Informative Voting in Large Elections

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Abstract. Recounting introduces multiple pivotal events in two-candidate elections. In addition to determining which candidate is elected, an individual's vote is pivotal when the vote margin is just at the levels that would trigger a recount. In large elections, the motive to avoid recounting cost can become the dominant consideration for rational voters, inducing them to vote informatively according to their private signals. In environments where elections without recounting fail to aggregate information efficiently, a modified election rule with small recounting cost can induce asymptotically efficient outcomes in the best equilibrium, with a vanishingly small probability of actually invoking a recount. In environments where efficient information aggregation obtains in elections without recounting, introducing recounting can reduce the size of the electorate needed for the equilibrium outcome to converge to an efficient outcome.

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1. Introduction

More than two centuries ago Condorcet (1875) first articulated the idea that voting groups with diverse information about their alternatives make a better choice the larger the group size. This celebrated Condorcet jury theorem is a statistical proposition based on an early application of the law of large numbers. It gives confidence to our belief that large elections can resolve conflicts due to dispersed information and produce good collective decisions. However, the underlying sincere-voting behavior by the electorate has been challenged by economists who study this topic. Austen-Smith and Banks (1996) point out how *informative voting*—that is, voting according to one's own private signal—is generally inconsistent with rationality (see also Feddersen and Pesendorfer, 1996). Since a non-pivotal vote does not affect the outcome and is thus payoff-irrelevant, rational voting behavior requires conditioning one's vote on the information inferred from the vote being pivotal as well as on one's own private information. In a large election, the information inferred from being pivotal can overwhelm one's own private information. Thus, informative voting generally fails in a large election except for a small fraction of informed voters.

The failure of informative voting can negatively impact information aggregation in large elections. Given any election rule, the outcome in a large two-candidate election would be determined by the corresponding decisive voter's preference if all private information about the candidates became common knowledge; for example, if the rule is simple majority the decisive voter has the median preference in the electorate. In contrast, under strategic voting, the failure of informative voting for most informed voters means that only the private information from a small fraction of informed voters gets aggregated and their votes determine the outcome. Although Feddersen and Pesendorfer (1997) show that the vanishing fraction of informative-voting electorate does not prevent a large election from achieving information efficiency in the sense that the outcome almost surely would remain the same even if all private information were perfectly aggregated, there are environments in which information efficiency fails under strategic voting. One such environment involves "aggregate uncertainty," where there are partisan voters who randomly split their votes between the two candidates, resulting in uncertainty in realized vote shares even when the number of voters becomes arbitrarily large (Feddersen and Pesendorfer, 1997). Another environment involves conflicting preferences, where the same change in the public belief about a candidate can increase his appeal to some voters but lower his appeal to other voters (Bhattacharya, 2013). In both these environments, information efficiency could obtain had all informed voters voted informatively.

In this paper, we resurrect informative voting as an equilibrium strategy in large two-candidate elections by introducing other pivotal events in addition to the standard one that determines the eventual winner. Although there are many ways to introduce additional pivotal events, we adopt a model of costly recounting. An election rule in this model is characterized by three thresholds of vote shares for a given candidate and a positive recounting cost. If the vote share for the candidate exceeds the largest threshold then that candidate is declared an outright winner; and symmetrically, if the vote share falls below the smallest threshold then the the opposing candidate is declared an outright winner. If the vote share falls between the smallest and the largest thresholds, a "recount" takes place after each voter incurs the recounting cost. The candidate is declared the winner if the vote share upon recount is above the middle threshold, and the opposing candidate wins otherwise. We study information aggregation in an environment with finitely many states and conditionally independent private signals. We explicitly model the presence of aggregate uncertainty by the presence of non-strategic uninformed voters, with the fraction of uninformed voters voting for a given candidate remaining random even in large elections. At the same time, we leave the description of strategic voters preferences as general as possible, to include the broadest set of environments including models such as that of Bhattacharya (2013), where preferences are non-monotone in the state.

Our main result establishes that, whenever efficient information aggregation is feasible, it is the asymptotic outcome of a sequence of equilibria in elections with recounting, in which every agent votes informatively. In our model, corresponding to the middle threshold is the standard pivotal event that votes for the two candidates are tied. Costly recounting creates two additional pivotal events: corresponding to the largest threshold is the pivotal event when one more vote for the given candidate would make him an outright winner and one more vote for the opponent would trigger a costly recount but would not change the winner, and corresponding to the smallest threshold is the symmetric pivotal event. Although the probabilities of the three pivotal events conditional on the state all vanish in the limit, one of them becomes dominant because its conditional probability goes to zero at the slowest rate.[1] In our equilibrium construction, the pivotal events where a recount can be triggered always dominate the pivotal event that determines who wins the election. At these pivotal events, the desire to save the recounting cost is the only motive, hence voters' incentives are entirely aligned. Each votes for one candidate or the other depending on which of the two pivotal events is more likely, as a function of his private signal. As a result, our election rule with recounting aggregates information efficiently whenever that is possible, including in environments where a standard election rule would fail to do so. Furthermore, we show that the probability of recount and thus incurring the cost in equilibrium is negligible in large elections. Hence, the improvement in information aggregation is achieved at no cost.

In environments where a standard election without recounting aggregates information efficiently, recounting still improves the outcome by increasing the rate of convergence to the informationally efficient outcome. In other words, the same desired level of information efficiency can be obtained by an electorate of smaller size under an election rule with recounting than under a standard election rule with no recounting. The reason is that in our equilibrium construction with recounting, the dominant pivotal event determines whether a recount is triggered, without changing the identity of the winner. In equilibrium the choice between the two candidates depends not on a voter's preferences, but on their signals. This makes the vote count more informative compared to an election without recounting.

Our construction of an efficient information aggregating equilibrium under recounting is robust to a number of modeling choices and assumptions we make.[2] We assume away any counting error in the main model, but if an initial vote count is subject to errors that are corrected in a recount, then as long as the errors are sufficiently small, for an election of a sufficiently large size, there is an equilibrium that aggregates information efficiently. Counting errors are in fact the main reason for elections to allow recounting in the first place. For example, in presidential elections in the United States, recounting is implemented in all 50 states and District of Columbia. In some states, a recount is automatically trigged by the initial vote count falls in a pre-specified close vote margin. In other states, there is no automatic

[1]This is a consequence of the theory of large deviations, which studies the limit behavior of rare events. See, for example, Dembo and Zeitouni (1998) for a textbook treatment.

[2]See Section 5 for a formal presentation of all our robustness claims.

recounting, but candidates, election officials and even voters are allowed to request a recount after the initial vote count, with states differing in terms of whether there are requirements of a close vote margin, and how the cost of recounting is allocated. Our model is precisely automatic recounting, but requested recounting leads to the same equilibrium outcome.[3] Our results therefore highlight an information aggregation role of recounting, regardless of how it is implemented in practice.

The modern version of Condorcet jury theorem with strategic voters relies on pivotal reasoning—a voter casts her vote with belief conditional on the event that she is pivotal. In high-stakes settings such as supreme court decisions (Iaryczower and Shum, 2012) or shareholder voting (Maud and Rydqvist, 2009), there is evidence that voters are strategically sophisticated: they do not simply vote according to their private information and cast their votes as if they are decisive. In a laboratory setting, on the other hand, Esponda and Vespa (2014) show that a sizable fraction of subjects behave nonoptimally because they have difficulty extracting information from hypothetical (pivotal) events. Sincere voting makes it more difficult to achieve information efficiency because such voting strategy is not type-independent: a sincere voter who receives an imperfect signal in favor of one candidate may still vote for the other candidate if her preference for the latter is strong enough. However, we show that as long as the fraction of sincere voters in the electorate is not too large, information efficiency can still be achieved through recounting.

2. A Model of Elections with Recounting

We study an election with a large number $n+1$ of voters to choose between two candidates: $\mathcal{R}$ and $\mathcal{L}$. Denote the share of votes for $\mathcal{R}$ as V. An "election rule" consists of three thresholds $v_{\mathcal{L}}$, $v_{\mathcal{C}}$ and $v_{\mathcal{R}}$, satisfying $v_{\mathcal{L}} < v_{\mathcal{C}} < v_{\mathcal{R}}$, and specifies:

1. candidate $\mathcal{R}$ is elected if $V > v_{\mathcal{R}}$;
2. candidate $\mathcal{L}$ is elected if $V < v_{\mathcal{L}}$;
3. a "recount" is triggered if $V \in [v_{\mathcal{L}}, v_{\mathcal{R}}]$; and after the recount, candidate $\mathcal{R}$ is elected if $V \geq v_{\mathcal{C}}$ and candidate $\mathcal{L}$ is elected otherwise.

We assume that there is no error in the initial vote count stage or in the recount

[3]This is true so long as the cost of requesting is insignificant compared with the potential benefit of changing the outcome. Then, in casting their votes, voters hold the belief about when a recount will be requested, which corresponds to the automatic triggers.

stage. Therefore the vote share for $\mathcal{R}$ in the recount stage will be exactly the same as that recorded in the initial count. We do not consider unanimity rules; both $v_{\mathcal{R}}$ and $v_{\mathcal{L}}$ are assumed to be strictly between 0 and 1.

Recount matters to voters because each voter incurs an additive payoff loss of $\delta > 0$ (in addition to their payoff from the election outcome specified below) when a recount is triggered. A standard election rule without recounting can be represented as a special case of election rule defined above, with $\delta = 0$.

Voters are independently drawn from a large population of potential voters. A fraction $1 - \alpha \in (0, 1]$ of potential voters are informed voters; the rest are uninformed. There is a finite number, M, of payoff-relevant states of the world $S = \{s_1, \ldots, s_M\}$. Voters' common prior belief over S is described by the distribution $\mu = (\mu_1, \ldots, \mu_M)$, with μ_i being the probability of state s_i. Each informed voter observes a conditionally independent signal σ about the state from the signal space $\Sigma = \{\sigma_1, \ldots, \sigma_J\}$.

Assumption 1. *The signal's conditional distributions, $\beta(\cdot|s)$, satisfy the monotone likelihood ratio property:*

$$\frac{\beta(\sigma_j|s_i)}{\beta(\sigma_{j'}|s_i)} > \frac{\beta(\sigma_j|s_{i'})}{\beta(\sigma_{j'}|s_{i'})} \quad \textit{for all } i > i' \textit{ and } j > j'.$$

For notational brevity, we write $s > s'$ when $s = s_i$ and $s' = s_{i'}$ with $i > i'$, and say the two states are adjacent if $i = i' + 1$. Similarly we write $\sigma > \sigma'$ when $\sigma = \sigma_j$ and $\sigma' = \sigma_{j'}$ with $j > j'$, and say they are adjacent if $j = j' + 1$. An immediate implication of Assumption 1 is that the posterior distributions over S after observing a signal realization, $\{\mu^\sigma\}_{\sigma \in \Sigma}$, are ordered with respect to first order stochastic dominance, so that the higher the signal observed, the more an informed voter revises his expectation about the realized state upward.

Uninformed voters are introduced to preserve uncertainty about the realized vote share in each given state s in large elections. They are non-strategic; a fraction θ of them vote for candidate $\mathcal{R}$ and the remaining fraction $1 - \theta$ vote for $\mathcal{L}$.[4] The

[4]The uninformed voters are partisan in the sense that they have preferences between the two candidates that cannot be swayed by any evidence. Otherwise, they may optimally choose to abstain from voting. See Feddersen and Pesendorfer (1996).

fraction θ is a random variable distributed on $[\underline{\theta}, \overline{\theta}] \subset [0, 1]$, with a continuous and positive density function f and corresponding distribution function F. The aggregate uncertainty state θ is independent of the payoff state s.

Informed voters are heterogeneous with respect to a preference type $t \in T$. The distribution of preference types among the population of potential informed voters is described by a probability measure P over T, and is independent of s and θ. A payoff function, $w : \{\mathcal{L}, \mathcal{R}\} \times S \times T \to \mathbb{R}$, describes the payoff (without considering recounting cost) to an informed voter as a function of the candidate elected, the realized payoff relevant state, and the voter's type. The voter's payoff is reduced by δ if the same election outcome is achieved after a recount. We make the following joint assumption over the payoff function and the distribution of preference types:

Assumption 2. *The payoff difference $u(s,t) \equiv w(\mathcal{R}, s, t) - w(\mathcal{L}, s, t)$, and the distribution of preference types P satisfy*

$$\int_{\{t \in T | u(s,t) > 0\}} \mathrm{d}P(t) > \int_{\{t \in T | u(s',t) > 0\}} \mathrm{d}P(t) \quad \textit{for all } s > s'.$$

Under Assumption 2, in a large election with only informed voters and perfect information, the unique equilibrium outcome in undominated strategies would be monotone in the state. This assumption implies that the full information outcome has a "threshold" structure, with the winner changing at most once as a function of the realized payoff state. A sufficient condition for Assumption 2 is the commonly used requirement (e.g., Federsen and Pesendorfer (1997)) of "state-monotone preferences" that $u(\cdot, t)$ is increasing. Monotone preferences are not, however, necessary. Assumption 2 is also satisfied, for example, in Bhattacharya's (2013) model where, for a majority of voters u is increasing in s, and for a minority of voters the opposite is true.

2.1. Strategy and equilibrium

For a given n, we consider a voting game with $n+1$ voters, Γ_n, described by: (i) the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$ and δ; (ii) the payoff relevant states S, the preference type space T, and the payoff function $u : S \times T \to \mathbb{R}$; and (iii) the prior belief μ over S, the probability measure P over T, the distribution function over the aggregate uncertainty state F, as well as the set of signals Σ and the conditional probability distributions over Σ, $\{\beta(\cdot|s)\}_{s \in S}$. These are all common knowledge.

A strategy describes the probability that an informed voter casts a vote in favor of $\mathcal{R}$ as a function of both his preference type t and the realization of his private signal σ. Thus a strategy is a function

$$k : T \times \Sigma \to [0,1].$$

We restrict the strategy space to functions k such that $k(\cdot,\sigma)$ is T-measurable for each σ, so that the integral of $k(\cdot,\sigma)$ with respect to the probability measure P is well defined. Since preferences and information are independent from each other, we can write such integral as

$$H(\sigma;k) \equiv \int_T k(t,\sigma)\,\mathrm{d}P(t).$$

Given a strategy k, the function $H(\cdot;k)$ describes the probability that a randomly drawn informed voter casts a vote for $\mathcal{R}$ as a function of the private signal σ he observes.

For any strategy k, we define $z(s,\theta;k)$ as the probability that a randomly drawn voter casts a vote for candidate $\mathcal{R}$ in the payoff state s and aggregate uncertainty state θ. Given the independence between s and θ, we have

$$z(s,\theta;k) = (1-\alpha)\sum_{j=1}^{J} H(\sigma_j;k)\beta(\sigma_j|s) + \alpha\theta. \tag{1}$$

We will refer to $z(s,\theta;k)$ as the vote share for candidate $\mathcal{R}$ in state (s,θ) given the strategy k.

Since signals are independent across voters conditional on the state, from the perspective of each individual informed voter, the probability of a vote share v for candidate $\mathcal{R}$ conditional on the payoff state s and the aggregate uncertainty state θ is given by:

$$g_n(v|s,\theta;k) = \binom{n}{nv} z(s,\theta;k)^{nv}\left(1-z(s,\theta;k)\right)^{n(1-v)}. \tag{2}$$

Define

$$g_n(v|s;k) = \int_{\underline{\theta}}^{\overline{\theta}} g_n(v|s,\theta;k)f(\theta)\,\mathrm{d}\theta.$$

To avoid dealing with integer problems, we have implicitly assumed in (2) that $nv_{\mathcal{L}}, nv_{\mathcal{C}}$ and $nv_{\mathcal{R}}$ are integers. Given our focus on the limit case for n large, it is equivalent to assuming that the voting rule thresholds are rational numbers.

An informed voter casts his vote assuming that his vote is pivotal. Upon observing a private signal realization $\sigma \in \Sigma$, an informed voter's private belief that the state is s becomes

$$\mu^{\sigma}(s) = \frac{\mu(s)\beta(\sigma|s)}{\sum_{s' \in S} \mu(s')\beta(\sigma|s')}.$$

There are three pivotal events:

1. $v = v_{\mathcal{L}}$: Regardless of the state, voting $\mathcal{R}$ instead of $\mathcal{L}$ triggers a recount, incurring a cost of δ.
2. $v = v_{\mathcal{C}}$: Voting $\mathcal{R}$ instead of $\mathcal{L}$ tilts the election outcome (after the recount) to $\mathcal{R}$. This changes the voter's payoff by $u(s,t)$ when the state is s.
3. $v = v_{\mathcal{R}}$: Regardless of the state, voting $\mathcal{R}$ instead of $\mathcal{L}$ determines the outcome of the election immediately, saving the recounting cost δ.

Therefore, the best response correspondence of informed voters $\tilde{k}$ to k satisfies

$$\left(\sum_{s \in S} \mu^{\sigma}(s)\left(g_n(v_{\mathcal{R}}|s;k)\delta + g_n(v_{\mathcal{C}}|s;k)u(s,t) - g_n(v_{\mathcal{L}}|s;k)\delta\right)\right)\tilde{k}(t,\sigma) \geq 0,$$
$$\left(\sum_{s \in S} \mu^{\sigma}(s)\left(g_n(v_{\mathcal{R}}|s;k)\delta + g_n(v_{\mathcal{C}}|s;k)u(s,t) - g_n(v_{\mathcal{L}}|s;k)\delta\right)\right)(1 - \tilde{k}(t,\sigma)) \leq 0. \quad (3)$$

Upon observing a private signal σ, an informed voter of preference type t must cast with probability one a vote that yields a strictly larger expected payoff, where expectations are taken with respect to the probability functions $g_n(\cdot|\cdot;k)$ obtained from the primitives of the game and from strategy k, and can randomize if he is indifferent.

Definition 1. *An equilibrium of the Bayesian game Γ_n is a fixed point k_n of the best response correspondence given by (3).*

The above definition allows both pure-strategy and mixed-strategy Bayesian Nash equilibrium in Γ_n.

2.2. Ranking of pivotal events

Fix a strategy k and the implied vote share functions $z(s,\theta;k)$. In a large election, the probability that the actual vote share equals a particular value v is vanishingly small. A key observation of this paper is that the *rates* at which the probabilities of different pivotal events go to zero are different, so that in the limit some pivotal events are infinitely more likely to occur than others. Calculating the rate of convergence is therefore an important part of the analysis of large elections with multiple pivotal events.

Using Stirling's approximation formula for the binomial coefficient, from (2) we have

$$\lim_{n\to\infty} g_n(v|s,\theta;k)\frac{\sqrt{2\pi v(1-v)n}}{I\left(v;z(s,\theta;k)\right)^n} = 1, \tag{4}$$

where

$$I(v;z) = \left(\frac{z}{v}\right)^v \left(\frac{1-z}{1-v}\right)^{1-v}.$$

The function $-\log I(v;z)$ is known as the "rate function" or "entropy function" in the theory of large deviations. It determines the rate at which the probability $g_n(v|s,\theta;k)$ for any $v \neq z(s,\theta;k)$ goes to zero. In particular, if $I(v;z(s,\theta;k)) < I(v';z(s',\theta';k))$, then

$$\lim_{n\to\infty} \frac{g_n(v|s,\theta;k)}{g_n(v'|s',\theta';k)} = \lim_{n\to\infty}\left(\frac{I\left(v;z(s,\theta;k)\right)}{I\left(v';z(s',\theta';k)\right)}\right)^n = 0.$$

A critical property of the function $I(v;z)$ is that it increases in z for $z < v$ and decreases in z for $z > v$, attaining the maximum value of 1 at $z = v$.

In our model an informed voter does not know the aggregate uncertainty state θ. The probability that he assigns to a certain pivotal event v to occur in the payoff relevant state s, $g_n(v|s;k)$, is the integral of $g_n(v|s,\theta;k)$ over all possible aggregate uncertainty states θ. The following lemma shows that in comparing the rates of convergence across different events and payoff states, only the aggregate uncertainty state θ which maximizes the function $I(v;z(s,\theta;k))$ matters. For any v, s and k, let

$$\theta(v,s;k) \equiv \arg\max_{\theta\in[\underline{\theta},\overline{\theta}]} I(v;z(s,\theta;k)).$$

Since $I(v;z)$ increases in z for $z < v$ and decreases in z for $z > v$, and since $z(s,\theta;k)$ increases in θ, by the critical property of the function I, we have

$$\theta(v,s;k) = \arg\min_{\theta\in[\underline{\theta},\overline{\theta}]} \left|z(s,\theta;k) - v\right|.$$

For example, when $z(s,\overline{\theta};k) < v \leq v' < z(s',\underline{\theta};k)$, we have $\theta(v,s;k) = \overline{\theta}$ and $\theta(v',s';k) = \underline{\theta}$.

Lemma 1. *For any v, v' and any s, s',*

$$\lim_{n\to\infty} \frac{g_n(v|s;k)}{g_n(v'|s';k)} = \frac{f(\theta(v,s;k))}{f(\theta(v',s';k))} \lim_{n\to\infty} \frac{g_n(v|s,\theta(v,s;k);k)}{g_n(v'|s',\theta(v',s';k);k)}.$$

Proof. Since $z(s,\cdot;k)$ is strictly increasing, $I(v;z(s,\theta;k))$ is increasing in θ for $\theta < \theta(v,s;k)$ and decreasing in θ for $\theta > \theta(v,s;k)$. Let $B_\epsilon(v,s) \subset [\underline{\theta}, \overline{\theta}]$ be a small interval of width ϵ that contains $\theta(v,s;k)$. Specifically, if $\theta(v,s;k) = \underline{\theta}$, choose $B_\epsilon(v,s) = [\underline{\theta}, \overline{b})$ where $\overline{b} = \underline{\theta} + \epsilon$; and if $\theta(v,s;k) = \overline{\theta}$, choose $B_\epsilon(v,s) = (\underline{b}, \overline{\theta}]$ where $\underline{b} = \overline{\theta} - \epsilon$. If $\theta(v,s;k)$ is interior, choose $B_\epsilon(v,s) = (\underline{b}, \overline{b})$ such that $\overline{b} - \underline{b} = \epsilon$ and $I(v;z(s,\underline{b};k)) = I(v;z(s,\overline{b};k))$. Denote $B^c_\epsilon(v,s) = [\underline{\theta}, \overline{\theta}] \backslash B_\epsilon(v,s)$ to be the complement of $B_\epsilon(v,s)$. Note that $I(v;z(s,\theta;k)) > I(v;z(s,\theta';k))$ for any $\theta \in B_\epsilon(v,s)$ and $\theta' \in B^c_\epsilon(v,s)$.

By (4), when n is sufficiently large, for any pivotal event v and any state s, we have

$$\int_{B^c_\epsilon(v,s)} g_n(v|s,\theta;k) f(\theta)\,\mathrm{d}\theta < g_n(v|s,\theta'_n;k)\Pr[\theta \in B^c_\epsilon(v,s)],$$

where θ'_n is equal to $\underline{b}$ or $\overline{b}$. Continuity of $g_n(v|s,\cdot;k)$ also implies that there is a $\hat{\theta}_n \in B_\epsilon(v,s)$ such that

$$\int_{B_\epsilon(v,s)} g_n(v|s,\theta;k) f(\theta)\,\mathrm{d}\theta = g_n(v|s,\hat{\theta}_n;k)\Pr[\theta \in B_\epsilon(v,s)].$$

We further claim that $\lim_{n\to\infty} \hat{\theta}_n = \theta(v,s;k)$. To see this, note that by definition

$$\lim_{n\to\infty} \int_{B_\epsilon(v,s)} \frac{g_n(v|s,\theta)}{g_n(v|s,\hat{\theta}_n)} f(\theta)\,\mathrm{d}\theta = \Pr[\theta \in B_\epsilon(v,s)],$$

which is only possible if $\hat{\theta}_n$ converges to $\theta(v,s;k)$, because from the fact that $\theta(v,s;k)$ maximizes $I(v;z(s,\theta;k))$, we must have $\lim_{n\to\infty} g_n(v|s,\theta;k)/g_n(v|s,\theta(v,s;k)) = 0$ for all $\theta \neq \theta(v,s;k)$.

From the two conditions above, we obtain that for any ϵ positive,

$$\lim_{n\to\infty} \frac{\int_{B^c_\epsilon(v,S)} g_n(v|s,\theta;k) f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v,s)} g_n(v|s,\theta;k) f(\theta)\,\mathrm{d}\theta} \leq \lim_{n\to\infty} \frac{g_n(v|s,\theta'_n;k)\Pr[\theta \in B^c_\epsilon(v,s)])}{g_n(v|s,\hat{\theta}_n;k)\Pr[\theta \in B_\epsilon(v,s)]} = 0, \tag{5}$$

where the equality follows because $\lim_{n\to\infty} g_n(v|s,\theta';k)/g_n(v|s,\theta;k) = 0$ whenever $\theta' \in B^c_\epsilon(v,s)$ and $\theta \in B_\epsilon(v,s)$, and because $\hat{\theta}_n$ is bounded away from θ'_n.

For any v, v' and s, s',

$$\lim_{n\to\infty} \frac{g_n(v|s;k)}{g_n(v'|s';k)} = \lim_{n\to\infty} \frac{\int_{\underline{\theta}}^{\overline{\theta}} g_n(v|s,\theta;k) f(\theta)\,\mathrm{d}\theta}{\int_{\underline{\theta}}^{\overline{\theta}} g_n(v'|s',\theta;k) f(\theta)\,\mathrm{d}\theta} = \lim_{n\to\infty} \frac{\int_{B_\epsilon(v,s)} g_n(v|s,\theta;k) f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v',s')} g_n(v'|s',\theta;k) f(\theta)\,\mathrm{d}\theta},$$

where the last equality follows from (5). The above holds for any ϵ positive and thus

$$\lim_{n\to\infty} \frac{g_n(v|s;k)}{g_n(v'|s';k)} = \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{\int_{B_\epsilon(v,s)} g_n(v|s,\theta;k) f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v',s')} g_n(v'|s',\theta;k) f(\theta)\,\mathrm{d}\theta}.$$

Reversing the limit order and calculating the inner limit using l'Hopital's rule, we obtain the desired result. ■

Lemma 1 implies that for given strategy k, the ratio $g_n(v|s;k)/g_n(v'|s';k)$ for any pair of pivotal events v, v' and any pair of payoff states s, s' can have a limit different from zero or infinity only if

$$I\left(v; z(s, \theta(v,s;k);k)\right) = I\left(v'; z(s', \theta(v',s';k);k)\right). \tag{6}$$

We refer to the above property as the *equal-rate condition.*

The following definition is a useful comparison of the rates at which two pivotal events vanish.

Definition 2. *Given a sequence of strategies $\{k_n\}$, a pivotal event v in state s* ***dominates*** *another pivotal event v' in state s', if*

$$\lim_{n\to\infty} \frac{g_n(v'|s';k_n)}{g_n(v|s;k_n)} = 0.$$

From Lemma 1, if for example $z(s, \overline{\theta}; k_n) < v < v'$ or $z(s, \underline{\theta}; k_n) > v > v'$ for all n sufficiently large, then v dominates v' in the same state s.

3. Informationally Efficient Equilibria

The central question of the paper is whether recounting can help achieve the same outcome as in an election with just informed voters and common knowledge of all the private signals about the payoff-relevant states. Our model of elections with recounting generally admits multiple equilibria. In this section and the next, we focus on the best equilibrium in terms of information efficiency. At the end this section we provide a discussion on other equilibria of the model.

In an election where private signals about the payoff-relevant state are perfectly aggregated (and with neither recounting nor aggregate uncertainty), Assumption 1 implies that either candidate $\mathcal{L}$ always wins, or candidate $\mathcal{R}$ always wins, or there is some state $s^* \in \{2, \ldots, M\}$ such that candidate $\mathcal{R}$ is elected for all $s \geq s^*$ while candidate $\mathcal{L}$ is elected for $s < s^*$. We focus on the last case, so that the full information outcome is not common knowledge, and efficiently aggregating the voters information is necessary to achieve it.[5] The full information outcome selects, in each state s, candidate $\mathcal{R}$ if a $v_{\mathcal{C}}$-majority of voter favors it in state s, and $\mathcal{L}$ otherwise.[6] Definition 3 below, adapted from Feddersen and Pesendorfer (1997), reflects the presence of aggregate uncertainty in our model, and requires that the election outcome is not affected by the aggregate uncertainty state realization.

Definition 3. *A sequence of strategies $\{k_n\}$ achieves **full information equivalence** if for all $\epsilon > 0$, there is an N such that for $n > N$ and for any realization of the uncertainty state θ, candidate $\mathcal{L}$ is chosen with probability greater than $1 - \epsilon$ when the payoff relevant state is $s < s^*$, and candidate $\mathcal{R}$ is chosen with probability greater than $1 - \epsilon$ if $s \geq s^*$.*

In the presence of aggregate uncertainty, full information equivalence may not be possible. While the aggregate information available to informed voters would always be sufficient to identify the payoff relevant state in a large election, the noise introduced in the voting outcome by the behavior of uninformed voters (modeled as the aggregate uncertainty state) may be large enough that no sequence of strategies ever satisfy the conditions in Definition 3.

Definition 3 does not require $\{k_n\}$ to converge. However, if there is a strategy k such that

$$z(s, \overline{\theta}; k) < v_{\mathcal{C}} < z(s', \underline{\theta}; k) \quad \text{for all } s < s^* \text{ and } s' \geq s^*, \tag{7}$$

then the constant sequence $\{k_n\}$ with $k_n = k$ for all n achieves full information equivalence. This follows directly from the weak law of large numbers and the

[5]The first part of the Proposition 1 remains valid if candidate $\mathcal{L}$ always wins or if candidate $\mathcal{R}$ always wins, but the equilibrium construction does not satisfy condition (9) of Lemma 3.

[6]When preferences are monotone in the preference type as well as in the state, as in Feddersen and Pesendorfer (1997), this coincide with the outcome preferred by the $v_{\mathcal{C}}$-median voter.

monotonicity of the vote share z in the aggregate uncertainty state θ. The share of uninformed votes for $\mathcal{R}$ is largest in the aggregate uncertainty state $\overline{\theta}$ and smallest in the aggregate uncertainty state $\underline{\theta}$. Full information equivalence requires that voting by informed voters generates a sufficiently large spread of $\mathcal{R}$'s vote share between "high states" (i.e., $s' \geq s^*$) and "low states" (i.e., $s < s^*$), so that the election outcome is determined by informed voters only and not by the aggregate uncertainty state. This is possible only if a non-negligible measure of types vote "informatively," i.e., change their vote as their private signal varies. Whether there exists a strategy k that satisfies (7) depends jointly on the outcome-determining threshold $v_{\mathcal{C}}$ together with the informativeness of the voters' signals and the distribution of aggregate uncertainty. It does not depend, however, on whether the election has recounting (i.e., $\delta > 0$), or on the values of the recounting thresholds $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$.

Conversely, if there is no strategy that satisfies (7), full information equivalence cannot be achieved. For sufficiently large n, under any strategy k_n, there is a payoff relevant state s and an aggregate uncertainty state θ such that the probability of yielding the outcome preferred by the $v_{\mathcal{C}}$-majority of informed voter is bounded away from 1.

Next we introduce a class of strategies that are "monotone in signals," and we show that whether a monotone strategy can distinguish between the threshold state s^* and the adjacent lower state s^*_- is a sufficient test for the feasibility of full information equivalence.

Definition 4. *A strategy k is **monotone** if, for all $t \in T$,*

$$\begin{aligned} k(t,\sigma) > 0 &\implies k(t,\sigma') = 1 \quad \textit{for all } \sigma' > \sigma; \\ k(t,\sigma) < 1 &\implies k(t,\sigma') = 0 \quad \textit{for all } \sigma' < \sigma. \end{aligned}$$

Given a monotone strategy profile, each type randomizes its vote for at most one signal realization, and the probability of casting a vote for $\mathcal{R}$ is non-decreasing in the signal realization. The following lemma provides the intuitive result that full information equivalence is feasible if and only if it is feasible in monotone strategies.

Lemma 2. *For any electoral rule, full information equivalence is feasible if and only if there exists a monotone strategy k such that*

$$z(s^*_-, \overline{\theta}; k) < v_{\mathcal{C}} < z(s^*, \underline{\theta}; k). \tag{8}$$

Proof. The if part of the statement follows from Assumption 1, because $z(s,\theta;k)$ is increasing in the payoff relevant state s for every monotone strategy k. The only if part follows because any k satisfying the feasibility condition (7) can be changed into a monotone strategy satisfying the same property by the following algorithm. Fixed any t with non-monotone $k(t,\sigma)$. Let $\underline{\sigma} = \max\{\sigma : k(t,\sigma') = 0\ \forall \sigma' < \sigma\}$ and $\overline{\sigma} = \min\{\sigma : k(t,\sigma') = 1\ \forall \sigma' > \sigma\}$. Since $k(t,\sigma)$ is non-monotone, we have $\underline{\sigma} < \overline{\sigma}$. Construct $\tilde{k}(t,\sigma)$ as follows. Let $\tilde{k}(t,\sigma) = k(t,\sigma)$ for all $\sigma \neq \underline{\sigma}, \overline{\sigma}$; if $k(t,\underline{\sigma})\beta(\underline{\sigma}|s^*) < (1-k(t,\overline{\sigma}))\beta(\overline{\sigma}|s^*)$, let $\tilde{k}(t,\underline{\sigma}) = 0$ and $\tilde{k}(t,\overline{\sigma}) = k(t,\overline{\sigma}) + k(t,\underline{\sigma})\beta(\underline{\sigma}|s^*)/\beta(\overline{\sigma}|s^*)$; otherwise, let $\tilde{k}(t,\underline{\sigma}) = k(t,\underline{\sigma}) - (1-k(t,\overline{\sigma}))\beta(\overline{\sigma}|s^*)/\beta(\underline{\sigma}|s^*)$ and $\tilde{k}(t,\overline{\sigma}) = 1$. By construction,

$$\sum_{j=1}^{J} \beta(\sigma_j|s^*)\tilde{k}(t,\sigma_j) = \sum_{j=1}^{J} \beta(\sigma_j|s^*)k(t,\sigma_j),$$

which by Assumption 1 implies

$$\sum_{j=1}^{J} \beta(\sigma_j|s^*_-)\tilde{k}(t,\sigma_j) < \sum_{j=1}^{J} \beta(\sigma_j|s^*_-)k(t,\sigma_j).$$

Further, $\overline{\sigma} - \underline{\sigma}$ is reduced. The algorithm stops when the resulting strategy becomes monotone. Integrating over all t, we have $z(s^*,\theta;\tilde{k}) = z(s^*,\theta;k)$ and $z(s^*_-,\theta;\tilde{k}) \leq z(s^*_-,\theta;k)$ for all θ. ∎

For monotone strategies, Assumption 1 implies that candidate $\mathcal{R}$'s vote share, $z(s,\theta;k)$, is an increasing function of the state. This observation, together with Lemma 1, immediately imply the following result, which will be critical in our equilibrium construction.

Lemma 3. *Let k_n be a monotone strategy for n sufficiently large such that*

$$z(s^*_-,\overline{\theta};k_n) \leq v_{\mathcal{L}} < v_{\mathcal{C}} < v_{\mathcal{R}} \leq z(s^*,\underline{\theta};k_n). \tag{9}$$

Then, the pivotal event $v_{\mathcal{L}}$ in state s^*_- dominates $v_{\mathcal{L}}$ in state $s < s^*_-$, and $v_{\mathcal{C}}$ and $v_{\mathcal{R}}$ in $s \leq s^*_-$, and the pivotal event $v_{\mathcal{R}}$ in state s^* dominates $v_{\mathcal{R}}$ in state $s > s^*$, and $v_{\mathcal{C}}$ and $v_{\mathcal{L}}$ in $s \geq s^*$.

A sequence of strategies satisfying condition (9) in Lemma 3 would achieve full information equivalence. The opposite is not true in general. Nevertheless, any sequence of monotone strategies that achieves full information equivalence would also satisfy (9) provided that the recounting thresholds $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ are close enough to $v_{\mathcal{C}}$.

Our main result will show that, whenever full information equivalence is feasible, in an election with recounting there is a sequence of equilibrium strategies that achieves it for all $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ sufficiently close to $v_{\mathcal{C}}$. The equilibrium construction will also satisfy condition (9). This implies that not only full information equivalence obtains, but also costly recounting is never triggered in the limit, and therefore the election with recounting achieves the first-best efficient outcome in the limit.

Proposition 1. *Suppose that full information equivalence is feasible for an electoral rule. If $\delta > 0$ and $v_{\mathcal{L}}$, $v_{\mathcal{R}}$ are sufficiently close to $v_{\mathcal{C}}$, there exists a sequence of strategies $\{k_n\}$ such that, for all n sufficiently large, k_n is an equilibrium of Γ_n and $\{k_n\}$ achieves full information equivalence. Further, the probability of recount goes to 0 as n goes to infinity.*

We establish Proposition 1 via a fixed-point argument in *type-independent* monotone strategies. Any type-independent monotone strategy k can be described by a real number $\psi \in [0, J]$, which we refer to as its "continuous representation," such that

$$k(t, \sigma_j) = \begin{cases} 0 & \text{if } j \leq J - \psi, \\ 1 & \text{if } j \geq J - \psi + 1, \\ \psi - (J - j) & \text{otherwise.} \end{cases}$$

The integer part of ψ represents the number of high signals for which all types vote for $\mathcal{R}$, the decimal part of ψ represents the probability of all types voting for $\mathcal{R}$ for the single signal for which randomization occurs (if at all), and for all still lower signals all types vote for $\mathcal{R}$.[7] For example, if $J = 5$, then $\psi = 2.5$ represents a type-independent monotone strategy where all types vote for $\mathcal{R}$ for σ_4 and σ_5, vote for $\mathcal{R}$ with probability 0.5 for σ_3, and vote for $\mathcal{L}$ for σ_2 and σ_1.

In any sufficiently large election, the best response of any type to a strategy k that satisfies condition (9) must be "almost type-independent," meaning that there is at most one signal realization such that the best responses of any two types may differ. In other words, if type t weakly prefers to vote for $\mathcal{R}$ at signal realization σ,

[7] Since there is a one-to-one relation between a type-independent monotone strategy and its continuous representation, we use the two interchangeably.

then any type t' must strictly prefer to vote for $\mathcal{R}$ at signal realization $\sigma' > \sigma$:

$$\sum_{s\in S} \mu^{\sigma}(s)\left(g_n(v_{\mathcal{R}}|s;k)\delta + g_n(v_{\mathcal{C}}|s;k)u(s,t) - g_n(v_{\mathcal{L}}|s;k)\delta\right) \geq 0 \implies$$
$$\sum_{s\in S} \mu^{\sigma'}(s)\left(g_n(v_{\mathcal{R}}|s;k)\delta + g_n(v_{\mathcal{C}}|s;k)u(s,t') - g_n(v_{\mathcal{L}}|s;k)\delta\right) > 0. \qquad (10)$$

To see why this is true, rewrite the first inequality of (10) as

$$\sum_{s\geq s^*} \mu^{\sigma}(s)\left(\frac{g_n(v_{\mathcal{R}}|s;k)}{g_n(v_{\mathcal{R}}|s^*;k)}\delta + \frac{g_n(v_{\mathcal{C}}|s;k)}{g_n(v_{\mathcal{R}}|s^*;k)}u(s,t) - \frac{g_n(v_{\mathcal{L}}|s;k)}{g_n(v_{\mathcal{R}}|s^*;k)}\delta\right) g_n(v_{\mathcal{R}}|s^*;k) \geq$$
$$\sum_{s<s^*} \mu^{\sigma}(s)\left(-\frac{g_n(v_{\mathcal{R}}|s;k)}{g_n(v_{\mathcal{L}}|s^*_-;k)}\delta - \frac{g_n(v_{\mathcal{C}}|s;k)}{g_n(v_{\mathcal{L}}|s^*_-;k)}u(s,t) + \frac{g_n(v_{\mathcal{L}}|s;k)}{g_n(v_{\mathcal{L}}|s^*_-;k)}\delta\right) g_n(v_{\mathcal{L}}|s^*_-;k).$$

For n large, by Lemma 3 the left-hand-side of the above becomes arbitrarily close to $\mu^{\sigma}(s^*)g_n(v_{\mathcal{R}}|s^*;k)\delta$ and the right-hand-side arbitrarily close to $\mu^{\sigma}(s^*_-)g_n(v_{\mathcal{L}}|s^*_-;k)\delta$. Because these two dominant terms do not depend on preference type t, the claim (10) then follows because $\mu^{\sigma}(s^*)/\mu^{\sigma}(s^*_-)$ is strictly increasing in σ by Assumption 1.

Next, for any almost type-independent strategy k, there is a type-independent strategy such that the set of best responses to the two strategies is the same. Let σ_j be the signal realization for which $k(t,\sigma_j)$ differs across t. The probability that a randomly drawn informed voter with a private signal σ_j would vote for $\mathcal{R}$ is $\int_T k(t,\sigma_j)\,\mathrm{d}P(t) = H(\sigma_j;k)$. Now, consider the type-independent strategy whose continuous representation is given by

$$\Psi(k) = J - j + H(\sigma_j;k).$$

The strategies k and $\Psi(k)$ are identical for every signal realization $\sigma < \sigma_j$ (every type votes for $\mathcal{L}$) and $\sigma > \sigma_j$ (every type votes for $\mathcal{R}$). They may differ for the realization σ_j; however the probability that a randomly drawn type with signal σ_j votes for $\mathcal{R}$ is the same for the two profiles. That is, $z(s,\theta;k) = z(s,\theta;\Psi(k))$ for all s and θ, and thus

$$g_n(v|s;k) = g_n(v|s;\Psi(k))$$

for all v and s. The set of best responses to k and to $\Psi(k)$ coincide.

Now we can complete the fixed point argument. Suppose that ψ_n is a continuous representation of a type-independent monotone strategy that satisfies (9). For n sufficiently large, if k_n is a best response to ψ_n and further $\Psi(k_n) = \psi_n$, then k_n is a best response to itself, because the set of best responses to k_n and to $\psi_n = \Psi(k_n)$ are the same. It follows that k_n is an equilibrium of Γ_n.

Proof of Proposition 1. Since full information equivalence is feasible, there is a strategy k such that $z(s^*_-, \overline{\theta}; k) < v_{\mathcal{C}} < z(s^*, \underline{\theta}; k)$. We first use a similar construction as in Lemma 2 to show that there is a type-independent monotone strategy $\hat{\psi}$ such that $z(s^*, \underline{\theta}; \hat{\psi}) = z(s^*, \underline{\theta}; k)$ and $z(s^*_-, \overline{\theta}; \hat{\psi}) \leq z(s^*_-, \overline{\theta}; k)$, with the latter a strict inequality if k is not almost type-independent and monotone. Define $\underline{\sigma} = \max\{\sigma : H(\sigma'; k) = 0 \ \forall \sigma' < \sigma\}$ and $\overline{\sigma} = \min\{\sigma : H(\sigma'; k) = 1 \ \forall \sigma' > \sigma\}$. By definition, either $\underline{\sigma} = \overline{\sigma}$, or $\underline{\sigma} < \overline{\sigma}$. In the first case, k is almost (or already) type-independent and monotone, and $\hat{\psi} = \Psi(k)$ is the continuous representation of a type-independent and monotone strategy, with $z(s^*, \underline{\theta}; \hat{\psi}) = z(s^*, \underline{\theta}; k)$ and $z(s^*_-, \overline{\theta}; \hat{\psi}) = z(s^*_-, \overline{\theta}; k)$. In the second case, construct a new strategy $\tilde{k}$ from k as follows. Let $\tilde{k}(t, \sigma) = k(t, \sigma)$ for all $\sigma \neq \underline{\sigma}, \overline{\sigma}$. If $H(\underline{\sigma}; k)\beta(\underline{\sigma}|s^*) < (1 - H(\overline{\sigma}; k))\beta(\overline{\sigma}|s^*)$, let $\tilde{k}(t, \underline{\sigma}) = 0$ and $\tilde{k}(t, \overline{\sigma}) = H(\overline{\sigma}; k) + H(\underline{\sigma}; k)\beta(\underline{\sigma}|s^*)/\beta(\overline{\sigma}|s^*)$ for all t. Otherwise, let $\tilde{k}(t, \underline{\sigma}) = H(\underline{\sigma}; k) - (1 - H(\overline{\sigma}; k))\beta(\overline{\sigma}|s^*)/\beta(\underline{\sigma}|s^*)$ and $\tilde{k}(t, \overline{\sigma}) = 1$ for all t. By construction, $z(s^*, \underline{\theta}; \tilde{k}) = z(s^*, \underline{\theta}; k)$ because

$$\sum_{j=1}^{J} \beta(\sigma_j|s^*) H(\sigma_j; \tilde{k}) = \sum_{j=1}^{J} \beta(\sigma_j|s^*) H(\sigma_j; k).$$

By Assumption 1,

$$\sum_{j=1}^{J} \beta(\sigma_j|s^*_-) H(\sigma_j; \tilde{k}) < \sum_{j=1}^{J} \beta(\sigma_j|s^*_-) H(\sigma_j; k),$$

which implies $z(s^*_-, \overline{\theta}; \tilde{k}) < z(s^*_-, \overline{\theta}; k)$. Compared with k, the number of signals σ for which $H(\sigma; \tilde{k})$ is not 0 or 1 is reduced by 1. The claim then follows by induction.

The above construction of $\hat{\psi}$ implies that $z(s^*_-, \overline{\theta}; \hat{\psi}) < v_{\mathcal{C}} < z(s^*, \underline{\theta}; \hat{\psi})$. Thus, condition (9) also holds for any $v_{\mathcal{L}} \in (z(s^*_-, \overline{\theta}; \hat{\psi}), v_{\mathcal{C}})$ and $v_{\mathcal{R}} \in (v_{\mathcal{C}}, z(s^*, \underline{\theta}; \hat{\psi}))$. Since $z(s, \theta; \psi)$ is continuous and strictly increases in ψ, there are unique values $\underline{\psi}$ and $\overline{\psi}$ with $\underline{\psi} < \hat{\psi} < \overline{\psi}$ such that $z(s^*_-, \overline{\theta}; \overline{\psi}) = v_{\mathcal{L}}$ and $z(s^*, \underline{\theta}; \underline{\psi}) = v_{\mathcal{R}}$. For any $\psi \in [\underline{\psi}, \overline{\psi}]$, condition (9) is satisfied.

We construct a correspondence A_n from the space of type-independent monotone strategies $[0, J]$ into itself as follows:

$$A_n(\psi) = \begin{cases} \{J\} & \text{if } \psi < \underline{\psi}, \\ \{\Psi(k) : k \in BR_n(\psi)\} & \text{if } \psi \in [\underline{\psi}, \overline{\psi}], \\ \{0\} & \text{if } \psi > \overline{\psi}, \end{cases}$$

where $BR_n(\cdot)$ is the best-response correspondence given by (3). Since $g_n(v|s;\psi)$ is a continuous function of ψ for all pivotal events and states, for any type t and any signal σ the set of best responses is upper hemicontinuous in ψ. It follows that $A_n(\cdot)$ is upper hemicontinuous for $\psi \in [\underline{\psi}, \overline{\psi}]$, as $\Psi(\cdot)$ is a summation of the best responses over all $\sigma \in \Sigma$ and an integration over all $t \in T$ and hence preserves upper hemicontinuity. Further, since $z(s^*_-, \overline{\theta}; \overline{\psi}) = v_{\mathcal{L}}$, for sufficiently large n, the pivotal event of $v_{\mathcal{L}}$ in state s^*_- dominates all other pivotal events in any state, and thus every type's best response to $\overline{\psi}$ is to vote for $\mathcal{L}$ regardless of the signal. It follows that $A_n(\overline{\psi}) = \{0\}$. Similarly, $A_n(\underline{\psi}) = \{J\}$, because the best response to $\underline{\psi}$ is to vote for $\mathcal{R}$ regardless of the signal. Therefore, $A_n(\cdot)$ is upper hemicontinuous for $\psi \in [0, J]$. The fact that A_n has a fixed point is an application of Kakutani's fixed point theorem. Let ψ_n be a fixed point of $A_n(\cdot)$. Then there exists a best response to ψ_n, denoted k_n, such that $\Psi(k_n) = \psi_n$. Such k_n is an equilibrium of Γ_n.

By construction, for all sufficiently large n, any equilibrium k_n is almost type-independent and monotone. Since $A_n(\underline{\psi}) = \{J\}$ and $A_n(\overline{\psi}) = \{0\}$, we have $\psi_n \in (\underline{\psi}, \overline{\psi})$. Therefore, $z(s^*_-, \overline{\theta}; k_n) = z(s^*_-, \overline{\theta}; \psi_n) < z(s^*_-, \overline{\theta}; \underline{\psi}) = v_{\mathcal{L}} < v_{\mathcal{C}}$, and similarly $z(s^*, \underline{\theta}; k_n) > v_{\mathcal{C}}$. By the weak law of large numbers, the sequence of equilibrium strategies $\{k_n\}$ achieves full information equivalence.

Finally, for any $s \leq s^*_-$ and any θ, for sufficiently large n, we have $z(s, \theta; k_n) = z(s, \theta; \psi_n) \leq z(s^*_-, \overline{\theta}; \psi_n) < z(s^*_-, \overline{\theta}; \overline{\psi}) = v_{\mathcal{L}}$. By the weak law of large numbers, the probability of recount in the equilibrium k_n in any state $s \leq s^*_-$, which is the probability that v falls between $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$, goes to 0 as n goes to infinity. Similarly, the probability of recount in any state $s \geq s^*$ also goes to 0 as n goes to infinity because $z(s, \theta; k_n) = z(s, \theta; \psi_n) \geq z(s^*, \underline{\theta}; \psi_n) > z(s^*, \underline{\theta}; \underline{\psi}) = v_{\mathcal{R}}$. ■

Since our equilibrium construction satisfies condition (9), the pivotal event $v_{\mathcal{L}}$ dominates for $s < s^*$, and $v_{\mathcal{R}}$ dominates for $s \geq s^*$. Further, it must also be the case that $g_n(v_{\mathcal{L}}|s^*_-; k_n)/g_n(v_{\mathcal{R}}|s^*; k_n)$ neither goes to zero nor to infinity, along the sequence of equilibrium strategies. Otherwise, eventually it becomes a unique best response for all types to vote for the same candidate regardless of the signal observed, which is not a fixed point of $A_n(\cdot)$. This implies that along any sequence of equilibrium strategies that satisfies (9):

$$\lim_{n\to\infty} I\left(v_{\mathcal{R}}, z(s^*, \underline{\theta}; k_n)\right) = \lim_{n\to\infty} I\left(v_{\mathcal{L}}, z(s^*_-, \overline{\theta}; k_n)\right). \tag{11}$$

As shown in the proof of Proposition 1, the equilibrium k_n in the above condition can be replaced by the type-independent strategy $\Psi(k_n)$ with the same implied vote share z. Since for any type-independent and monotone profile ψ, the vote share function $z(s, \theta; \psi)$ is strictly increasing in ψ, the equal-rate condition (11) is satisfied by a unique $\psi^I \in (\underline{\psi}, \overline{\psi})$. This means that, while the existence result of Proposition 1 does not exclude that there may be multiple equilibria for each game Γ_n, all equilibrium strategies that satisfy (9) correspond to the same type-independent strategy:

$$\lim_{n\to\infty} \Psi(k_n) = \psi^I.$$

In the limit, all equilibrium strategies k_n generate the same distribution over vote shares—a result that we formally state in the next proposition.

Proposition 2. *Let $\{k_n\}$ be a sequence of strategy satisfying condition (9) such that k_n is an equilibrium of Γ_n for each n. Then, $\lim_{n\to\infty} g_n(v|s; k_n) = \lim_{n\to\infty} g_n(v|s; \psi^I)$ for all v and s.*

Proposition 1 establishes that, with an election rule that allows for costly recounting, full information equivalence can indeed be achieved as an equilibrium outcome whenever it is feasible. We provide two examples to show that in environments where full information equivalence is not an equilibrium outcome with standard elections, allowing for costly recounting will resurrect efficiency.

Example 1: Inefficiency with aggregate uncertainty

Consider a binary-state (R and L), binary-signal (r and l) model with $\beta(r|R) = \beta(l|L) = q \in (1/2, 1)$. The preferences of an informed voter of type $t \in [0, 1]$ are described by the payoff difference function:

$$u(s, t) = \begin{cases} 1 - t & \text{if } s = R, \\ -t & \text{if } s = L. \end{cases}$$

Both the informed voters' preference types and the fraction θ of uninformed voters voting for candidate $\mathcal{R}$ are uniformly distributed on $[0, 1]$. Let the common prior beliefs assign equal probability to the two states, implying that $\mu^r(R) = \mu^l(L) = q$, and the election outcome be determined by a simple majority rule (i.e., $v_{\mathcal{C}} = 1/2$).

For any n, in an election without recounting, the following strategy $\hat{k}$ for an informed voter is an equilibrium: after receiving signal r, choose candidate $\mathcal{R}$ if

$t \leq q$ and $\mathcal{L}$ if $t > q$; after receiving l, choose candidate $\mathcal{R}$ if $t \leq 1-q$ and $\mathcal{L}$ if $t > 1-q$. To see this, note that given the assumed strategy profile, the vote share functions are:

$$z(R,\theta;\hat{k}) = \alpha\theta + (1-\alpha)\left(q^2 + (1-q)^2\right),$$
$$z(L,\theta;\hat{k}) = \alpha\theta + (1-\alpha)2q(1-q).$$

Thus, the probability that candidate $\mathcal{R}$ receives exactly half of the votes of n randomly selected voters satisfies

$$\begin{aligned} g_n\left(1/2\middle|R;\hat{k}\right) &= \binom{n}{n/2}\int_0^1 \left(z(R,\theta;\hat{k})(1-z(R,\theta;\hat{k}))\right)^{n/2} \mathrm{d}\theta \\ &= \binom{n}{n/2}\int_0^1 \left(z(L,\tilde{\theta};\hat{k})(1-z(L,\tilde{\theta};\hat{k}))\right)^{n/2} \mathrm{d}\tilde{\theta} = g_n\left(1/2\middle|L;\hat{k}\right), \end{aligned}$$

where the second equality follows from the substitution $z(R,\theta;\hat{k}) = 1 - z(L,1-\theta;\hat{k})$ and a change of variable $\tilde{\theta} = 1-\theta$. It follows immediately from (3) that every informed voter is best-responding.

The above strategy is essentially the unique equilibrium of the game. To see this, for any n consider a strategy k_n such that $g_n(1/2|R;k_n) > g_n(1/2|L;k_n)$. From (3), the best response k'_n to k_n satisfies $H(r;k'_n) > q$ and $H(l;k'_n) > 1-q$. But this implies that $g_n(1/2|R;k'_n) < g_n(1/2|L;k'_n)$, and therefore k_n cannot be an equilibrium strategy.[8] Similarly, any k_n such that $g_n(1/2|R;k_n) < g_n(1/2|L;k_n)$ cannot be an equilibrium strategy either.

If $z(R,0;\hat{k}) = (1-\alpha)\left(q^2 + (1-q)^2\right) < 1/2$, which is equivalent to $z(L,1;\hat{k}) > 1/2$, the unique equilibrium does not achieve full information equivalence. For example, if $\alpha = 1/5$ and $q = 2/3$, then for n sufficiently large and with probability arbitrarily close to 1, candidate $\mathcal{L}$ is elected in state R when the uncertainty state satisfies $\theta < 5/18$. and candidate $\mathcal{R}$ is elected in state L when $\theta > 13/18$. In the same environment, for an electoral rule with recounting and thresholds $v_{\mathcal{L}} = 1/2 - \epsilon$ and $v_{\mathcal{R}} = 1/2 + \epsilon$, a strategy profile, $\tilde{k}$, where informed voters "vote their signal" (i.e., vote for $\mathcal{R}$ when observing r and $\mathcal{L}$ when observing l) is an equilibrium for

[8]Since $H(r;k'_n) > q$ and $H(l;k'_n) > 1-q$, we have $z(R,\theta;k'_n) > 1 - z(L,1-\theta;k'_n)$. It is straightforward to show that $z(R,\theta;k'_n) > 1 - z(R,\theta;k'_n)$ if and only if $1 - z(L,1-\theta;k'_n) > z(L,1-\theta;k'_n)$. As a result, $z(R,\theta;k'_n)(1-z(R,\theta;k'_n)) < (1-z(L,1-\theta;k'_n))z(L,1-\theta;k'_n)$. It then follows that $g_n(1/2|R;k'_n) < g_n(1/2|L;k'_n)$.

n sufficiently large. The expected vote share for $\mathcal{R}$ is at least $8/15$ in state R, and at most $7/15$ in state L. For ϵ small, the pivotal event $v_{\mathcal{L}}$ dominates the other two pivotal events in state L; the pivotal event $v_{\mathcal{R}}$ dominates in state R; and $g_n(v_{\mathcal{L}}|L;\tilde{k}) = g_n(v_{\mathcal{R}}|R;\tilde{k})$ for all n. These conditions imply that for n large the strategy $\tilde{k}$ is an equilibrium, and achieves full information equivalence in the limit.

More generally, by Proposition 1, full information equivalence is feasible with recounting if (7) is satisfied when informed voters "vote their signal," or

$$z(L,1;\tilde{k}) = (1-\alpha)(1-q) < 1/2 < (1-\alpha)q = z(R,0;\tilde{k}).$$

In contrast, in the equilibrium without recounting, the vote share of $\mathcal{R}$ in state R is smaller than $1/2$ for some aggregate uncertainty state if

$$(1-\alpha)(q^2 + (1-q)^2) < 1/2.$$

For any $q \in (1/2, 1)$, the above two conditions are both satisfied if

$$1/(2q) < 1-\alpha < 1/(2(q^2 + (1-q)^2)).$$

For q close to 1, the above holds for α close to 0. Thus, when the signals are very accurate, even a tiny fraction of uninformed voters can make an election without recounting go wrong, while with recounting there is still an equilibrium that achieves full information equivalence as the electorate becomes large. □

Example 2: Inefficiency with non-monotone preferences

Consider a model with the same equally likely states R and L and binary signals r and l with $\beta(r|R) = \beta(l|L) = q$ as in Example 1, but with voters divided into two groups that favor opposite candidates similar to Bhattacharya (2013). All voters are informed. A majority voter with preference type t has the same preferences as an informed voter of type t in Example 1, but the preferences of a minority voter with preference type t are instead described by the payoff difference function:

$$\tilde{u}(s,t) = \begin{cases} -(1-t) & \text{if } s = R, \\ t & \text{if } s = L. \end{cases}$$

When the the payoff relevant state s changes from L to R, the favored candidate of majority voters switches from $\mathcal{L}$ to $\mathcal{R}$, while for minority voters the opposite is true.

With a slight abuse of notation, we denote as $P(t)$ the probability that a randomly drawn informed voter is a majority voter with preference type $t' \leq t$, and similarly $\tilde{P}(t)$ the probability that a randomly drawn voter is a minority voter with $t' \leq t$. Assumption 2 is satisfied so long as $P(1) > \tilde{P}(1)$.

Consider first a simple majority election without recounting. With only two states, it becomes convenient to represent an equilibrium with the induced probability γ_n of state R conditional on the two candidates having exactly half of n votes. By Bayes' rule, in equilibrium

$$\frac{g_n(1/2|R;\gamma_n)}{g_n(1/2|L;\gamma_n)} = \frac{\gamma_n}{1-\gamma_n}.$$

It then follows from (3) that for each signal $\sigma = r, l$, there is a threshold preference type t_n^σ such that, upon observing signal σ, a majority voter with t votes for R if and only if $t \leq t_n^\sigma$ while a minority voter with t votes R if and only if $t \geq t_n^\sigma$, where

$$t_n^r = \frac{q\gamma_n}{q\gamma_n + (1-q)(1-\gamma_n)},$$
$$t_n^l = \frac{(1-q)\gamma_n}{(1-q)\gamma_n + q(1-\gamma_n)}.$$

Thus, given γ_n, the equilibrium probability that a randomly drawn voter votes for R in state s, $z(s;\gamma_n)$, is given by:

$$z(R;\gamma_n) = q(P(t_n^r) + \tilde{P}(1) - \tilde{P}(t_n^r)) + (1-q)(P(t_n^l) + 1 - \tilde{P}(t_n^l)),$$
$$z(L;\gamma_n) = (1-q)(P(t_n^r) + \tilde{P}(1) - \tilde{P}(t_n^r)) + q(P(t_n^l) + \tilde{P}(1) - \tilde{P}(t_n^l)).$$

The limit value, γ, of any sequence of equilibrium values $\{\gamma_n\}$ must satisfy

$$|z(R;\gamma) - 1/2| = |z(L;\gamma) - 1/2|. \tag{12}$$

Otherwise, if $g_n(1/2|R;\gamma_n)/g_n(1/2|L;\gamma_n)$ converges to 0, then the limit of γ_n is 0. This implies that t_n^r and t_n^l both converge to 0, and as a result $z(R;\gamma_n)$ and $z(L;\gamma_n)$ both converge to $\tilde{P}(1)$. This is a contradiction to the assumption that (12) is violated. Similarly, $g_n(1/2|L;k_n)/g_n(1/2|R;k_n)$ converging to 0 leads to a contradiction to condition (12).

Figure 1 illustrates how condition (12) determines the limit equilibrium outcomes. Here we assume that t is distributed uniformly on $[0,1]$ for majority voters,

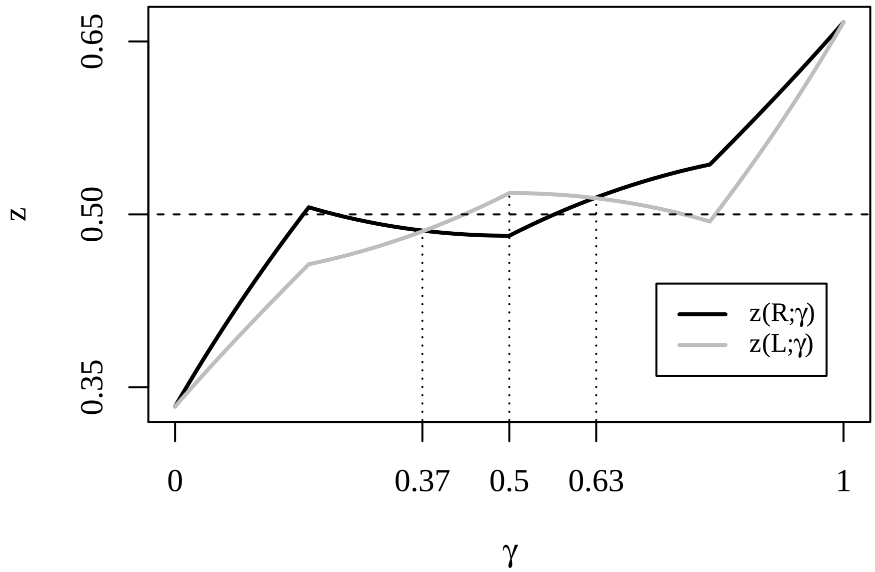

Figure 1. *The limit of equilibrium outcomes are all inefficient in simple majority elections. In the limit equilibrium with $\gamma = 0.37$, candidate $\mathcal{L}$ wins in both states. In the limit equilibrium with $\gamma = 0.63$, candidate $\mathcal{R}$ wins in both states. In the limit equilibrium with $\gamma = 0.5$, candidate $\mathcal{L}$ wins in state R and candidate $\mathcal{R}$ wins in state L.*

and uniformly on $[1/3, 2/3]$ for minority voters, with $P(1) = 1 - \tilde{P}(1) = 2/3$. With $q = 2/3$, Figure 1 plots $z(R;\gamma)$ and $z(L;\gamma)$. There are five values of γ that satisfy (12). The three interior values are all limits of equilibria. They are inefficient, because either the same candidate is elected in both states, or the candidate favored by minority voters is elected.

In an election with recounting, full information equivalence is feasible in equilibrium. For example, for $v_{\mathcal{L}} = 1/2 - \epsilon$ and $v_{\mathcal{R}} = 1/2 + \epsilon$, consider the strategy $\tilde{k}$ where all majority and minority voters vote their signal. The induced vote shares are $z(R;\tilde{k}) = q$ and $z(L;\tilde{k}) = 1 - q$. For n sufficiently large and ϵ sufficiently small, the pivotal event $v_{\mathcal{L}}$ dominates the other two pivotal events in state L; the pivotal event $v_{\mathcal{R}}$ dominates in state R; and $g_n(v_{\mathcal{L}}|L;\tilde{k}) = g_n(v_{\mathcal{R}}|R;\tilde{k})$ for all n. These conditions imply that for n large the strategy $\tilde{k}$ is an equilibrium, and achieves full information equivalence in the limit. □

So far we have focused on the best equilibrium of the game Γ_n. From the equilibrium construction described in Proposition 1, the key properties of the strategy k_n in the best equilibrium are that it is almost type-independent and that it distinguishes between states s^* and s^*_-. Because the two dominant pivotal events ($v_\mathcal{L}$ and $v_\mathcal{R}$) do not determine the identity of the winner, the fact that the full information equivalent outcome would select R in all states $s \geq s^*$ and would select $\mathcal{L}$ in all states $s \leq s^*_-$ is not used in our construction. In other words, suppose we take an arbitrary state s_i, and suppose there exists a strategy k such that it is feasible to separate state s_i from its preceding state s_{i-1}:

$$z(s, \overline{\theta}; k) < v_\mathcal{L} < v_\mathcal{R} < z(s', \underline{\theta}; k) \quad \text{for all } s < s_i \text{ and } s' \geq s_i.$$

Then, the same logic that leads to Proposition 1 allows us to show that there exists a sequence of monotone strategies $\{\hat{k}_n\}$ such that $\mathcal{L}$ is elected in all states lower than s_i and $\mathcal{R}$ is elected in all other states, and such that $\hat{k}_n$ is an equilibrium of the game Γ_n.

This multiplicity of equilibria is generally not a problem in a model with binary signals (i.e., $\Sigma = \{\sigma_1, \sigma_2\}$). However, when $J > 2$, full information equivalence is achieved only when the best equilibrium that separates s^* from s^*_- is played. For example, if an equilibrium strategy $\hat{k}_n$ that separates s_i from s_{i-1} is played, and if $s_i < s^*$, then this equilibrium would choose the wrong candidate $\mathcal{R}$ in states $s_i, s_{i+1}, \ldots, s^*_-$, when $\mathcal{L}$ should have been chosen for full information equivalence. Notice that all these other equilibria still have a monotone structure. In the example above with $s_i < s^*$, even when the "wrong" equilibrium strategy $\hat{k}_n$ is adopted, an election with recounting would still produce the "right" winner in states $s \geq s^*$ or $s \leq s_{i-1}$.

4. The Rate of Convergence to Information Efficiency

Our main result establishes that recounting can restore full information equivalence in the best equilibrium when such an outcome is not an equilibrium in a standard election without recounting. But even when full information equivalence obtains in the limit of equilibria in standard elections, introducing recounting can still improve on the equilibrium outcome. In this section, we show that recounting improves the "informativeness" of the equilibrium strategy profile. That is, recounting generates a larger spread of the expected vote share for $\mathcal{R}$ across the two critical states s^* and s^*_-. Indeed, this property is consistent with our conclusion in the previous section that, with recounting, full information equivalence is more robust to aggregate uncertainty. As a result of this property, the informationally efficient outcome will be approximated at a faster rate as the size of the electorate grows.

The key to understand why recounting improves the informativeness of the equilibrium strategy is that the dominant pivotal event only determines whether a recount is triggered, but does not change the identity of the winner. As a result, the incentives to vote are type-independent. This allows us to construct an equilibrium where all types vote informatively (i.e., their votes change across signals). Without recounting, at the only pivotal event v_C, the vote determines the election outcome. Thus a voter's belief over payoff relevant states and his preference type matter for his voting decision. For types whose beliefs are not significantly changed by their private signals conditional on the pivotal event being realized, their votes must be uninformative.

In the following proposition, we compare a sequence of equilibria of $\{\Gamma_n^\delta\}$, where it is assumed that the recounting cost is $\delta > 0$, to a sequence of equilibria of $\{\Gamma_n^0\}$, which differ only by the absence of recounting cost (i.e., $\delta = 0$). We introduce the following assumption.

Assumption 3. *For any probability distribution γ over S, there is a subset of types $T' \subseteq T$, such that $P(T') > 0$ and, for all $t \in T'$,*

$$\left(\sum_{s\in S}\gamma(s)\beta(\sigma|s)u(s,t)\right)\left(\sum_{s\in S}\gamma(s)\beta(\sigma'|s)u(s,t)\right) > 0 \quad \textit{for all } \sigma,\sigma' \in \Sigma.$$

If γ is the posterior belief conditional on the pivotal event v_C, then all types in T' vote uninformatively in any equilibrium of Γ_n^0 as their expected payoff difference between voting $\mathcal{R}$ and voting $\mathcal{L}$ does not change sign with the voter's private information. Assumption 3 says that there is always a positive mass of such types. Assumption 3 is both a requirement that preferences vary enough with types, and that private signals are not too informative. It is satisfied in most strategic voting models, such as, Feddersen and Pesendorfer (1997) and Bhattacharya (2013).[9] The

[9]In Example 1 of Section 3, types lower than $1-q$ and higher than q do not vote informatively in a

proof of Proposition 3 below makes it clear that the role of Assumption 3 is to ensure that any equilibrium strategy k_n^0 of a standard election game Γ_n^0 is not almost type-independent and monotone.[10]

Proposition 3. *Let $\{k_n^\delta\}$ and $\{k_n^0\}$ be two sequences of monotone strategies such that: (i) both achieve full information equivalence; (ii) the sequence $\{k_n^\delta\}$ satisfies condition (9); and (iii) k_n^δ is an equilibrium of Γ_n^δ and k_n^0 is an equilibrium of Γ_n^0 for all n. Then, for all $v_{\mathcal{R}}$ and $v_{\mathcal{L}}$ sufficiently close to $v_{\mathcal{C}}$,*

$$\lim_{n\to\infty} z(s_-^*,\overline{\theta};k_n^\delta) \leq \lim_{n\to\infty} z(s_-^*,\overline{\theta};k_n^0) < \lim_{n\to\infty} z(s^*,\underline{\theta};k_n^0) \leq \lim_{n\to\infty} z(s^*,\underline{\theta};k_n^\delta),$$

with all strict inequalities if Assumption 3 holds.

Proof. The limit of the equilibrium in the game Γ_n^δ must satisfy the equal-rate condition that

$$\lim_{n\to\infty} I(v_{\mathcal{L}}, z(s_-^*,\overline{\theta};k_n^\delta)) = \lim_{n\to\infty} I(v_{\mathcal{R}}, z(s^*,\underline{\theta};k_n^\delta)).$$

Similarly, the limit of the equilibrium in the game Γ_n^0 must satisfy an analogous equal-rate condition:

$$\lim_{n\to\infty} I(v_{\mathcal{C}}, z(s_-^*,\overline{\theta};k_n^0)) = \lim_{n\to\infty} I(v_{\mathcal{C}}, z(s^*,\underline{\theta};k_n^0)).$$

Otherwise, either the ratio $g_n(v_{\mathcal{C}}|s;k_n^0)/g_n(v_{\mathcal{C}}|s^*;k_n^0)$ goes to 0 for all $s \neq s^*$, or the ratio $g_n(v_{\mathcal{C}}|s;k_n^0)/g_n(v_{\mathcal{C}}|s_-^*;k_n^0)$ goes to 0 for all $s \neq s_-^*$. In either case, every type's best response to k_n^0 is independent of his signal, as conditional on the realization of the pivotal event the state is known, which contradicts the hypothesis that the $\{k_n^0\}$ achieves full information equivalence.

For all recounting thresholds sufficiently close to $v_{\mathcal{C}}$, if $z(s^*,\underline{\theta};k_n^0) > z(s^*,\underline{\theta};k_n^\delta)$, then the two equal-rate conditions can be satisfied only if $z(s_-^*,\overline{\theta};k_n^0) < z(s_-^*,\overline{\theta};k_n^\delta)$. To prove the proposition it is then sufficient to establish the following claim:

$$z(s^*,\underline{\theta};k_n^0) \geq z(s^*,\underline{\theta};k_n^\delta) \implies z(s_-^*,\overline{\theta};k_n^0) \geq z(s_-^*,\overline{\theta};k_n^\delta), \tag{13}$$

with the second inequality being strict if Assumption 3 holds.

standard election. More generally, in a model with a continuum of states, Feddersen and Pesendorfer (1997) show that the fraction of informed voters who do not vote informatively in a large election without recounting goes to one.

[10]Formally, this requires that there is no σ_j such that $H(\sigma;k_n^0)$ is 0 for all $\sigma < \sigma_j$ and 1 for all $\sigma > \sigma_j$.

Suppose that $z(s^*, \underline{\theta}; k_n^0) \geq z(s^*, \underline{\theta}; k_n^\delta)$. In the proof of Proposition 1, we have shown that given the strategy k_n^0 we can construct a type-independent monotone strategy $\tilde{\psi}$ such that $z(s^*, \underline{\theta}; \tilde{\psi}) = z(s^*, \underline{\theta}; k_n^0)$ and $z(s_-^*, \overline{\theta}; \tilde{\psi}) \leq z(s_-^*, \overline{\theta}; k_n^0)$, with the latter holding strictly if k_n^0 is not (almost) type-independent and monotone. Further, given an equilibrium strategy k_n^δ that satisfies (9), there is a type-independent strategy $\Psi(k_n^\delta)$ such that $z(s, \theta; k_n^\delta) = z(s, \theta; \Psi(k_n^\delta))$ for all s, θ. It follows that $z(s^*, \underline{\theta}; \tilde{\psi}) \geq z(s^*, \underline{\theta}; \Psi(k_n^\delta))$. Since the vote share function $z(s, \theta; \cdot)$ is strictly increasing, we have $\tilde{\psi} \geq \Psi(k_n^\delta)$, and thus

$$z(s_-^*, \overline{\theta}; k_n^0) \geq z(s_-^*, \overline{\theta}; \tilde{\psi}) \geq z(s_-^*, \overline{\theta}; \Psi(k_n^\delta)) = z(s_-^*, \overline{\theta}; k_n^\delta),$$

where the first inequality is strict if k_n^0 is not (almost) type-independent and monotone. When Assumption 3 holds, the equilibrium k_n^0 of Γ_n^0 involves a positive mass of voters whose votes are uninformative, and is therefore not (almost) type-independent. Claim (13) follows immediately. ■

Proposition 3 establishes that, under Assumption 3, the probability of a "mistake," meaning an election outcome different from the full information outcome, is smaller in both states s_-^* and in state s^*, when the electoral rules mandate a costly recount for a sufficiently tight election. For example, the probabilities of electing $\mathcal{R}$ in state s_-^* in an election with recounting and without satisfy[11]

$$\lim_{n\to\infty} \frac{\int_{v_\mathcal{C}}^1 g_n(v|s_-^*; k_n^\delta)\,\mathrm{d}v}{\int_{v_\mathcal{C}}^1 g_n(v|s_-^*; k_n^0)\,\mathrm{d}v} = \lim_{n\to\infty} \frac{g_n(v_\mathcal{C}|s_-^*; k_n^\delta)}{g_n(v_\mathcal{C}|s_-^*; k_n^0)} = 0,$$

because $I(v_\mathcal{C}, z(s_-^*, \overline{\theta}; k_n^\delta)) < I(v_\mathcal{C}, z(s_-^*, \overline{\theta}; k_n^0))$. Furthermore, if the vote share function $z(\cdot, \overline{\theta}; k_n^0)$ is increasing,[12] we have the same ranking of total probabilities of electing the wrong candidate $\mathcal{R}$ in state lower than or equal to s_-^* in an election with recounting and without:

$$\lim_{n\to\infty} \frac{\sum_{s\leq s_-^*} \int_{v_\mathcal{C}}^1 g_n(v|s; k_n^\delta)\,\mathrm{d}v}{\sum_{s\leq s_-^*} \int_{v_\mathcal{C}}^1 g_n(v|s; k_n^0)\,\mathrm{d}v} = \lim_{n\to\infty} \frac{g_n(v_\mathcal{C}|s_-^*; k_n^\delta)}{g_n(v_\mathcal{C}|s_-^*; k_n^0)} = 0.$$

Thus, whenever an electoral rule without recounting achieves full information equivalence, adding recounting is still beneficial by providing a faster rate of convergence to the same informationally efficient outcome.

[11] The first equality follows from a similar argument as in Lemma 1.

[12] This is true if all informed voters have state-monotone preferences in that $u(\cdot, t)$ is weakly increasing for all t.

5. Discussion

5.1. Counting errors

Our model does not allow for counting errors, so that the vote count in the initial stage is identical to the vote count in the recount stage. There are different ways to introduce counting errors. We consider two alternatives.

In the first version of a model with counting error, we assume that each vote for candidate $\mathcal{R}$ has an independent probability $\zeta < 1/2$ of being miscounted as a vote for candidate $\mathcal{L}$, and likewise each vote for $\mathcal{L}$ has an independent probability ζ of being miscounted as a vote for $\mathcal{R}$. Further assume that if there is a recount, all the counting errors are corrected. Under these assumptions, if the true vote share for candidate $\mathcal{R}$ is v, the initial vote count for $\mathcal{R}$ will be

$$v_e = (1-\zeta)v + \zeta(1-v).$$

Note that $v_e > v$ if and only if $v < 1/2$, which is due to regression to the mean. Define

$$v'_{\mathcal{L}} \equiv \frac{v_{\mathcal{L}} - \zeta}{1-2\zeta}, \qquad v'_{\mathcal{R}} \equiv \frac{v_{\mathcal{R}} - \zeta}{1-2\zeta}.$$

Then, under the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$, the election would go into the recount stage if the true vote share v for $\mathcal{R}$ is between $v'_{\mathcal{L}}$ and $v'_{\mathcal{R}}$.

With counting errors, whether full information equivalence is feasible now depends on the specifics of the electoral rule as well as the probability of miscounting. Specifically, full information equivalence is feasible if there exists a strategy k such that

$$z(s,\overline{\theta};k) < \min\{v'_{\mathcal{L}}, v_{\mathcal{C}}\} \leq \max\{v'_{\mathcal{R}}, v_{\mathcal{C}}\} < z(s',\underline{\theta};k) \text{ for all } s < s^* \text{ and } s' \geq s^*. \quad (14)$$

Condition (14) and the original condition (7) for feasibility coincide whenever

$$v'_{\mathcal{L}} < v_{\mathcal{C}} < v'_{\mathcal{R}}. \quad (15)$$

Unless $v_{\mathcal{C}} = 1/2$, in which case (15) holds for any pair of $(v_{\mathcal{L}}, v_{\mathcal{R}})$, it is possible that (15) may not hold.[13]

[13]Even if $v_{\mathcal{C}} = 1/2$, condition (15) may not hold if the error rate depends on whether a vote is for $\mathcal{R}$ or for $\mathcal{L}$. A similar conclusion to Proposition 4 holds in this case: full information equivalence is achievable with recounting if the errors are small enough.

To replicate the equilibrium construction of Proposition 1, condition (15) is necessary. In the original argument to establish Proposition 1, the requirement (14) is satisfied by taking $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ sufficiently close to $v_{\mathcal{C}}$. In the presence of counting errors this is not possible, as recounting thresholds that are too tight may lead to inefficiencies through identifying the true loser to be the winner. However, for any pair $(v_{\mathcal{L}}, v_{\mathcal{R}})$, if the counting error ζ is sufficiently small, then (15) holds. Further, for all pairs $(v_{\mathcal{L}}, v_{\mathcal{R}})$ sufficiently close to $v_{\mathcal{C}}$, if a strategy satisfies (7), then there exists ζ sufficiently small such that this strategy also satisfies (14). Thus, the equilibrium construction of Proposition 1 can be replicated provided the counting error is small enough. We can summarize this discussion in the following statement.

Proposition 4. *Suppose full information equivalence is feasible for an electoral rule. If $\delta > 0$, for all $v_{\mathcal{L}}$, $v_{\mathcal{R}}$ sufficiently close to $v_{\mathcal{C}}$ there is a $\overline{\zeta}(v_{\mathcal{L}}, v_{\mathcal{R}}) > 0$ such that, for all miscounting probabilities $\zeta < \overline{\zeta}(v_{\mathcal{L}}, v_{\mathcal{R}})$, there exists a sequence of monotone strategies $\{k_n\}$ that achieves full information equivalence, and such that k_n is an equilibrium of the game Γ_n for each n.*

Our second model of counting errors assumes they are systemic instead of independent mistakes in counting each ballot. For example, such correlated errors may occur when a certain counting protocol (how to deal with hanging chads, etc.) is not properly followed, so that all the votes in the same polling station or even the entire election are miscounted is a specific way. To model these errors, we assume that if the true vote share for candidate $\mathcal{R}$ is v, then upon the initial count the vote share is recorded as

$$v_e = \begin{cases} 1 & \text{if } v + \xi > 1, \\ 0 & \text{if } v + \xi < 0, \\ v + \xi & \text{otherwise.} \end{cases}$$

In the above, ξ is a random variable with positive and continuous density on the support $[\underline{\xi}, \overline{\xi}]$. Upon recounting, all errors are detected so that the election outcome is based on the true vote share v. The effect of the systemic counting error ξ is similar to the effect of aggregate uncertainty θ, except that ξ only influences the initial vote share but not the final tally. Specifically, if $z(s, \underline{\theta}; k_n) + \underline{\xi} > v_{\mathcal{R}}$, then in state s the pivotal event $v_e = v_{\mathcal{R}}$ dominates the other pivotal events $v = v_{\mathcal{C}}$, and $v_e = v_{\mathcal{L}}$ for sufficiently large n. Proposition 1 continues to hold if there exists a strategy k such that

$$z(s^*_-, \overline{\theta}; k) + \overline{\xi} < v_{\mathcal{C}} < z(s^*, \underline{\theta}; k) + \underline{\xi}. \tag{16}$$

While (16) is stronger than the requirement that full information equivalence is feasible, it is implied by the latter whenever the distribution of the systemwide error is sufficiently concentrated (i.e. $\overline{\xi} - \underline{\xi}$ is sufficiently small). Thus, similarly to the first model of counting errors, the result of Proposition 1 is robust to the introduction of small counting errors.

5.2. Sincere voters

Our results can be extended to a model where a fraction $x \in (0, 1)$ of informed voters are sincere. These voters cast their votes based on their private signals and preferences as if their vote determines the outcome of the election. In place of (3), their voting strategy k^x satisfies

$$\left(\sum_{s \in S} \mu^{\sigma}(s) u(s,t)\right) k^x(t,\sigma) \geq 0; \; \left(\sum_{s \in S} \mu^{\sigma}(s) u(s,t)\right) (1 - k^x(t,\sigma)) \leq 0.$$

Under Assumption 1, strategy k^x is monotone in signals if preferences are monotone in states. However, for sufficiently rich signal space and type space, k^x is not almost type-independent.

Fix a strategy k used by all strategic voters, who now make up the fraction of $(1 - \alpha)(1 - x)$ of the electorate. Instead of (1), the probability $z^x(s, \theta; k)$ of a randomly drawn voter choosing $\mathcal{R}$ in payoff state s and aggregate uncertainty state θ becomes

$$z^x(s,\theta;k) = (1-\alpha) \sum_{j=1}^{J} ((1-x) H(\sigma_j; k) + x H(\sigma_j; k^x)) \beta(\sigma_j | s) + \alpha\theta.$$

With the above expression of $z^x(s, \theta; k)$ replacing $z(s, \theta; k)$, the analysis of strategic voters proceeds as before. We have the following counterpart of Proposition 1.

Proposition 5. *Fix any $x \in (0, 1)$, and suppose that there exists a strategy k for strategic voters such that*

$$z^x(s^*_-, \overline{\theta}; k) < v_{\mathcal{C}} < z^x(s^*, \underline{\theta}; k). \tag{17}$$

If $\delta > 0$ and $v_{\mathcal{L}}$, $v_{\mathcal{R}}$ are sufficiently close to $v_{\mathcal{C}}$, there exists a sequence of strategies $\{k_n\}$ such that, for all n sufficiently large, k_n is an equilibrium of Γ_n and $\{k_n\}$ achieves full information equivalence.

It is straightforward to compare Proposition 5 with Proposition 1. The difference boils down to condition (17) versus condition (8). By assumption, sincere voters have the same information as strategic voters. As a result, whether (8) is satisfied or not is independent of x. In contrast, a smaller value of x makes condition (17) more likely to be satisfied. As shown in the proof of Proposition 1, if (17) is satisfied by some strategy k for informed voters, then it is also satisfied by a type-independent strategy $\hat{\psi}$. Since a type-independent strategy maximizes the contribution by strategic voters to the difference $z^x(s^*, \underline{\theta}; \hat{\psi}) - z^x(s^*_-, \overline{\theta}; \hat{\psi})$, and since k^x is not (almost) type-independent, if (17) is satisfied for some x, then it is also satisfied for a smaller value of x. Thus, our main result of full information equivalence through costly recounting is robust to limited presence of sincere voters.

There is another sense that our main result is robust to sincere voting. Under Assumption 1 and Assumption 2, the expected contribution to $z^x(s, \theta; k)$ by sincere voters, given by

$$(1-\alpha)x \sum_{j=1}^{J} H(\sigma_j; k^x)\beta(\sigma_j|s) = (1-\alpha)x \sum_{j=1}^{J} \beta(\sigma_j|s) \int_T k^x(t, \sigma_j)\,\mathrm{d}P(t),$$

is increasing in s. This implies that compared to uninformed voters represented by aggregate uncertainty θ, sincere voters contribute positively to the difference $z^x(s^*, \underline{\theta}; k) - z^x(s^*_-, \overline{\theta}; k)$ for any strategy k of strategic voters. More precisely, if (17) is satisfied for some x and α, then it is satisfied for any $x' > x$ and $\alpha' < \alpha$ such that $(1-\alpha')(1-x') = (1-\alpha)(1-x)$. Thus, replacing non-strategic voters with sincere voters helps achieve full information efficiency.

5.3. Recounting cost

Our model of election with recounting does not depend on the magnitude of the recounting cost δ. We only assume that δ is positive and fixed as n goes to infinity. This restriction can be relaxed by assuming instead that recounting costs a fixed amount of $\Delta > 0$, and that in an election with $n+1$ voters each voter bears a cost of $\delta_n = \Delta/(n+1)$.

Since the equilibrium strategy k_n satisfies $z(s^*, \underline{\theta}; k_n) > v_{\mathcal{R}}$ for sufficiently large n and is monotone, Lemma 1 implies that the ratio $g_n(v|s; k_n)/g_n(v_{\mathcal{R}}|s^*; k_n)$ goes to 0 as n goes to infinity for $v = v_{\mathcal{L}}, v_{\mathcal{C}}$ and every $s \geq s^*$. Similarly, the ratio $g_n(v|s; k_n)/g_n(v_{\mathcal{L}}|s^*_-; k_n)$ goes to 0 as n goes to infinity for $v = v_{\mathcal{R}}, v_{\mathcal{C}}$ and every $s \leq$

s_{-}^{*}. Moreover, these ratios go to 0 at an exponential rate because the rate functions of the different pivotal events are ranked. From (10), the incentives of an informed voter observing a signal realization σ are now described by the inequality

$$\sum_{s\geq s^*}\mu^{\sigma}(s)\left(\frac{g_n(v_{\mathcal{R}}|s;k)}{g_n(v_{\mathcal{R}}|s^*;k)}\delta_n+\frac{g_n(v_{\mathcal{C}}|s;k)}{g_n(v_{\mathcal{R}}|s^*;k)}u(s,t)-\frac{g_n(v_{\mathcal{L}}|s;k)}{g_n(v_{\mathcal{R}}|s^*;k)}\delta_n\right)g_n(v_{\mathcal{R}}|s^*;k)\geq$$
$$\sum_{s<s^*}\mu^{\sigma}(s)\left(-\frac{g_n(v_{\mathcal{R}}|s;k)}{g_n(v_{\mathcal{L}}|s_{-}^*;k)}\delta_n-\frac{g_n(v_{\mathcal{C}}|s;k)}{g_n(v_{\mathcal{L}}|s_{-}^*;k)}u(s,t)+\frac{g_n(v_{\mathcal{L}}|s;k)}{g_n(v_{\mathcal{L}}|s_{-}^*;k)}\delta_n\right)g_n(v_{\mathcal{L}}|s_{-}^*;k).$$

For n large, by Lemma 3 the left-hand-side of the above still becomes arbitrarily close to $\mu^{\sigma}(s^*)g_n(v_{\mathcal{R}}|s^*;k)\delta_n$, and the right-hand-side is still arbitrarily close to $\mu^{\sigma}(s_{-}^*)g_n(v_{\mathcal{L}}|s_{-}^*;k)\delta_n$. This is because, even though the recounting cost δ_n goes to 0 as n goes to infinity, it goes to 0 only at the rate $1/n$. The remainder of the proof of Proposition 1 goes through with no change.

5.4. Uncertain size of electorate

The analysis presented here can be generalized to the case with an uncertain electorate size if we assume that the number of voters is N, with N being a Poisson random variable with mean n. Myerson (1998; 2000) develops the tools to study such Poisson games.

Recall that from Stirling's approximation to the binomial probability in equation (4), the rate at which the pivotal probability that the vote share equals v goes to 0 is given by:

$$\lim_{n\to\infty}\frac{\log g_n(v|s,\theta;k_n)}{n}=\log I(v;z(s,\theta;k_n)).$$

In contrast, Myerson (2000) shows that in a Poisson model, the corresponding rate is:

$$\lim_{n\to\infty}\frac{\log g_n(v|s,\theta;k_n)}{n}=I(v;z(s,\theta;k_n))-1$$

Since $I-1$ is an increasing transformations of $\log I$, given any v, s and k_n, the θ that maximizes $\log I$ in the model with no population uncertainty also maximizes $I-1$ in the Poisson model. Lemma 1 then implies that if $z(s^*,\underline{\theta};k_n)>v_{\mathcal{R}}$, then the event $v=v_{\mathcal{R}}$ dominates the events $v=v_{\mathcal{C}}$ and $v=v_{\mathcal{L}}$ in every state $s\geq s^*$. Likewise, if $z(s_{-}^*,\overline{\theta};k_n)<v_{\mathcal{L}}$, then the event $v=v_{\mathcal{L}}$ dominates the events $v=v_{\mathcal{C}}$ and $v=v_{\mathcal{R}}$ in every state $s\leq s_{-}^*$. All the results in the current paper remains intact in the Poisson model.

6. Concluding Remarks

This paper is an outgrowth of our earlier papers (Damiano, Li and Suen, 2010; 2012) that use costly delay to improve information aggregation in a two-agent negotiation problem. Here, we introduce multiple pivotal events to resurrect informative voting in large elections. The key to our equilibrium construction relies on the fact that, while the probabilities of different pivotal events are all vanishingly small in large elections, the rate at which they go to zero can be ranked. Since the desire to avoid recounting cost is preference-independent, and since pivotal events triggering a recount dominate the pivotal event involving a tie between the candidates, we demonstrate how informative voting by all types can be an equilibrium in large elections with recounting, producing asymptotically informationally efficient outcomes which may otherwise be infeasible in standard elections. The analysis of elections with multiple pivotal events also features in Razin (2003) in the context of signaling policy preference by voters, and in Bouton and Castanheira (2012) and Ahn and Oliveros (2012) in models of multi-candidate and multi-issue voting.

In this paper we have considered the Condorcet jury theorem in large elections. In a jury setting, Feddersen and Pesendorfer (1998) show that a unanimous conviction rule in jury decisions may lead to higher probability of false conviction as well as false acquittal than the simple majority rule, and the probability of convicting an innocent defendant may increase with the size of the jury. More relevant to the present paper is a recent literature that asks whether the Condorcet jury theorem continues to hold when acquiring information is costly to individual agents. Mukhopadhaya (2005) shows that in a symmetric mixed strategy equilibrium, as the number of committee members increases, each member chooses to collect information with a smaller probability. He finds examples in which, using the majority rule, a larger committee makes the correct decision with a lower probability than does a smaller one. Koriyama and Szentes (2009) consider a model in which agents choose whether or not to acquire information in the first stage, and then the decision is made according to an ex post efficient rule in the second stage. They show that there is a maximum group size such that in smaller groups each member will choose to collect evidence, and the Condorcet jury theorem fails for larger groups. However, in a model with the quality of information as a continuous choice variable, Martinelli (2006) shows that if the marginal cost of information is near zero for nearly irrelevant information, then there will be effective information aggregation despite the fact that each individual voter will choose to be very poorly informed. In another paper, Krishna and Morgan (2012) show that when participation in an election is costly but voluntary, those who choose to participate will vote informatively even in a standard election. However the fraction of participating voters is vanishingly small in a large election, rendering asymptotic information efficiency difficult to achieve if there is aggregate uncertainty in the model.

References

Ahn, David; and Oliveros, Santiago (2012). "Combinatorial Voting." *Econometrica*, 80, 89–141.

Austin-Smith, David; and Banks, Jeffrey (1996). "Information Aggregation, Rationality and the Condorcet Jury Theorem." *American Political Science Review*, 90, 34–45.

Bhattacharya, Sourav (2013). "Preference Monotonicity and Information Aggregation in Elections."*Econometrica*, 81, 1229–1247.

Bouton, Laurent; and Castanheira, Micael (2012). "One Person, Many Votes: Divided Majority and Information Aggregation." *Econometrica*, 80, 43–87.

Condorcet, M.J.A.N. de Caritat (1875). "An Essay on the Application of Analysis to the Probability of Decisions Rendered by a Plurality of Voters," abridged and translated in I. McLean and A.B. Urken, eds., *Classics of Social Choice*, 1995 (Ann Arbor: University of Michigan Press).

Damiano, Ettore; Li, Hao; and Suen, Wing (2010). "Delay in Information Aggregation." Working paper, University of Toronto.

Damiano, Ettore; Li, Hao; and Suen, Wing (2012). "Optimal Deadlines for Agreements." *Theoretical Economics*, 7, 357–393.

Degan, Arianna; and Antonio Merlo (2009). "Do Voters Vote Ideologically?" *Journal of Economic Theory*, 144, 1868–1894.

Dembo, Amir; and Zeitouni, Ofer (1998). *Large Deviations Techniques and Applications*, 2d. ed. New York: Springer.

Esponda, Ignacio; and Vespa, Emanuel (2014). "Hypothetical Thinking and Information Extraction in the Laboratory." *American Economic Journal: Microeconomics*, 6, 180–202.

Feddersen, Timothy; and Pesendorfer, Wolfgang (1996). "The Swing Voter's Curse." *American Economic Review*, 86, 408–424.

Feddersen, Timothy; and Pesendorfer, Wolfgang (1997). “Voting Behavior and Information Aggregation in Elections with Private Information.” *Econometrica*, 65, 1029–1058.

Feddersen, Timothy; and Pesendorfer, Wolfgang (1998). “Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting.” *American Political Science Review*, 92, 23–35.

Iaryczower, Matias; and Shum, Matthew (2012). “The Value of Information in the Court: Get it Right, Keep it Tight.” *American Economic Review*, 102, 202–237.

Koriyama, Yukio; and Szentes, Balazs (2009). “A Resurrection of the Condorcet Jury Theorem.” *Theoretical Economics*, 4, 227–252.

Krishna, Vijay; and Morgan, John (2012). “Voluntary Voting: Costs and Benefits.” *Journal of Economic Theory*, 147, 2083–2123.

Martinelli, Cesar (2006). “Would Rational Voters Acquire Costly Information?” *Journal of Economic Theory*, 129, 225–251.

Maud, Ernst; and Rydqvist, Kristian (2009). “Do Shareholders Vote Strategically? Voting Behavior, Proposal Screening, and Majority Rules.” *Review of Finance*, 13, 47–79.

Mukhopadhaya, Kaushik (2005). “Jury Size and the Free Rider Problem.” *Journal of Law, Economics and Organization*, 19, 24–45.

Myerson, Roger (1998). “Population Uncertainty and Poisson Games.” *International Journal of Game Theory*, 27, 375–392.

Myerson, Roger (2000). “Large Poisson Games.” *Journal of Economic Theory*, 94, 7–45.

Razin, Ronny (2003). “Signaling and Election Motivations in a Voting Model with Common Values and Responsive Candidates.” *Econometrica*, 71, 1083–1119.