution, the seller's revenue from participants is

$$n(1-\alpha)\left(\int_{r^*}^{\overline{r}(r^*)} (1-\alpha)^{n-1} F^{n-1}(w)\phi(w)f(w)dw + \int_{\overline{r}(r^*)}^1 ((1-\alpha)F(w)+\alpha)^{n-1}\phi(w)f(w)dw\right),$$

and the revenue from non-participants is

$$\sum_{m=1}^{n} B_m^n(\alpha) F^{n-m}(\overline{r}(r^*)) \pi(r^*) = (((1-\alpha)F(\overline{r}(r^*)) + \alpha)^n - (1-\alpha)^n F^n(\overline{r}(r^*))) \pi(r^*).$$

Compare the above with the revenue of an equal-priority mechanism given by (10), (7) and (6).<sup>11</sup> In response to the incentive compatibility constraint (3), the seller not only increases the fixed-price offer to non-non-participants to above the monopoly price  $r^*$ , but also decreases the reserve price to below the optimum of  $r^*$  and creates a pooling interval above  $r^*$  in the auction among participants even though the virtual valuation function is strictly increasing. Relative to the unconstrained solution, under the optimal equal-priority mechanism the seller may obtain a greater revenue from participants, but the gain is outweighed by the loss in revenue from non-participants.<sup>12</sup> Overall the seller faces a revenue loss under the optimal equal-priority mechanism compared to the unconstrained solution.

Participants are better off under the optimal equal-priority auction than in the benchmark. Under the unconstrained solution, the interim expected payoff a participant with valuation w is given by

$$\begin{cases} 0 & \text{if } w < r^* \\ \int_{r^*}^w (1-\alpha)^{n-1} F^{n-1}(x) dx & \text{if } w \in [r^*, \overline{r}(r^*)] \\ \int_{r^*}^{\overline{r}(r^*)} (1-\alpha)^{n-1} F^{n-1}(x) dx + \int_{\overline{r}(r^*)}^w ((1-\alpha)F(x) + \alpha)^{n-1} dx & \text{if } w > \overline{r}(r^*). \end{cases}$$

Compare the above with  $U^{\sigma}(w)$  given by (2) and (7). The incentive compatibility constraint

<sup>&</sup>lt;sup>11</sup>We can obtain the revenue formula under the unconstrained solution from (10), as well as the formulas for the interim payoffs of participants and non-participants below from (2) and from (9) respectively, by replacing  $Q^{\sigma}(w)$  with 0 for  $w < r^*$ ,  $(1 - \alpha)^{n-1}F^{n-1}(w)$  for  $w \in [r^*, \overline{r}(r^*)]$  and  $((1 - \alpha)F(w) + \alpha)^{n-1}$  for  $w > \overline{r}(r^*)$ , and replacing  $\chi(\underline{w}, \overline{w})$  with  $(((1 - \alpha)F(\overline{r}(r^*)) + \alpha)^n - (1 - \alpha)^nF^n(\overline{r}(r^*)))/(n\alpha)$ .

<sup>&</sup>lt;sup>12</sup>As we show below, the entire equal-priority interval  $[\underline{w}, \overline{w}]$  of participants is below the reserve price of  $\overline{r}(r^*)$  in the unconstrained solution. This implies that under the optimal equal-priority auction the good is allocated to the participants with greater probabilities than under the constrained solution. See the example of uniform valuation distribution below.

(3) leads to the seller setting the reserve price r below  $r^*$ . Moreover, for all  $w \in [\underline{w}, \overline{w}]$  we have

$$\chi(\underline{w}, \overline{w}) > (1 - \alpha)^{n-1} F^{n-1}(w).$$

As we have shown in section 3.2, the optimal equal-priority mechanism has  $\phi(\overline{w}) < \pi(t)$  and thus  $\overline{w} < \overline{r}(r^*)$ . It follows that all participants with valuations above r are strictly better off under the optimal equal-priority mechanism than in the benchmark. The gain in the interim payoff of participants represents the information rent that the seller pays as a result of the incentive compatibility constraint (3).

In contrast to participants who earn information rents, non-participants are hurt by stochastic participation. Under the unconstrained solution, the interim expected payoff a non-participant with valuation w is given by

$$\sum_{m=0}^{n-1} B_m^{n-1}(\alpha) F^{n-m}(\overline{r}(r^*)) \frac{\max\{w - r^*, 0\}}{m+1}$$
$$= (((1-\alpha)F(\overline{r}(r^*)) + \alpha)^n - (1-\alpha)^n F^n(\overline{r}(r^*))) \frac{\max\{w - r^*, 0\}}{n\alpha}$$

Compare the above with  $U^{\mu}(w)$  given by (9) and (6). The incentive compatibility constraint (3) induces the seller to charge a price t to non-participants higher than  $r^*$  under in the benchmark. Moreover, since  $\underline{w} < \overline{w} < \overline{r}(r^*)$ , we have<sup>13</sup>

$$n\alpha\chi(\underline{w},\overline{w}) < ((1-\alpha)F(\overline{r}(r^*)) + \alpha)^n - (1-\alpha)^n F^n(\overline{r}(r^*)).$$

Since non-participants face a higher price and have to share the equal-priority pool with participants who have valuations from the interval  $[\underline{w}, \overline{w}]$ , all non-participants with valuations above  $r^*$  are strictly worse off under the optimal equal-priority mechanism than in the benchmark.

The relative simplicity of optimal equal-priority mechanisms allows us to gauge the significance of stochastic participation relative to the benchmark. We use an explicit example to illustrate. Suppose that the valuation distribution F is uniform on [0, 1], with  $r^* = \frac{1}{2}$ 

<sup>&</sup>lt;sup>13</sup>The right-hand side below is equal to  $\alpha \sum_{j=0}^{n-1} ((1-\alpha)F(\overline{r}(r^*)) + \alpha)^{n-1-j}((1-\alpha)F(\overline{r}(r^*)))^j$ , while  $n\chi(\underline{w},\overline{w}) = \sum_{j=0}^{n-1} ((1-\alpha)F(\overline{w}) + \alpha)^{n-1-j}((1-\alpha)F(\underline{w}))^j$ .

and  $\overline{r}(r^*) = \frac{5}{8}$ . We can solve for the optimal equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  as follows. Define  $\Lambda = (\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w})$ ; in this example, we have  $\Lambda = \overline{w} - \underline{w}$ .<sup>14</sup> Since the three first-order conditions (11), (12) and (13) are all independent of n, we can then proceed to use the three conditions to solve for  $r, \underline{w}, \overline{w}$  and t, all as functions of  $\Lambda$ . The range of the values of  $\Lambda$  is from the lower-bound of 0, which by (11), (12) and (13) implies that  $r = t = \frac{1}{2}$  and  $\overline{w} = \underline{w} = \frac{5}{8}$ , to an upper-bound  $\overline{\Lambda} = \alpha/(2\sqrt{5-\alpha}+4)$ , at which point  $\underline{w} - t$  becomes 0. Plugging in these functions in the binding constraint (8), we can find a unique value of  $\widehat{\Lambda} \in (0, \overline{\Lambda})$  for each  $n \geq 2$ , and hence the optimal equal-priority mechanism.

In this example, the value of  $\hat{\Lambda}$  represents the optimal size of the equal-priority pool for participants due to stochastic participation. For any fixed Bernoulli probability  $\alpha$  that a given buyer is a non-participant,  $\hat{\Lambda}$  is increasing in the number of potential buyers n. The maximum size is  $\overline{\Lambda}$ , which increases with  $\alpha$  and is maximized at  $\frac{1}{8}$  when  $\alpha = 1$ .<sup>15</sup> For  $\alpha = \frac{1}{2}$ , the optimal size for n = 5 is approximately 0.059, already approaching the maximum size of  $\overline{\Lambda} \approx 0.061$ . These numbers suggest that optimal equal-priority pools can be significant under stochastic participation. As mentioned at the end of the introduction, ignoring the impact of stochastic participation can significantly bias structural estimations of valuation distributions in auctions.

In section 3.2, we have characterized the optimal equal-priority mechanisms when the Bernoulli probability  $\alpha$  goes to either 0 or 1. Although the mechanism does not converge to the optimal auction of Myerson (1981) at  $\alpha = 0$  or the monopoly pricing at  $\alpha = 1$ , the seller's revenue from the optimal equal-priority mechanism does converge to the corresponding revenue. As a result, the seller faces little revenue loss under the optimal equal-priority mechanism relative to the unconstrained solution when the value of  $\alpha$  is extreme. In the uniform example, it turns out that the percentage revenue loss relative to the unconstrained solution is small even for intermediate values of  $\alpha$ . For  $\alpha = \frac{1}{2}$ , when n = 5, under the optimal equal-priority mechanism the revenue from non-participants is about 0.081, compared to about 0.088 under the unconstrained solution, while the revenue from participants is about

<sup>&</sup>lt;sup>14</sup>The variable  $\Lambda$  has the interpretation of an arbitrary non-negative value of the multiplier associated with the constraint (8), normalized by dividing by n.

<sup>&</sup>lt;sup>15</sup>As we have shown in section 3.2, as  $\alpha$  approaches 1, the upper-bound  $\overline{w}$  of the optimal equal-priority pool approaches  $\overline{r}(r^*)$  and the lower-bound  $\underline{\omega}$  approaches  $r^*$ . For the uniform example,  $\overline{r}(r^*) - r^* = \frac{1}{8}$ .

0.438 versus about 0.437. The overall percentage revenue loss relative to the unconstrained solution is just over 1%.

Our example suggests that equal-priority mechanisms are quite effective responses to the incentive problem in stochastic participation. Another way to see this is to consider changes in the seller's revenue under the optimal equal-priority auction for the same total number of buyers n when the value of  $\alpha$  decreases. For n = 5, we have seen that the seller's optimal revenue is 0.519 when  $\alpha = \frac{1}{2}$ ; it increases to 0.559 when  $\alpha = \frac{2}{5}$ . This amounts to an almost 8% increase, in contrast to a just above 7% increase under the unconstrained solution for the same decrease in  $\alpha$ . The percentage increase in the revenue is greater under the optimal equal-priority auction, because not only participants are more profitable for the seller to trade with than non-participants, which is also true under the unconstrained solution, but also the seller can reduce the information rent paid to participants. An interpretation of a decrease in  $\alpha$  is that seller invests resources to turn some non-participants into participants, perhaps through "educating" potential buyers about how auctions work or making it more convenient to participate in auctions. As suggested by our numerical example, the rate of return for such investment can be substantial.

Stochastic participation can have significant effects on the welfare of participants and non-participants. For n = 5 and  $\alpha = \frac{1}{2}$ , the interim payoff of participants under the optimal equal-priority mechanism, averaged over all valuations on [0, 1], is about 0.046. This is about 8% more than the average interim payoff of around 0.042 under the unconstrained solution, representing a sizable information rent due to the incentive compatibility constraint (3). For non-participants, the average interim payoff under the optimal equal-priority mechanism is about 0.013, compared with about 0.018 under the unconstrained solution. The negative impact of stochastic participation on the welfare of an average non-participant is similar to the information rent to an average participant in magnitude, but in percentage terms it is much higher, about 28%. This is because non-participants have a much lower average interim payoff than participants under both the optimal equal-priority auction and under the unconstrained solution. Indeed, if a non-participant could invest some resources to become a participant, the rate of return for such investment would be quite high.

# 5 Discussion

To generalize the notion of mechanism design with stochastic participation, it is useful to think of an "outside market" where the seller can sell the good to buyers that do not participate in the seller's mechanism. Correspondingly, we can think of a direct mechanism as specifying, for any profile of reported valuations by participants, whether the good is allocated to a participant or sold in the outside market, and in case of the latter, a set of participants to join the outside market along with all non-participants, together with a price.<sup>16</sup> The seller's option of selling the good to the outside market creates an incentive problem because would-be participants could instead join the outside market. As shown in Theorem 1, an equal-priority auction is a solution to this problem, where participants with valuations from an optimally chosen interval do indeed join the outside market, even though the seller's mechanism can make the interval depend on the size and the composition of the outside market.

The optimality of an equal-priority auction does not rely on the specific structural assumptions on the outside market that participants and non-participants are independently drawn from a fixed number of potential buyers through a Bernoulli distribution, and have valuations independently drawn from the same distribution. Imagine instead that there are two independent Poisson processes that determine the realized numbers of participants and non-participants, and these two groups of buyers draw their private valuations from different distributions. As in the model of section 2, the seller faces the same incentive problem by selling to the outside market, so long as the unconstrained solution similarly defined as in section 4.2 – charging the monopoly price to a non-participant whenever the revenue is higher than the highest virtual valuation of a participant – attracts deviations by participants to the outside market. Since the numbers of participants and non-participants are both random, there is no "forcing mechanism" that would allow the seller to solve the incentive problem without paying an information rent to participants.<sup>17</sup> Similar to Theorem 1, in

 $<sup>^{16}</sup>$ As we have commented in section 2.1, for any direct mechanism that is incentive compatible with respect to valuations, it is sufficient to specify probabilities of getting the good when the good is allocated to a participant.

<sup>&</sup>lt;sup>17</sup>If either or both numbers are deterministic, the seller could detect any deviations by participants and commit to not selling the good when such a deviation occurs.

the optimal mechanism the information rent takes the form of an interval of valuations so that participants with any valuation on the interval have an equal allocation priority as nonparticipants. Participants with valuations just above the interval have a higher allocation priority than non-participants, even though their virtual valuation is lower than the revenue the seller could obtain from the outside market.

In our model we have assumed that when the good is sold in the outside market it is done through a take-it-or-leave-it offer to a randomly selected buyer in the market. Whether the effect of inducing a participant to join the outside market, in terms of the revenue and the information rent, is positive or negative depends neither on the size nor on the composition of the current market.<sup>18</sup> This feature of take-it-or-leave-it offers turns out to critical to the optimality of equal-priority auctions.

Instead of making a take-it-or-leave-it offer, imagine that the seller uses a first-comefirst-served rule in the outside market. Under this rule, if a participant is included in the outside market for some profile of reported valuations, then for any price chosen by the seller that is below the participant's valuation, the participant gets the good with the same probability as all non-participants with valuations above the price. As a result, whether a participant with a given valuation should be included in the outside market can depend both on the size and the composition of the current market. To see this, suppose that the current market consists of only non-participants, and thus the probability that the offer is accepted increases in the size of the market. With a participant included in the market, the offer is accepted with probability one. This implies that the effect of including a participant in the market depends on it size. Further, for the same size of the current market, whether it already includes at least one participant determines whether the probability that the offer is accepted is one or less than one, and thus the effect of including an additional participant in the market depends on its composition. Equal-priority auctions are generally not optimal under a first-come-first-served rule in the outside market.

In the present model, we can restore the optimality of equal-priority auctions under a

<sup>&</sup>lt;sup>18</sup>More precisely, in equation (15), for any multiplier function  $\lambda(\cdot)$ , and for any realized number of nonparticipant m and any set of participants already included in the outside market, by inducing some participant i with valuation  $v_i$  to join the market, the sign of the effect on the Lagrangian depends only on whether  $K^{\sigma}(v_i)$  is greater or smaller than the value of the Lagrangian before i joins.

first-come-first-served rule in the outside market by restricting the outside market to possibly include only a single interval of valuations for participants and use only a single price, regardless of the profile of reported valuations. Such restriction is more natural in the aforementioned alternative model where the numbers of participants and non-participants are determined by two independent Poisson distributions. One can imagine that the seller cannot condition the decision of whether to sell the outside market or the choice of the firstcome-first-served price on the realized number of non-participants, because the seller never observes the size of the market. Furthermore, in the alternative model the assumption that the seller does not observe the size of the outside market may be sufficient to imply that it is optimal to have a single equal-priority pool and a single price. We leave this conjecture for future research.

# 6 Appendix: Omitted Proofs

### Proof of Lemma 1

We verify that the expected payoff of a participant with valuation w matches  $U^{\sigma}(w)$  given by (2) and (7). There are four cases.

(i) By truthfully reporting his valuation, a participant with w < r never wins the object, and thus the expected payoff is 0, matching  $U^{\sigma}(w)$  in (7) and (2) for w < r.

(ii) By truthful reporting, a participant with  $w \in [r, \underline{w})$  wins the object only when m = 0and all n - 1 other participants have valuation at most w, pays the maximum of r and the second highest valuation. Thus, the expected payoff is

$$w(1-\alpha)^{n-1}F^{n-1}(w) - \left(r(1-\alpha)^{n-1}F^{n-1}(r) + \int_r^w x \ d\left((1-\alpha)^{n-1}F^{n-1}(x)\right)\right).$$

By integration by parts, the above matches  $U^{\sigma}(v)$  in (2) and (7) for  $v \in [r, \underline{w})$ .

(iii) By truthful reporting, a participant with  $w \in [\underline{w}, \overline{w}]$  wins the object with probability one when m = 0 and all n - 1 other participants have valuation at most  $\underline{w}$ , and pays the maximum of r and the second highest valuation. The contribution of this event to the buyer's expected payoff is

$$w(1-\alpha)^{n-1}F^{n-1}(\underline{w}) - \left(r(1-\alpha)^{n-1}F^{n-1}(r) + \int_{r}^{\underline{w}} x \ d\left((1-\alpha)^{n-1}F^{n-1}(x)\right)\right)$$
  
= $U^{\sigma}(\underline{w}) + (w-\underline{w})(1-\alpha)^{n-1}F^{n-1}(\underline{w}).$ 

The buyer also wins the object with probability 1/(m + k + 1) when there are m nonparticipants, all n - m - 1 other participants have valuation at most  $\overline{w}$ , and m + k is at least 1 (where k is the number of participants with valuation on  $[\underline{w}, \overline{w}]$ ), and pays  $\underline{w}$ . The contribution of this event to the buyer's expected payoff is

$$(w - \underline{w}) \left( \chi(\underline{w}, \overline{w}) - (1 - \alpha)^{n-1} F^{n-1}(\underline{w}) \right).$$

The sum of the above two expressions matches  $U^{\sigma}(w)$  in (2) and (7) for  $w \in [\underline{w}, \overline{w}]$ .

(iv) By truthful reporting, a participant with  $w > \overline{w}$  wins the object with probability one when m = 0 and the second highest bid is below  $\underline{w}$ , and the participant pays the maximum of the second highest bid and the reserve price r. The contribution to the expected payoff is

$$U^{\sigma}(\underline{w}) + (w - \underline{w})(1 - \alpha)^{n-1}F^{n-1}(\underline{w}).$$

The participant also wins with probability one when the second highest bid is below  $\overline{w}$  and when  $m + k \ge 1$ , and pays  $(\underline{w} + \overline{w}(m + k))/(m + k + 1)$ . The contribution to the expected payoff is

$$\sum_{m=0}^{n-1} B_m^{n-1}(\alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(\underline{w}, \overline{w}) \left( w - \frac{\underline{w} + \overline{w}(m+k)}{m+k+1} \right) - (w - \underline{w})(1-\alpha)^{n-1} F^{n-1}(\underline{w})$$
$$= (w - \overline{w})((1-\alpha)F(\overline{w}) + \alpha)^{n-1} + (\overline{w} - \underline{w})\chi(\underline{w}, \overline{w}) - (w - \underline{w})(1-\alpha)^{n-1}F^{n-1}(\underline{w}).$$

Finally, the participant with  $w > \overline{w}$  wins with probability one and pays the second highest bid x when it is above  $\overline{w}$ , which occurs with probability

$$\sum_{m=0}^{n-1} B_m^{n-1}(\alpha) (F^{n-1-m}(x) - F^{n-1-m}(\overline{w})).$$

By integration by parts, the contribution to the expected payoff is

$$\int_{\overline{w}}^{w} \sum_{m=0}^{n-1} B_{m}^{n-1}(\alpha) (F^{n-1-m}(x) - F^{n-1-m}(\overline{w})) dx$$
$$= \int_{\overline{w}}^{w} \sum_{m=0}^{n-1} B_{m}^{n-1}(\alpha) F^{n-1-m}(x) dx - (w - \overline{w}) ((1 - \alpha)F(\overline{w}) + \alpha)^{n-1}.$$

The sum of the three expressions for the contributions to the expected payoff matches  $U^{\sigma}(w)$ in (2) and (7) for  $w > \overline{w}$ .

## Proof of Lemma 2

Fix an incentive compatible, optimal equal-priority mechanism  $\{r, \underline{w}, \overline{w}; t\}$  with  $r \leq \underline{w} \leq \overline{w}$ . When  $r \leq t \leq \underline{w}$ , define

$$D = U^{\sigma}(\underline{w}) - U^{\mu}(\underline{w}) = \int_{r}^{\underline{w}} (1 - \alpha)^{n-1} F^{n-1}(w) dw - \chi(\underline{w}, \overline{w})(\underline{w} - t),$$

and let R be the revenue, given by (10). If  $0 < r < \underline{w}$ , or if  $0 = r < \underline{w}$  and dr > 0, or if  $0 < r = \underline{w}$  and dr < 0, we have

$$\frac{\partial D}{\partial r} = -(1-\alpha)^{n-1} F^{n-1}(r);$$
  
$$\frac{\partial R}{\partial r} = -n(1-\alpha)^n F^{n-1}(r)\phi(r)f(r).$$

If  $0 < t < \underline{w}$ , or  $0 = t < \underline{w}$  and dt > 0, or  $0 < t = \underline{w}$  and dt < 0, we have

$$\frac{\partial D}{\partial t} = \chi(\underline{w}, \overline{w});$$
$$\frac{\partial R}{\partial t} = n\alpha\chi(\underline{w}, \overline{w})\pi'(t).$$

If  $t < \underline{w} < \overline{w}$ , or if  $t = \underline{w} < \overline{w}$  and  $d\underline{w} > 0$ , or  $t < \underline{w} = \overline{w}$  and  $d\underline{w} < 0$ , we have

$$\begin{split} \frac{\partial \chi(\underline{w},\overline{w})}{\partial \underline{w}} &= \frac{(1-\alpha)f(\underline{w})}{(1-\alpha)(F(\overline{w})-F(\underline{w}))+\alpha} \left(\chi(\underline{w},\overline{w}) - ((1-\alpha)F(\underline{w}))^{n-1}\right);\\ \frac{\partial D}{\partial \underline{w}} &= (1-\alpha)^{n-1}F^{n-1}(\underline{w}) - \chi(\underline{w},\overline{w}) - \frac{\partial \chi(\underline{w},\overline{w})}{\partial \underline{w}}(\underline{w}-t);\\ \frac{\partial R}{\partial \underline{w}} &= n(1-\alpha)((1-\alpha)^{n-1}F^{n-1}(\underline{w}) - \chi(\underline{w},\overline{w}))\phi(\underline{w})f(\underline{w}) \\ &+ n((1-\alpha)(\pi(\underline{w})-\pi(\overline{w})) + \alpha\pi(t))\frac{\partial \chi(\underline{w},\overline{w})}{\partial \underline{w}}. \end{split}$$

If  $\underline{w} < \overline{w} < 1$ , or if  $\underline{w} = \overline{w} < 1$  and  $d\overline{w} > 0$ , or if  $\underline{w} < \overline{w} = 1$  and  $d\overline{w} < 0$ , we have

$$\begin{aligned} \frac{\partial \chi(\underline{w},\overline{w})}{\partial \overline{w}} &= \frac{(1-\alpha)f(\overline{w})}{(1-\alpha)(F(\overline{w})-F(\underline{w}))+\alpha} \left( \left((1-\alpha)F(\overline{w})+\alpha\right)^{n-1} - \chi(\underline{w},\overline{w}) \right); \\ \frac{\partial D}{\partial \overline{w}} &= -\frac{\partial \chi(\underline{w},\overline{w})}{\partial \overline{w}} (\underline{w}-t); \\ \frac{\partial R}{\partial \overline{w}} &= n(1-\alpha) \left( \chi(\underline{w},\overline{w}) - \left((1-\alpha)F(\overline{w})+\alpha\right)^{n-1} \right) \phi(\overline{w})f(\overline{w}) \\ &+ n((1-\alpha)(\pi(\underline{w})-\pi(\overline{w}))+\alpha\pi(t)) \frac{\partial \chi(\underline{w},\overline{w})}{\partial \overline{w}}. \end{aligned}$$

The proof of the lemma is divided into seven steps.

(i) We claim that  $r \leq t \leq \underline{w}$ . We can rule out t < r right away, because it violates (8). To rule out  $t > \underline{w}$ , note that in this case (8) is slack. From the expression of  $\partial R/\partial t$ , concavity of  $\pi(\cdot)$  and the optimality of  $\{r, \underline{w}, \overline{w}; t\}$  together imply that  $t = r^*$ . If  $r < \underline{w}$ , then since  $\underline{w} < t = r^*$ , we have  $r < r^*$ . From the expression of  $\partial R/\partial r$ , a marginal increase in r would increase the first term in (10), contradicting the optimality of  $\{r, \underline{w}, \overline{w}; t\}$ . Thus,  $r = \underline{w}$ . If  $\underline{w} < \overline{w}$ , then from the expression of  $\partial R/\partial \underline{w}$ , a marginal increase in  $\underline{w}$  would increase the revenue, contradicting the assumption of optimality. Thus,  $r = \underline{w} = \overline{w} < t = r^*$ . From the expressions of  $\partial R/\partial \underline{w}$  and  $\partial R/\partial \overline{w}$ , a increase in  $\underline{w}$  and  $\overline{w}$  by the same marginal amount would increase the revenue, a contradiction. Thus,  $t \leq \underline{w}$ .

(ii) We claim that  $r < t < \underline{w}$ . We can rule out  $r = t < \underline{w}$  right away, because it violates (8). To rule out  $r < t = \underline{w}$ , note that in this case (8) is slack. Since r < t, either  $r < r^*$  or  $t > r^*$ , or both. If  $r < r^*$ , then by raising r marginally, the seller could increase the revenue because  $\partial R/\partial r > 0$ . If  $t > r^*$ , then by lowering t marginally, the seller

could increase the revenue because  $\partial R/\partial t < 0$ . Either way, we have a contradiction to the assumption of optimality. Finally, we rule out  $r = t = \underline{w}$ . If  $r = t = \underline{w} < r^*$ , then by raising t marginally, the seller relaxes (8), and increases the revenue because  $\partial R/\partial t > 0$ . If  $r = t = \underline{w} > r^*$ , then by lowering r marginally, the seller relaxes (8), and increases the revenue because  $\partial R/\partial r < 0$ . If  $r = t = \underline{w} = r^*$ , then by lowering r marginally, the seller relaxes (8), and increases the revenue because  $\partial R/\partial r < 0$ . If  $r = t = \underline{w} = r^*$ , then by lowering r marginally, the seller relaxes (8) because  $\partial D/\partial r < 0$ , without changing the revenue because  $\partial R/\partial r = 0$ . With (8) slack, the seller could then increase the revenue by either further raising  $\underline{w}$  marginally if  $\underline{w} = r^* < \overline{w}$ , because  $\phi(\underline{w}) = 0$  implies  $\partial R/\partial \underline{w} > 0$ , or by raising both  $\underline{w}$  and  $\overline{w}$  by the same infinitesimal amount if  $\underline{w} = \overline{w} = r^*$ , because  $\partial R/\partial w + \partial R/\partial \overline{w} > 0$ . In each case, we have a contradiction to the assumption of optimality.

(iii) We claim that  $r < t < \underline{w} < \overline{w}$ . Suppose instead  $\underline{w} = \overline{w} = \hat{w}$ , and consider decreasing both  $\underline{w}$  and  $\overline{w}$  by the same marginal amount. We have  $\partial D/\partial \underline{w} + \partial D/\partial \overline{w} < 0$ , and  $\partial R/\partial \underline{w} + \partial R/\partial \overline{w}$  has the same sign as  $\pi(t) - \phi(\hat{w})$ . Thus, we must have  $\pi(t) > \phi(\hat{w})$ : otherwise, the seller relaxes (8) without decreasing the revenue, which would then allow the seller to increase the revenue by either raising r or lowering t, as r < t implies  $r < r^*$  or  $t > r^*$ , or both. Since  $\phi(1) = 1$ , it follows from  $\pi(t) > \phi(\hat{w})$  that  $\hat{w} < 1$ . Consider perturbing the equal-priority mechanism by reducing w from  $\hat{w}$  and raising  $\overline{w}$  from  $\hat{w}$  such that

$$-(\chi(\hat{w},\hat{w}) - (1-\alpha)^{n-1}F^{n-1}(\hat{w}))d\underline{w} = (((1-\alpha)F(\hat{w}) + \alpha)^{n-1} - \chi(\hat{w},\hat{w}))d\overline{w}.$$

By construction,

$$-\frac{\partial \chi(\hat{w}, \hat{w})}{\partial w} d\underline{w} = \frac{\partial \chi(\hat{w}, \hat{w})}{\partial \overline{w}} d\overline{w}.$$

This implies that (8) is relaxed, because

$$\frac{\partial D}{\partial \underline{w}} d\underline{w} + \frac{\partial D}{\partial \overline{w}} d\overline{w} = ((1-\alpha)^{n-1} F^{n-1}(\hat{w}) - \chi(\hat{w}, \hat{w})) d\underline{w},$$

which is strictly positive. The seller's revenue is unchanged, because

$$\begin{aligned} \frac{\partial R}{\partial \underline{w}} d\underline{w} + \frac{\partial R}{\partial \overline{w}} d\overline{w} = n(1-\alpha)f(\hat{w}) \left(\chi(\hat{w},\hat{w}) - (1-\alpha)^{n-1}F^{n-1}(\hat{w})\right)(\pi(t) - \phi(\hat{w}))d\underline{w} \\ + n(1-\alpha)f(\hat{w}) \left(((1-\alpha)F(\hat{w}) + \alpha)^{n-1} - \chi(\hat{w},\hat{w})\right)(\pi(t) - \phi(\hat{w}))d\overline{w}, \end{aligned}$$

which is equal to 0 by construction. The seller could now increase the revenue by either raising r or lowering t, as r < t implies  $r < r^*$  or  $t > r^*$ , or both. This contradicts the assumption of optimality.

(iv) We claim that (8) binds,  $r < r^* < t$ , and  $\pi(t) > \phi(\overline{w})$ . If (8) is slack, then since r < t implies that  $r < r^*$  or  $t > r^*$ , or both, the seller could increase the revenue by either raising r or lowering t, a contradiction to the assumed optimality. If  $r^* \leq r < t$ , the seller could relax (8) by lowering r marginally without decreasing the revenue, which then would allow the seller to increase the revenue by lowering t. Similarly, if  $r < t \leq r^*$ , the seller could relax (8) by raising t marginally without decreasing the revenue, which then would allow the seller to increase the revenue by lowering the revenue, which then would allow the seller to increase the revenue by raising the revenue, which then would allow the seller to increase the revenue by raising r. Finally, we show that  $\pi(t) > \phi(\overline{w})$ . Otherwise, since  $\partial R/\partial \overline{w}$  has the same sign as

$$\begin{aligned} &\alpha(\pi(t) - \phi(\overline{w})) + (1 - \alpha)(\pi(\underline{w}) - \pi(\overline{w})) - \phi(\overline{w})(F(\overline{w}) - F(\underline{w})) \\ &= &\alpha(\pi(t) - \phi(\overline{w})) - \int_{\underline{w}}^{\overline{w}} (\phi(\overline{w}) - \phi(w))f(w)dw, \end{aligned}$$

which is strictly less than  $\alpha(\pi(t) - \phi(\overline{w}))$ , the seller can lower  $\overline{w}$  marginally. This relaxes (8) because  $\partial D/\partial \overline{w} < 0$ , and increases the revenue, contradicting the assumed optimality. Note that  $\pi(t) > \phi(\overline{w})$  implies  $\overline{w} < 1$ .

(v) To obtain (11), consider perturbations  $d\underline{w}$  and  $d\overline{w}$ , while keeping r and t unchanged. An optimality condition is that

$$\frac{\partial R}{\partial \underline{w}} d\underline{w} + \frac{\partial R}{\partial \overline{w}} d\overline{w} = 0$$

for all perturbations  $d\underline{w}$  and  $d\overline{w}$  satisfying

$$\frac{\partial D}{\partial \underline{w}} d\underline{w} + \frac{\partial D}{\partial \overline{w}} d\overline{w} = 0.$$

Thus we have

$$\frac{\partial R/\partial \underline{w}}{\partial D/\partial \underline{w}} = \frac{\partial R/\partial \overline{w}}{\partial D/\partial \overline{w}}.$$

Using the expressions for  $\chi(\underline{w}, \overline{w})$ ,  $\partial \chi(\underline{w}, \overline{w}) / \partial \underline{w}$  and  $\partial \chi(\underline{w}, \overline{w}) / \partial \overline{w}$ , straightforward algebra

lead us to the first-order condition (11) for an optimal equal-priority mechanism with respect to  $\underline{w}$  and  $\overline{w}$ .

(vi) To obtain (12), consider perturbations dt and  $d\overline{w}$ . The optimality condition is

$$\frac{\partial R/\partial t}{\partial D/\partial t} = \frac{\partial R/\partial \overline{w}}{\partial D/\partial \overline{w}}.$$

By (11), we have

$$\frac{\partial R/\partial \overline{w}}{\partial D/\partial \overline{w}} = -n(1-\alpha)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}).$$

Combining the above two equations, we have the first order condition (12).

(vii) Lastly, to obtain (13), consider perturbations dr and  $d\overline{w}$ , while keeping t and  $\underline{w}$  unchanged. The resulting optimality condition is

$$\frac{\partial R/\partial r}{\partial D/\partial r} \ge \frac{\partial R/\partial \overline{w}}{\partial D/\partial \overline{w}},$$

and  $r \ge 0$ , with complementary slackness. Using (11), we have the following first-order condition

$$-\phi(r)f(r) \le (\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}),$$

and  $r \ge 0$ , with complementary slackness. Since  $-\phi(0)f(0) = 1$ , and since (11) implies that  $\phi(\overline{w}) < \pi(t) < t < \underline{w}$ ,

$$(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) = (\phi(\overline{w}) - \underline{w})f(\underline{w}) + 1 - F(\underline{w}) < 1.$$

It follows that the optimal r is interior and so (13) holds.

### Proof of Lemma 3

Fix a direct mechanism  $(q_m^{\sigma}, p_m^{\sigma})_{m=0}^{n-1}$  and  $(q_m^{\mu}, p_m^{\mu})_{m=1}^n$ . Define  $p^{\mu} \in [0, 1]$  to be the expected offer to non-participants, given by

$$\sum_{m=0}^{n-1} B_m^{n-1}(\alpha) \mathbb{E}_v[q_{m+1}^{\mu}(v)(p^{\mu} - p_{m+1}^{\mu}(v))] = 0.$$

Since  $\max\{w - p, 0\}$  is convex in p for any w,

$$U^{\mu}(w) = \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) \mathbb{E}_v \left[ q_{m+1}^{\mu}(v) \max\{w - p_{m+1}^{\mu}(v), 0\} \right]$$
$$\geq \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) \mathbb{E}_v [q_{m+1}^{\mu}(v)] \max\{w - p^{\mu}, 0\}.$$

Thus, replacing all functions  $\{p_m^{\mu}(\cdot)\}_{m=1}^n$  with a single offer  $p^{\mu}$  reduces the deviation payoff of a participant. The seller's revenue from non-participants is

$$\sum_{m=1}^{n} B_m^n(\alpha) \mathbb{E}_v \left[ m q_m^{\mu}(v) \pi \left( p_m^{\mu}(v) \right) \right] = n \alpha \sum_{m=0}^{n-1} B_m^{n-1}(\alpha) \mathbb{E}_v \left[ q_{m+1}^{\mu}(v) \pi \left( p_{m+1}^{\mu}(v) \right) \right].$$

The lemma then follows from the strict concavity of  $\pi(\cdot)$ .

### Proof of Theorem 1

Suppose that  $\{r, \underline{w}, \overline{w}; t\}$  is an optimal equal-priority mechanism. By Lemma 2, the first order conditions (11)-(13) are satisfied. We construct a non-negatively valued multiplier function  $\lambda(w)$  for all  $w \in [0, 1]$  such that the allocative rule  $(q_m^{\sigma}(v))_{m=0}^{n-1}$  and  $(q_m^{\mu}(v))_{m=1}^n$ defined by  $\{r, \underline{w}, \overline{w}; t\}$ , together with  $p^{\mu} = t$ , solves the Lagrangian relaxation. By Lemma 1, the offer rule  $(p_m^{\sigma}(v))_{m=0}^{n-1}$  we have specified for an equal-priority mechanism supports a truthful reporting equilibrium among participants. The conclusion then follows immediately. The proof is divided into four steps.

(i) Construction of the multiplier function. Let  $\lambda(w) = 0$  for all  $w \notin [\underline{w}, \overline{w}]$ , and let

$$\lambda(w) = n(1-\alpha)\frac{d}{dw}(f(w)(\phi(w) - \phi(\overline{w}))) = n(1-\alpha)(2f(w) + f'(w)(w - \phi(\overline{w})))$$

for all  $w \in [\underline{w}, \overline{w}]$ . Since  $f(w)\phi(w)$  is strictly increasing in w by Assumption 1 and since  $\phi(\overline{w}) > 0$ , using the first expression of  $\lambda(w)$  above we have  $\lambda(w) > 0$  at any  $w \in [\underline{w}, \overline{w}]$  such that  $f'(w) \leq 0$ . Since  $\phi(\overline{w}) < \pi(t)$  by (11), and since  $w \geq \underline{w} > t > \pi(t)$ , using the second expression we have  $\lambda(w) > 0$  at any  $w \in [\underline{w}, \overline{w}]$  such that f'(w) > 0. Thus,  $\lambda(w)$  as constructed is non-negative for any w.

(ii) We claim that  $p^{\mu} = t$  maximizes  $K^{\mu}$ , and hence the Lagrangian (15). For any  $w \in [\underline{w}, \overline{w}]$ , by construction

$$\int_{w}^{1} \lambda(x) dx = n(1-\alpha)f(w)(\phi(\overline{w}) - \phi(w)).$$

Using integration by parts, we have

$$\int_{0}^{1} \lambda(w) \max\{w - p^{\mu}, 0\} dw$$
  
=  $-\int_{\underline{w}}^{\overline{w}} (w - p^{\mu}) d\left(\int_{w}^{1} \lambda(x) dx\right)$   
=  $n(1 - \alpha) \left((\underline{w} - p^{\mu}) f(\underline{w})(\phi(\overline{w}) - \phi(\underline{w})) + \int_{\underline{w}}^{\overline{w}} f(w)(\phi(\overline{w}) - \phi(w)) dw\right)$ 

By (11), we have

$$K^{\mu} = n\alpha\phi(\overline{w}) + n\alpha(\pi(p^{\mu}) - \pi(t)) + (p^{\mu} - t)n(1 - \alpha)f(\underline{w})(\phi(\overline{w}) - \phi(\underline{w})).$$

The above is strictly concave in  $p^{\mu}$ . By (12), it is maximized at  $p^{\mu} = t$ , with the maximum

$$K_t^{\mu} = n\alpha\phi(\overline{w}).$$

(iii) Comparison of  $K^{\sigma}(\cdot)$  and  $K^{\mu}_t$ . First, for  $w \in [\underline{w}, \overline{w}]$ , by construction we have

$$K^{\sigma}(w) = n(1-\alpha)\phi(w) + \int_{w}^{1} \lambda(w)dw/f(x) = n(1-\alpha)\phi(\overline{w}).$$

Thus,

$$\frac{B_m^{n-1}(\alpha)}{n-m} K^{\sigma}(w) = \frac{B_{m-1}^{n-1}(\alpha)}{m} K_t^{\mu}.$$

Second, for all  $w > \overline{w}$ , we have

$$K^{\sigma}(w) = n(1-\alpha)\phi(w).$$

Since  $\overline{w} > r^*$  and Assumption 1 implies that  $\phi(w)$  is strictly increasing for all  $w > r^*$ , we

have that  $K^{\sigma}(w)$  is strictly increasing. As a result,

$$K^{\sigma}(w) > n(1-\alpha)\phi(\overline{w}) = K^{\sigma}(\overline{w}),$$

and so

$$\frac{B_m^{n-1}(\alpha)}{n-m} K^{\sigma}(w) > \frac{B_{m-1}^{n-1}(\alpha)}{m} K_t^{\mu}.$$

Third, for all  $w < \underline{w}$ ,

$$K^{\sigma}(w) = n(1-\alpha)\phi(w) + \frac{1}{f(w)} \int_{\underline{w}}^{\overline{w}} \lambda(x)dx$$
$$= n(1-\alpha)\phi(w) + n(1-\alpha)\frac{f(\underline{w})(\phi(\overline{w}) - \phi(\underline{w}))}{f(w)}.$$

By Assumption 1,  $K^{\sigma}(w)$  can cross 0 only once and only from below. Taking derivatives, we have that  $dK^{\sigma}(w)/dw$  has the same sign as

$$-\pi''(w)f(w) - f'(w)(f(w)\phi(w) + f(\underline{w})(\phi(\overline{w}) - \phi(\underline{w})).$$

Thus, at any  $w < \underline{w}$  such that  $K^{\sigma}(w) > 0$ , we have  $K^{\sigma}(w)$  is strictly increasing if f'(w) < 0. At any  $w < \underline{w}$  such that  $f'(w) \ge 0$ , since

$$\phi(w) + \frac{f(\underline{w})(\phi(\overline{w}) - \phi(\underline{w}))}{f(w)} = w - \frac{1 - F(w) - f(\underline{w})(\phi(\overline{w}) - \phi(\underline{w}))}{f(w)},$$

and since in part (vii) of the proof of Lemma 2 we have shown that  $f(\underline{w})(\phi(\overline{w}) - \phi(\underline{w})) < 1$ , again we have  $K^{\sigma}(w)$  is strictly increasing. It follows that for all  $w < \underline{w}$ 

$$K^{\sigma}(w) < n(1-\alpha)\phi(\overline{w}) = K^{\sigma}(\underline{w}),$$

and so

$$\frac{B_m^{n-1}(\alpha)}{n-m}K^{\sigma}(w) \le \frac{B_{m-1}^{n-1}(\alpha)}{m}K_t^{\mu}.$$

(iv) We claim that the allocations  $(q_m^{\sigma}(v))_{m=0}^{n-1}$  and  $(q_m^{\mu}(v))_{m=1}^n$  specified by  $\{r, \underline{w}, \overline{w}; t\}$  are a point-wise maximizer of the Lagrangian relaxation (15), with  $K^{\mu}$  replaced by its maximized

value of  $K_t^{\mu}$  from step (ii).

First, consider poin-wise maximization of the middle term in (15) for any fixed realized number m of non-participants, with  $1 \le m \le n-1$ . Suppose that for some realized valuation profile v we have  $v_i > \overline{w}$  for some  $i = 1, \ldots, n-m$ , but  $q_m^{\mu}(v) > 0$ . By (1), we can decrease  $q_m^{\mu}(v)$  marginally by  $dq_m^{\mu}(v) > 0$  and increase  $q_m^{\sigma}(\rho_m^i(v))$  by  $mdq_m^{\mu}(v)$ . Since

$$\frac{m}{n-m}B_m^{n-1}(\alpha)K^{\sigma}(v_i) > B_{m-1}^{n-1}(\alpha)K_t^{\mu},$$

the effect on the seller's revenue is strictly positive. Therefore,  $q_m^{\mu}(v) = 0$  for any v such that  $v_i > \overline{w}$  for some  $i = 1, \ldots, n - m$ . By the same argument, it is point-wise maximizing to set  $q_m^{\sigma}(\rho_m^i(v)) = 0$  for any  $v_i < \underline{w}$ , and  $q_m^{\mu}(v) = q_m^{\sigma}(\rho_m^i(v))$  for any  $v_i \in [\underline{w}, \overline{w}]$ . The claim regarding point-wise maximization of the middle term in (15) follows immediately, because by (iii),  $K^{\sigma}(w)$  is equal to the positive constant  $K_t^{\mu}$  for  $w \in [\underline{w}, \overline{w}]$ , is strictly less than  $K^{\sigma}(\underline{w})$  for  $w < \underline{w}$ , and is strictly increasing for any  $w > \overline{w}$ .

Second, consider the first term in the Lagrangian (15), with m = 0. By (iii),  $K^{\sigma}(w)$ crosses 0 only once and only from below, and is strictly increasing for  $w < \underline{w}$  such that  $K^{\sigma}(w) > 0$ . Further,  $K^{\sigma}(w)$  is equal to the positive constant  $K_t^{\mu}$  for  $w \in [\underline{w}, \overline{w}]$ , and is strictly increasing for any  $w > \overline{w}$ . Thus, for r that satisfies (13), it is point-wise maximizing to set  $q_0^{\sigma}(\rho_0^i(v)) = 1$  if  $v_i = \max\{v_1, \ldots, v_n\}$  and  $v_i > \overline{w}$ , or if  $v_i = \max\{v_1, \ldots, v_n\}$  and  $v_i \in [r, \underline{w}]$ ; set  $q_0^{\sigma}(\rho_0^i(v)) = 1/k$  if  $v_i \in [\underline{w}, \overline{w}]$ ,  $\max\{v_1, \ldots, v_n\} \in [\underline{w}, \overline{w}]$  and  $\#\{j : v_j \in [\underline{w}, \overline{w}]\} = k$ ; and set  $q_0^{\sigma}(\rho_0^i(v)) = 0$  otherwise.

For m = n and the last term in the Lagrangian (15), it is optimal to set  $q_n^{\mu} = 1/n$  because  $K_t^{\mu} > 0$ .

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