Unobserved Auctions

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Abstract

A seller of an indivisible good faces buyers who draw their private values independently from the same distribution. The seller commits to an "auction" that maps a profile of bids to a take-it-or-leave-it offer to a selected buyer, but this mapping is unobserved by buyers. In any equilibrium where the seller makes a finite number of given offers, each offer corresponds to a bid, with the lowest being the monopoly price. A buyer with any value above the monopoly price randomizes over the highest bid below his value and all higher bids, while the seller selects the highest bid and randomizes over this and all lower offers. The bidding strategy for buyers with value below the monopoly price determines the seller's equilibrium revenue. It is minimized, to the monopoly revenue, when they bid independently of their values, and maximized when their bids generate a first order stochastic dominated distribution of the highest bid among all buyers. The maximal revenue increases with the number of offers. In the most profitable equilibrium, all offers above the monopoly price are made.

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1 Introduction

When an internet user types a keyword on Google.com, Google Ads runs an auction to determine what ads the user sees on the pages displayed by Google's search engine. According to the help page of Google Ads, five factor together determine the outcome for an ad bidder in the auction: the bid, the quality of the ad, the expected impact from the assets and other formats of the ad, the Ad Rank of the ad, and the context of the ad. Among them, the only factor controlled by the ad bidder is the bid. The other four are produced by the auction algorithm employed by Google Ads, and are unknown to the ad bidder. From the perspective of the ad bidder, the rule of the ads auction is unobserved.

In this paper, we consider a stylized class of unobserved auction problems. We study the simplest auction setting with a seller of a single indivisible object and a fixed number of potential buyers whose values for the object are independently drawn from the same distribution.[1] The only departure from this well-understood setting is that buyers do not observe the rule of the auction. Specifically, in our unobserved auction game, the seller commits to a mapping from a profile of acceptable bids to a take-it-or-leave-it offer to a selected buyer. Buyers know that acceptable bids are restricted to the support of values, but do not know the mapping.

We make one simplifying assumption that the monopoly revenue function from making a take-it-or-leave-it offer to any buyer is strictly concave. This implies that the virtual value function is strictly increasing whenever it is positive. By Myerson (1981), the monopoly price is the optimal reserve price regardless of the number of buyers. In our unobserved auction game, optimal auctions are no longer an equilibrium. In the meantime, there is always a "babbling" equilibrium where buyers randomize over all bids and the seller randomly randomly selects a buyer to offer the monopoly price regardless of the bid profile. We investigate equilibria where bidding is informative of buyer values and the seller responds with different offers.

We start with equilibria where the seller's auction maps bid profiles to a finite number

[1] Athey and Ellison (2011) study position auctions that are a better model of Google Ads auctions than the simple setting studied here, but as in standard auction models, they assume that bidders know the auction rule.

of fixed offers. Since buyers do not observe the mapping, the seller has to randomize over the offers after selecting a buyer. We show that each offer corresponds to a bid, and the lowest is equal to the monopoly price. A buyer with any value above the monopoly price randomizes over the highest bid below his value and all higher bids, in such a way that upon selecting any bid the seller is indifferent among all offers less than or equal to the bid. The seller in turn randomizes over the offers in such a away that any buyer with a value above the monopoly price is indifferent between the highest bid below his value and all higher bids. We show that this requires the seller's maximum revenue upon selecting a bid to be weakly increasing in the bid.

In any equilibrium with any finite number of fixed offers, the seller's revenue depends only on how bidders with values below the monopoly price randomize among all bids. When they randomize independently of their values, the seller is indifferent among all bids, and the equilibrium revenue is minimized and is equal to the monopoly revenue. Even though the equilibrium bidding strategy of buyers with values above the monopoly price is informative, the seller does not gain from making different offers. In any equilibrium where the seller's maximum revenue upon selecting the highest bid is strictly greater than that upon selecting the lowest bid, the seller gets strictly more than the monopoly revenue. The equilibrium revenue is maximized, when bids placed by buyers with valuations below the monopoly price generate a first order stochastic dominated distribution of the highest bid. This is achieved by iteratively maximize the ex ante probabilities of buyers making lower bids, starting from the lowest bid, subject to the equilibrium condition that the seller's revenue upon selecting each bid is maximized by the monopoly price.

For a seller who can make up to a fixed number of offers, we characterize how the offers are positioned in the most profitable equilibrium. The revenue-maximizing offers are all distinct, and interior to the support of valuations. An increase in the number of offers that can be made by the seller strictly increases the revenue in the most profitable equilibrium. We construct an equilibrium that attains the limit of the seller's revenue in the most profitable equilibrium as the number of offers goes to infinity. In this equilibrium, buyers with any value above the monopoly price place a mass point on the bid equal to their value and continuously randomize over all higher bids. Buyers with any value below the monopoly price continuously randomize over all bids between the monopoly price and a maximum bid that is a strictly increasing function of their value. Upon selecting any bid, the seller is indifferent among all offers between the lowest value of buyer that makes the bid and the highest value that makes the bid, an interval that includes the monopoly price, and in equilibrium randomizes among all these bids in such a way that the offer distributions of any two bids are identical for all offers between the monopoly price and the lower bid. The distribution for the highest bid is first order stochastically dominated, and the seller's revenue is maximized among all equilibria.

This paper is organized as follows. Section 2 introduces the unobserved auction game. In Section 3, we consider equilibria with a finite number of offers. We first show that there is one-to-one correspondence between equilibrium bids and the offers, so that each bid generates the corresponding offer and all lower offers, and a higher bid generates the same offers as a lower bid with the same probabilities. The lowest offer is shown to be equal to the monopoly price. After the equilibrium conditions on bidding by buyers with values above the monopoly price are imposed, the revenue of the seller depends only on bidding by buyers with values below the monopoly price. We show that the seller's revenue is weakly increasing in the selected bid. The equilibrium revenue of the seller is minimized and is equal to the monopoly revenue when buyers with values below the monopoly price bid independently of their values. It is maximized when bidding by buyers with values below the monopoly price makes the distribution of the highest bid among buyers first order stochastically dominated. The maximal revenue increases with the number of offers. In Section 4, we construct an equilibrium with a continuum of offers that achieves the limit of finite offers as the number of offers goes to infinity. The seller makes all offers greater than or equal to the monopoly price. Correspondingly, buyers with values above the monopoly price place a mass point on the bid equal to their value, and continuously mix among all higher bids. Buyers with values below the monopoly price continuously randomize between the monopoly price and a maximum bid, with the maximum bid increasing in the value. We show that it is the most profitable equilibrium. Section 5 concludes with some remarks on our research agenda.

2 Model

A seller has an indivisible good with zero reservation value. There are $n \geq 2$ potential buyers, with their values independently drawn from the same distribution $F(\cdot)$ over $[0,1]$, with a continuous and positive density $f(\cdot)$. We assume that the monopoly revenue function

$$\pi(p) = p(1 - F(p))$$

is strictly concave. Let p^* be the unique maximizer of $\pi(p)$, and denote $\pi(p^*)$ as π^*.

Consider the following game. The seller first commits to a symmetric auction, to be described in detail below. After learning their values, all buyers simultaneously each submit a bid without observing the auction.[2] The allocation and offer rules of the auction are implemented. A buyer, say i, who receives the offer p decides either to accept it, with a payoff of $v_i - p$ where v_i is his value, or reject it, with a payoff of 0. The seller's realized payoff is p if the buyer accepts it, and 0 otherwise.

We model a symmetric auction as a mapping from a profile of n bids $b = (b_1, \ldots, b_n)$ to n pairs of functions $(q_i(b), G_i(p,b))_{i=1}^n$, where $q_i(b)$ is the probability that bidder i wins the object, and $G_i(p,b)$ represents the probability that the offer to the winner i is smaller than or equal to p. Each bid b_i is assumed to be a number between 0 and 1. Symmetry requires $q_i(b_i, b_{-i})$ and $G_i(b_i, b_{-i})$ to be invariant to permutations of b_{-i} for all i and $b = (b_i, b_{-i})$. Feasibility requires

$$\sum_{i=1}^{n} q_i(b) \leq 1$$

for all b.

A strategy of each bidder i is a mapping $H(\cdot, v_i)$ from the support $[0,1]$ of his value v_i to a distribution over bids b_i in $[0,1]$. That is, $H(b_i, v_i)$ gives the probability that the bid by bidder i with value v_i is smaller than or equal to b_i. We assume that the ex ante probability

[2] We assume that all bidders participate in the auction. Since the auction produces take-it-or-leave-it offers, the ex post individual rationality condition is satisfied for bidders.

that a bidder makes any bid b_i, given by

$$\int_0^1 h(b_i, v_i) f(v_i) \mathrm{d}v_i$$

is well defined, where $h(b_i, v_i)$ is the derivative of $H(\cdot, v_i)$ with respect to the first argument, whenever it exists.

Since buyers do not observe the seller's auction, the above game of unobserved auctions can be reduced to a simultaneous-move game played by the seller and the n unobservant buyers. A feasible, symmetric auction $(G_i(\cdot, b), q_i(b))_{i=1}^n$ of the seller and a buyer strategy $H(\cdot, v_i)$ form a symmetric Bayesian Nash equilibrium of the unobserved auction game, if

- for all $b = (b_1, \ldots, b_n)$ such that for some i, $b_i \notin \mathrm{supp}H(\cdot, v_i)$ for any v_i, $q_j(b) = 0$ for each $j = 1, \ldots, n$;
- for each v_i and each $b_i \in \mathrm{supp}H(\cdot, v_i)$,

$$\begin{aligned} &\mathbb{E}_{v_{-i}}\left[\int_p \max\{v_i - p, 0\} q_i(b_i, b_{-i}) \mathrm{d}G_i(p, b_i, b_{-i})\right] \\ \geq &\mathbb{E}_{v_{-i}}\left[\int_p \max\{v_i - p, 0\} q_i(b_i, b_{-i}) \mathrm{d}G_i(p, \tilde{b}_i, b_{-i})\right] \end{aligned} \tag{1}$$

  for all $\tilde{b}_i \in [0, 1]$, where b_{-i} is the profile of bids by bidders other than i with each b_j, $j \neq i$, distributed according to $H(b_j, v_j)$ for each value v_j;
- for each profile b of bids, where for each $j = 1, \ldots, n$, $b_j \in \mathrm{supp}H(\cdot, v_j)$ for some v_j, $q_i(b) > 0$ implies

$$\frac{p \int_p^1 h(b_i, v_i) f(v_i) \mathrm{d}v_i}{\int_0^1 h(b_i, v_i) f(v_i) \mathrm{d}v_i} \geq \frac{\tilde{p} \int_{\tilde{p}}^1 h(b_j, v_i) f(v_j) \mathrm{d}v_j}{\int_0^1 h(b_j, v_j) f(v_j) \mathrm{d}v_j} \tag{2}$$

  for all $p \in \mathrm{supp}G_i(\cdot, b)$, all j, and all $\tilde{p} \in [0, 1]$.

The seller's auction is defined for all possible profiles of bids b. The first equilibrium condition above requires the seller not to make any offer if any bidder i makes an "out-of-equilibrium" bid b_i not in the support of $H(\cdot, v_i)$ for any v_i. Condition (1) requires each bidder i with any value v_i makes an optimal bid. In particular, if bidder i randomizes over his bids in equilibrium, he is indifferent among all bids he uses. Condition (2) is a combination of two requirements. First, if $q_i(b)$ is positive, i.e., b_i is selected with a positive probability, then conditional on the selection, the seller's offer p is optimal. This is ensured by setting $j = i$ on the right-hand side of (2). In particular, if the seller randomizes over offers upon selecting b_i, the seller is indifferent among all bids in the support of $G_i(\cdot, b_i, b_{-i})$. Second, the seller cannot do strictly better by selecting a different bid b_j.

3 Finite-Offer Equilibria

In this section we restrict to equilibria $((G_i(\cdot, b), q_i(b))_{i=1}^n, H(\cdot, v_i))$ where a finite number of offers are made by the seller. That is, we assume that the union of the supports of $G_i(\cdot, b)$ over all i and all profiles of bids b is finite. We use $g_i(p, b)$ to denote the probability that each p in the support of $G_i(\cdot, b)$ is offered. We will later use the characterization of finite-offer equilibria to ultimately construct an equilibrium where the seller makes a continuum of offers.

3.1 Equilibrium offer distributions

We say that a bid m by any bidder i generates an offer t if the probability χ, defined below, is positive:

$$\chi = \mathbb{E}_{v_{-i}} \left[q_i(m, b_{-i}) g_i(t, m, b_{-i}) \right],$$

where b_{-i} is the profile of bids by bidders other than i with each b_j, $j \neq i$, distributed according to $H(b_j, v_j)$ for each value v_j. We assume that each equilibrium offer t is "serious," in that there exists a bid m that generates t, such that $m \in \text{supp}H(\cdot, w)$ for a positive measure of values $w \geq t$. Any non-serious offer can be removed from the self of equilibrium offers without any payoff implication to the seller or the bidders.

First, we show that the distributions of offers generated in an equilibrium are linearly ordered by their supports. In particular, for any two distinct offer distributions corresponding to two equilibrium bids, the support of one distribution is a truncation of the support of the other distribution from above. We refer to the former as the lower distribution generated by the lower bid, and the latter as the higher distribution generated by the higher bid. Further, the two distributions agree on the support of the lower distribution in that the lower bid generates each offer in the support with the same probability as the higher bid. That is, any two equilibrium bids must generate the same low offers with the same probabilities, so that the two offer distributions can differ only because the higher bid generates highe offers that are generated with probability zero by the lower bid.[3]

Lemma 1 *Suppose that in some equilibrium a bid m generates offers $t_1 < \ldots < t_l$ with probabilities $\chi_1, \ldots, \chi_l$, and another bid $\tilde{m}$ generates offers $\tilde{t}_1 < \ldots < \tilde{t}_{\tilde{l}}$ with probabilities $\tilde{\chi}_1, \ldots, \tilde{\chi}_{\tilde{l}}$. Then, $t_k = \tilde{t}_k$ and $\chi_k = \tilde{\chi}_k$ for each $j = 1, \ldots, \min\{l, \tilde{l}\}$.*

Proof. Consider the lowest offer t_1 generated by m. We claim that

$$\inf \{w : m \in \text{supp}H(\cdot, w) \text{ and } w \geq t_1\} = t_1.$$

If this were false, the seller could raise the revenue by increasing his price offer t_1 after selecting bid m, contradicting equilibrium condition (2). A similar result holds for $\tilde{m}$.

Now, if $t_1 > \tilde{t}_1$, then since t_1 is the lowest offer generated by m, any bidder i with value t_1 receives an expected payoff of zero by bidding m. However, by deviating and bidding $\tilde{m}$, the bidder would receive a strictly positive payoff of $\tilde{\chi}_1(t_1 - \tilde{t}_1)$. This contradicts the equilibrium condition (1). A symmetric argument rules out $t_1 < \tilde{t}_1$. Thus, $t_1 = \tilde{t}_1$.

Next, if $\chi_1 < \tilde{\chi}_1$, then given $t_1 = \tilde{t}_1$, a bidder i with any value strictly between t_1 and $\min\{t_2, \tilde{t}_2\}$ strictly prefers $\tilde{m}$ to m, contradicting the result that the infimum of the values above t_1 for which buyers bid m is t_1. A symmetric argument rules out $\chi_1 > \tilde{\chi}_1$. Thus, $\chi_1 = \tilde{\chi}_1$.

The lemma then follows from induction. ■

By Lemma 1, we can order distinct equilibrium bids in increasing order as follows. Each bid m_{k+1} generates at least 1 more offer than m_k, and the former generates all offers generated

[3] Lemma 1 does not depend on the assumption that the revenue function $\pi(\cdot)$ is concave, while Lemma 2 does. Thus, without the concavity assumption, there is still a linear order of equilibrium offer distributions, but the number of distributions may be smaller than the number of offers.

by the latter, with the same probabilities. There is no equilibrium bid m_0 that generates no offers; otherwise, not making an offer when all bidders bid m_0 violates equilibrium condition (2) for the seller. Our next characterization result uses the assumption that $\pi(\cdot)$ is strictly concave to show that m_{k+1} generates exactly 1 more offer than m_k. Thus, in an equilibrium with l offers, there are exactly l bids that generate l offer distributions.

Lemma 2 *In any equilibrium where the seller makes l offers $t_1 < \ldots < t_l$, there are l distinct equilibrium bids, $m_1, \ldots, m_l$, such that for each $k = 1, \ldots, l$, bids $m_k, \ldots, m_l$ generate offer t_k with probabilities χ_k.*

Proof. Fix an equilibrium where the seller makes l offers $t_1 < \ldots < t_l$. Suppose that for some k, all bids m_k and higher generate offers $t_k < t_{k+1}$, with probabilities χ_k and χ_{k+1} respectively, while bids m_{k-1} and lower do not generate either offer. Modify the auction by marginally increasing t_k and marginally decreasing t_{k+1} for all bids m_k and higher, so that

$$\chi_k \mathrm{d}t_k + \chi_{k+1}\mathrm{d}t_{k+1} = 0.$$

The effect of the modifications on the seller's expected revenue from making an offer to any bid m_j, $j = k, \ldots, l$, is given by

$$\begin{aligned} & n\chi_k \mathrm{d}t_k \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(t\int_t^1 h(m_j,w)f(w)\mathrm{d}w\right)\Big|_{t=t_k} + n\chi_{k+1}\mathrm{d}t_{k+1} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(t\int_t^1 h(m_j,w)f(w)\mathrm{d}w\right)\Big|_{t=t_{k+1}} \\ =& n\chi_k \mathrm{d}t_k \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(t\int_t^1 h(m_j,w)f(w)\mathrm{d}w\right)\Big|_{t=t_k} - \frac{\mathrm{d}}{\mathrm{d}t}\left(t\int_t^1 h(m_j,w)f(w)\mathrm{d}w\right)\Big|_{t=t_{k+1}}\right). \end{aligned}$$

Since t_k and t_{k+1} are offered only to bids m_j, $j = k, \ldots, l$, and since by Lemma 1 each message m_j generates all offers lower than t_k with the same probability as each message lower than m_k, bidders with values above t_k strictly prefer m_j to any message lower than m_k. Thus,

$$\sum_{j=k}^{l} h(m_j, w) = 1$$

for all $w > t_k$. With the same probability χ_k for all $j = k, \ldots, l$, the total effect on the seller's expected revenue is given by

$$n\chi_k \mathrm{d}t_k \cdot \left(\pi'(t_k) - \pi'(t_{k+1})\right).$$

Since $\mathrm{d}t_k > 0$, the concavity of $\pi(\cdot)$ implies that the above is strictly positive. This contradicts the equilibrium condition (2). The lemma follows immediately. ■

From now on, in an equilibrium with offers $t_1 < \ldots < t_l$, we refer to the bid m_k, $k = 1, \ldots, l$, that generates offers $t_1, \ldots, t_k$, simply as t_k, the highest offer that it generates. Let $\emptyset$ denote the outcome of no offer is made. The l equilibrium offer distributions are given by l positive probabilities, $\chi_1, \ldots, \chi_l$ that sum up to less than or equal to 1, such that for each $k = 1, \ldots, l$, the offer generated by bid t_k is a discrete random variable with the probability function $\left(1 - \sum_{j=1}^{k} \chi_j, \chi_1, \ldots, \chi_k, 0, \ldots, 0\right)$ over the support $\{\emptyset, t_1, \ldots, t_l\}$.

3.2 Equilibrium conditions

Using Lemma 1 and Lemma 2, we can provide necessary and sufficient conditions for an equilibrium with l offers. We first introduce more convenient notation for bidding strategies and for the seller's auction.

For bidders, for all $w \in [0,1]$, let $h_k(w)$ be the probability that a bidder with value w submits bid t_k, $k = 1, \ldots, l$, with $\sum_{k=1}^{l} h_k(w) = 1$. Let Φ_k be the ex ante probability that a bidder submits t_k, given by

$$\Phi_k = \int_0^1 h_k(w) f(w) \mathrm{d}w,$$

with $\sum_{k=1}^{l} \Phi_k = 1$.

Conditional on selecting a bid t_k to make an offer to, the seller's revenue function is

$$R_k(p) = \frac{p}{\Phi_k} \int_p^1 h_k(w) f(w) \mathrm{d}w.$$

For any profile of bids $(b_i, \ldots, b_n)$, for each $k = 1, \ldots, n$ let θ_k be the cardinality of the set $\{i : b_i = t_k\}$, and denote the profile of bids as $\theta = (\theta_1, \ldots, \theta_l)$, with $\sum_{k=1}^{l} \theta_k = n$. Given the strategy of bidders $\{h_k(w)\}_{k=1}^{l}$, the random variable θ has a multinomial distribution, with the probability given by

$$\frac{n!}{\theta_1! \times \cdots \times \theta_l!} \Phi_1^{\theta_1} \times \cdots \times \Phi_l^{\theta_l}.$$

Let θ_{-1} be the multinomial random variable induced by $n-1$ bidders independently using the bidding strategy.

The seller's auction is represented as follows. Let $q_k(\theta)$ be the probability that a given bidder with bid t_k is selected for an offer; by symmetry, the probability that the offer goes to some bidder with bid t_k is $q_k(\theta)\theta_k$. We have $\sum_{k=1}^{l} q_k(\theta)\theta_k = 1$. Conditional on a bidder with bid t_k is selected, let $g_{j,k}(\theta)$ be the probability the offer is equal to t_j, with $\sum_{j=1}^{l} g_{j,k}(\theta) = 1$. Then, the ex ante probability $\chi_{j,k}$ that bid t_k generates t_j, $j = 1, \ldots, l$, is given by

$$\chi_{j,k} = \sum_{\{\theta | \theta_1 + \cdots + \theta_k + \cdots + \theta_l = n-1\}} \frac{(n-1)!}{\theta_1! \times \cdots \times \theta_k! \times \cdots \times \theta_l!} \Phi_1^{\theta_1} \times \cdots \times \Phi_k^{\theta_k} \times \cdots \times \Phi_l^{\theta_l}$$
$$\cdot\, q_k(\theta_1, \ldots, \theta_k + 1, \ldots, \theta_l) g_{j,k}(\theta_1, \ldots, \theta_k + 1, \ldots, \theta_l).$$

Now we give a set of necessary and sufficient conditions for an equilibrium with l offers to replace (1) and (2).

Lemma 3 *For any $t_1 < \ldots < t_l$, buyers' bidding strategy $\{h_k(w)\}_{k=1}^{l}$ and seller's auction $\{q_k(\theta), \{g_{j,k}(\theta)\}_{j=1}^{l}\}_{k=1}^{l}$ form an equilibrium, if only if for $k = 1, \ldots, l$, (i) $h_j(w) = 0$ for all $j = 1, \ldots, k-1$ and $w > t_k$, (ii) $t_j \in \arg\max_p R_k(p)$ for each $j = 1, \ldots, k$, (iii) $R_k(t_1) \geq R_j(t_1)$ for all θ such that $q_k(\theta) > 0$ and $\theta_j > 0$, (iv) $g_{j,k}(\theta) = 0$ for all $j = k+1, \ldots, l$ and all θ such that $q_k(\theta) > 0$, and (v) $\chi_{k,j} = \chi_{k,k}$ for $j = k+1, \ldots, l$.*

Proof. For necessity of conditions (i)-(v), observe the following. Condition (i) requires that bidders with value w strictly higher than t_k strictly prefer bid t_k and higher to any bid t_{k-1} and lower. This follows from Lemma 1, as bid t_k and higher generate offers $t_1, \ldots, t_{k-1}$ with the same probabilities $\chi_1, \ldots, \chi_{k-1}$ as bid t_{k-1} and lower, but in addition generates offer t_k with a positive probability. Condition (ii) requires that the bidding strategy $\{h_k(w)\}_{k=1}^{l}$ is such that each t_j, $j = 1, \ldots, k$ is a maximizer of the condition revenue function $R_k(p)$. This follows from Lemma 2. Given condition (ii), condition (iii) requires that the seller's auction not to select bid t_k if it is more profitable to select bid t_j instead. This follows from equilibrium condition (2). Condition (iv) requires the seller's auction to not to make offer t_{k+1} and higher after selecting bid t_k. This follows from Lemma 2, as each bid t_k generates exactly k offers $t_1, \ldots, t_k$. Condition (v) requires that bid t_k and higher generate offer t_k with the same probability. This follows from Lemma 1.

For sufficiency, observe first that condition (iv) implies that $\chi_{j,k} = 0$ for all $k = 1, \ldots, l-1$ and $j = k+1, \ldots, l$. Together with condition (iv), we have that a bidder is indifferent among all bids if his value $w \leq t_1$, and is indifferent among all bids t_k and higher and strictly prefers any such bid to all bids t_{k-1} and lower, if his value $w \in (t_k, t_{k+1}]$ for each $k = 1, \ldots, l$ (with $t_{l+1} = 1$). Thus, condition (i) ensures that equilibrium condition (1) is satisfied for all value $w \in [0, 1]$. Next, conditions (ii) implies that conditional on selecting any bid t_k, $k = 1, \ldots, l$, the seller is indifferent among all offers $t_1, \ldots, t_k$. Together with condition (iii), it ensures that equilibrium condition (2) is satisfied. The sufficiency of conditions (i)-(v) follows immediately. ■

The five conditions together are suggestive of how we may construct equilibria of our unobserved auction game. Condition (i) says that bidders mix "upward" by bidding above the highest bid above their value, while condition (iv) says that the seller mix "downward" after selecting a bid. Further, conditions (ii) and (iii) require bidders to miix in such a way to ensure that each conditional revenue function $R_k(\cdot)$ for each $k = 1, \ldots, l$ peaks at each $t_1, \ldots, t_k$, and that a bid t_k is selected only when there is no other bid with a higher peak. At the same time, condition (v) requires the seller to mix in such a way that each bid t_k to generate all offers $t_1, \ldots, t_k$ with the same probabilities as higher bids $t_{k+1}, \ldots, t_l$ do.

We are primarily interested in comparing the seller's revenue across different equilibria. We do so by characterizing equilibrium bidding and equilibrium auction. Before we begin, we establish the first result that the lowest offer t_1 is equal to p^*. This is not surprising. By condition (ii) of Lemma 3, the lowest offer t_1 is a maximizer of each conditional revenue function $R_k(\cdot)$, $k = 1, \ldots, l$. Since in equilibrium the seller makes an offer with probability one, the lowest offer is equal to the monopoly price p^*.[4]

Proposition 1 *In any equilibrium with l offers $t_1, \ldots, t_l$, the lowest offer $t_1 = p^*$.*

[4]For notational convenience we will often continue to write t_1 instead of p^* for the lowest equilibrium offer.

Proof. Fix an equilibrium with l offers $t_1 < \ldots < t_l$. By condition (ii) of Lemma 3,

$$t_1 \in \arg\max_p R_k(p).$$

Thus,

$$t_1 \in \arg\max_p \sum_{k=1}^{l} p \int_p^1 h_k(w) f(w) \mathrm{d}w.$$

Since $\sum_{k=1}^{l} h_k(w) = 1$ for all w, we have

$$t_1 \in \arg\max_p \pi(p),$$

implying that $t_1 = p^*$. ■

An implication of Proposition 1 is

$$\pi^* = \sum_{k=1}^{l} R_k(p^*) \Phi_k. \tag{3}$$

That is, the weighted average of equilibrium revenues associated with the l bids is π^*, with the weight equal to the probability that bidders use each bid.

3.3 Equilibrium bidding

First, we use an induction argument to restrict any equilibrium bidding strategy $\{h_k(w)\}_{k=1}^{l}$ for all $w > t_1$, starting from $k = l$.

Lemma 4 *In any equilibrium with offers $t_1 < \ldots < t_l$, the buyers' bidding strategy $\{h_k(w)\}_{k=1}^{l}$ satisfies, for each $k = 1, \ldots, l$, with $t_{l+1} = 1$,*

$$t_j \int_{t_j}^1 h_k(w) f(w) \mathrm{d}w = \pi(t_k) - \pi(t_{k+1}),$$

for any $j = 1, \ldots, k$.

Proof. By condition (i) of Lemma 3, we have $h_l(w) = 1$ for all $w > t_l$, and therefore

$$t_l \int_{t_l}^{1} h_l(w) f(w) \mathrm{d}w = \pi(t_l).$$

By condition (ii) of Lemma 3,

$$t_j \int_{t_j}^{1} h_l(w) f(w) \mathrm{d}w = \pi(t_l),$$

for all $j = 1, \ldots, l-1$. Thus, the condition stated in the lemma holds for $k = l$.

Suppose that the condition holds for $k, \ldots, l$. By condition (i) of Lemma 3, we have $h_{k-1}(w) = 1 - \sum_{j=k}^{l} h_j(w)$ for all $w > t_{k-1}$, and therefore

$$t_{k-1} \int_{t_{k-1}}^{1} h_{k-1}(w) f(w) \mathrm{d}w = t_{k-1}(1 - F(t_{k-1})) - \sum_{j=k}^{l} t_{k-1} \int_{t_{k-1}}^{1} h_j(w) f(w) \mathrm{d}w.$$

Since the condition holds for all $k, \ldots, l$, we have

$$\sum_{j=k}^{l} t_{k-1} \int_{t_{k-1}}^{1} h_j(w) f(w) \mathrm{d}w = \sum_{j=k}^{l} (\pi(t_j) - \pi(t_{j+1})) = \pi(t_k).$$

Thus,

$$t_{k-1} \int_{t_{k-1}}^{1} h_{k-1}(w) f(w) \mathrm{d}w = \pi(t_{k-1}) - \pi(t_k).$$

By condition (ii) of Lemma 3,

$$t_j \int_{t_j}^{1} h_{k-1}(w) f(w) \mathrm{d}w = \pi(t_{k-1}) - \pi(t_k),$$

for all $j = 1, \ldots, k-1$. Thus, the condition stated in the lemma holds for $k-1$. ■

By Proposition 1, $t_1 = p^*$. Since the equation in Lemma 4 holds at t_1 for each $k = 1, \ldots, l$, we have

$$R_k(p^*)\Phi_k = \pi(t_k) - \pi(t_{k+1}), \tag{4}$$

with $t_{l+1} = 1$. By conditions (iv) and (v) of Lemma 3, the expected revenue of the seller in any equilibrium $\{q_k(\theta), \{g_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$ is given by

$$\pi = n \sum_{k=1}^{l} \Phi_k \sum_{j=1}^{k} \chi_{j,j} R_k(t_j).$$

By condition (ii) of Lemma 3, for any $k = 1, \ldots, l$,

$$R_k(t_j) = R_k(t_1)$$

for all $j = 2, \ldots, k$. By (4), the seller's revenue π in any equilibrium is given by

$$\pi = n \sum_{k=1}^{l} (\pi(t_k) - \pi(t_{k+1})) \sum_{j=1}^{k} \chi_{j,j}. \tag{5}$$

For given equilibrium offers $t_1 < \ldots < t_l$, the seller's revenue in any equilibrium depends only on the sequence of offer generating probabilities $\left\{ \sum_{j=1}^{k} \chi_{j,j} \right\}_{k=1}^{l}$.

Lemma 4 does not pin down the bidding strategy point-wise for values $w > t_1$. The conditions in the lemma are necessary for (i) and (ii) in Lemma 3. Now we construct a bidding strategy with piece-wise constant mixing $\{\hat{h}_k(w)\}_{k=1}^l$ for $w > t_1$ that not only satisfies Lemma 4 but are also sufficient for (i) and (ii) in Lemma 3.

For each $k = 1, \ldots, l$, by the concavity of $\pi(\cdot)$ we can uniquely define $z_k \in (t_k, t_{k+1})$ (we set $t_{l+1} = 1$) such that

$$(1 - F(z_k))t_{k+1} = \pi(t_k).$$

For each $k = 1, \ldots, l$, define

$$\hat{h}_k(w) = \begin{cases} (\pi(t_k) - \pi(t_{k+1}))/\pi(t_j) & \text{if } w \in (t_j, z_j], j = 1, \ldots, k \\ 0 & \text{if } w \in (z_j, t_{j+1}], j = 1, \ldots, k-1 \\ 1 & \text{if } w \in (z_k, t_{k+1}] \\ 0 & \text{if } w > t_{k+1} \end{cases}$$

By construction, for any $w \in (t_j, z_j]$, $j = 1, \ldots, l$, we have

$$\sum_{k=1}^{l} \hat{h}_k(w) = \sum_{k=j}^{l} \frac{\pi(t_k) - \pi(t_{k+1})}{\pi(t_j)} = \frac{\pi(t_j) - \pi(t_{l+1})}{\pi(t_j)} = 1.$$

That is, bidders with values on each interval $(t_j, z_j]$ mix with constant probabilities to bids t_j and higher. For any $w \in (z_j, t_{j+1}]$, $j = 1, \ldots, l$, we have $\hat{h}_j(w) = 1$, so bidders with values on each interval $(z_j, t_{j+1}]$ send message t_j with probability 1.

By construction, for each $k = 1, \ldots, l$, and each $j = 1, \ldots, k$, we have

$$\begin{aligned}\int_{t_j}^{1} \hat{h}_k(w) f(w) \mathrm{d}w &= \sum_{\tilde{j}=j}^{k} (F(z_{\tilde{j}}) - F(t_{\tilde{j}})) \frac{\pi(t_k) - \pi(t_{k+1})}{\pi(t_{\tilde{j}})} + F(t_{k+1}) - F(z_k) \\ &= \left(\frac{1}{t_j} - \frac{1}{t_{k+1}} \right) (\pi(t_k) - \pi(t_{k+1}) + \frac{\pi(t_k)}{t_{k+1}} - \frac{\pi(t_{k+1})}{t_{k+1}},\end{aligned}$$

where the second equality follows from the definition of $z_{\tilde{j}}$, $\tilde{j} = j, \ldots, k$. Thus, Lemma 4 is satisfied.

By construction, $\{\hat{h}_k(w)\}_{k=1}^{l}$ satisfies condition (i) in Lemma 3. For condition (ii), we establish the following result.

Lemma 5 *If the bidding strategy is given by* $\{\hat{h}_k(w)\}_{k=1}^{l}$ *for* $w > p^*$*, then for each* $k = 1, \ldots, l$*, we have* $t_j \in \arg\max_{p \geq t_1} R_k(p)$ *for each* $j = 1, \ldots, k$*.*

Proof. It suffices to show that

$$t_j \in \arg\max_{p \geq t_1} \; p \int_{p}^{1} \hat{h}_k(w) f(w) \mathrm{d}w$$

for each $k = 1, \ldots, l$, and for each $j = 1, \ldots, k$.

First, consider any $p \in [t_j, z_j]$ for any $j = 1, \ldots, k-1$. Since $\hat{h}_k(w) = 0$ for any $w \in (z_j, t_{j+1}]$, we have

$$\begin{aligned}\int_{p}^{1} \hat{h}_k(w) f(w) \mathrm{d}w &= \int_{p}^{z_j} \hat{h}_k(w) f(w) \mathrm{d}w + \int_{t_{j+1}}^{1} \hat{h}_k(w) f(w) \mathrm{d}w \\ &= \frac{\pi(t_k) - \pi(t_{k+1})}{\pi(t_j)} (F(z_j) - F(p)) + \frac{\pi(t_k) - \pi(t_{k+1})}{t_{j+1}},\end{aligned}$$

where the second equality follows because $\{\hat{h}_k(w)\}_{k=1}^{l}$ for $w > t_1$ satisfies Lemma 4. Then, using the definition of z_j we have

$$p \int_p^1 \hat{h}_k(w) f(w) \mathrm{d}w = (\pi(t_k) - \pi(t_{k+1})) \frac{\pi(p)}{\pi(t_j)},$$

and therefore

$$\max_{p \in [t_j, z_j]} p \int_p^1 \hat{h}_k(w) f(w) \mathrm{d}w = \pi(t_k) - \pi(t_{k+1}).$$

Second, consider any $p \in [z_j, t_{j+1}]$ for all $j = 1, \ldots, k-1$. Since $\hat{h}_k(w) = 0$ for all $w \in [z_j, t_{j+1}]$, we have

$$\max_{p \in [z_j, t_{j+1}]} p \int_p^1 \hat{h}_k(w) f(w) \mathrm{d}w = t_{j+1} \int_{t_{j+1}}^1 \hat{h}_k(w) f(w) \mathrm{d}w = \pi(t_k) - \pi(t_{k+1}),$$

where the second equality follows because $\{\hat{h}_k(w)\}_{k=1}^{l}$ for $w > t_1$ satisfies Lemma 4.

Third, consider any $p \in [t_k, z_k]$. We have

$$\begin{aligned} \int_p^1 \hat{h}_k(w) f(w) \mathrm{d}w &= \int_p^{z_k} \hat{h}_k(w) f(w) \mathrm{d}w + \int_{z_k}^{t_{k+1}} \hat{h}_k(w) f(w) \mathrm{d}w \\ &= \frac{\pi(t_k) - \pi(t_{k+1})}{\pi(t_k)} (F(z_k) - F(p)) + F(t_{k+1}) - F(z_k). \end{aligned}$$

Using the definition of z_k, we have

$$p \int_p^1 \hat{h}_k(w) f(w) \mathrm{d}w = (\pi(t_k) - \pi(t_{k+1})) \frac{\pi(p)}{\pi(t_k)},$$

and therefore

$$\max_{p \in [t_k, z_k]} p \int_p^1 \hat{h}_k(w) f(w) \mathrm{d}w = \pi(t_k) - \pi(t_{k+1}).$$

Fourth, consider any $p \in [z_k, t_{k+1}]$. We have

$$\int_p^1 \hat{h}_k(w) f(w) \mathrm{d}w = \int_p^{t_{k+1}} \hat{h}_k(w) f(w) \mathrm{d}w = F(t_{k+1}) - F(p).$$

Since $p > t_1 = p^*$, the strict concavity of $\pi(\cdot)$ implies that that

$$\max_{p\in[z_k,t_{k+1}]} p\int_p^1 \hat{h}_k(w)f(w)\mathrm{d}w = z_k(F(t_{k+1}) - F(z_k))$$

Fifth, consider any $p \geq t_{k+1}$. Since $\hat{h}_k(w) = 0$ for all $w > t_{k+1}$, we have

$$p\int_p^1 \hat{h}_k(w)f(w)\mathrm{d}w = 0.$$

The second part of the lemma follows immediately by combining the above five cases. ■

For convenience, we make the following definition.

Definition 1 *A bidding strategy* $\{h_k(w)\}_{k=1}^l$ *is a feasible extension of* $\{\hat{h}_k(w)\}_{k=1}^l$*, if for all* $k = 1, \dots, l$*,* $h_k(w) = \hat{h}_k(w)$ *for* $w > t_1$*, and* $h_k(w) \geq 0$ *with* $\sum_{k=1}^l h_k(w) = 1$ *for* $w \leq t_1$*.*

Since $\{\hat{h}_k(w)\}_{k=1}^l$ satisfies Lemma 4, in any extension we have

$$\Phi_k = \int_0^{p^*} h_k(w)f(w)\mathrm{d}w + \frac{\pi(t_k) - \pi(t_{k+1})}{p^*}. \tag{6}$$

The above depends only on the extension $\{h_k(w)\}_{k=1}^l$ for $w \leq p^*$. The constructed bidding strategy $\{\hat{h}_k(w)\}_{k=1}^l$ for $w > t_1$ is not the only one that satisfies the equilibrium conditions (i) and (ii) in Lemma 3.[5] So long as these conditions hold, however, all constructions are irrelevant to the seller's equilibrium revenue. What Lemma 5 gives us is that, to characterize how the seller's equilibrium revenue depends on bidders' strategy, it is sufficient to focus on feasible extensions $\{h_k(w)\}_{k=1}^l$ for $w \leq p^*$ such that $t_1 \in \arg\max_{p\leq p^*} R_k(p)$ for each $k = 1, \dots, l$.

3.4 Monotone auction

By condition (ii) of Lemma 3, the seller's maximum revenue from bid t_k is given by $R_k(p^*)$. We first use conditions (iii), (iv) and (v) to show that in any equilibrium $R_k(p^*)$ is weakly increasing in kj.

[5] This is not true anymore when we consider equilibria where all offers in the interval $[p^*, 1]$ are made.

Lemma 6 *In any equilibrium with offers* $t_1 = p^* < t_2 < \ldots < t_l$, *we have* $R_k(p^*) \geq R_{k-1}(p^*)$ *for each* $k = 2, \ldots, l$.

The proof of the above lemma is in the appendix. The idea is quite simple. Imagine that $R_k(p^*)$ for some $k \geq 2$ is strictly lower than all other maximum conditional revenues $R_j(p^*)$, $j \neq k$. Then, condition (iii) of Lemma 3, $q_k(\theta) = 0$ unless $\theta_k = n$. By condition (v), $\chi_{j,k} = \chi_{j,j}$ for all $j = 1, \ldots, k-1$, while by condition (iv), $g_{j+1,j} = \ldots = g_{l,j} = 0$. The proof in the appendix shows that, since t_k is only selected when it is made by all bidders, conditions (iv) and (v) cannot be satisfied.

By Lemma 6, the seller's revenue in any equilibrium can be achieved by a "monotone auction" where the seller's selection strategy always picks out the highest bid.

Definition 2 *For any* $t_1 < \ldots < t_l$, $\{q_k(\theta), \{g_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$ *is a monotone auction if* $q_k(\theta) > 0$ *for any* $k = 1, \ldots, l$ *implies that* $\theta_j = 0$ *for all* $j = k+1, \ldots, l$.

We now show by construction that for any $\{\Phi_k\}_{k=1}^l$ satisfying $\Phi_k > 0$ and $\sum_{k=1}^l \Phi_k = 1$, there is a monotone auction $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$ that satisfies conditions (iv) and (v) in Lemma 3. Define $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$ recursively, starting from $k = 1$. Let $\hat{q}_1(\theta) = 0$ for all bid profile $\theta = (\theta_1, \ldots, \theta_l)$ such that $\theta_1 \leq n-1$; otherwise, $\hat{q}_1(\theta) = 1/n$ and $\hat{g}_{1,1}(\theta) = 1$. The probability $\hat{\chi}_{1,1}$ that t_1 generates t_1 is

$$\hat{\chi}_{1,1} = \frac{\Phi_1^{n-1}}{n}.$$

For each $k = 2, \ldots, l$, let $\hat{q}_k(\theta) = 0$ for all bid profile $\theta = (\theta_1, \ldots, \theta_l)$ such that $\sum_{j=1}^k \theta_j \leq n-1$. Otherwise, unless $\theta_k = 0$, let $\hat{q}_k(\theta) = 1/\theta_k$. For each θ such that $\hat{q}_k(\theta) > 0$, for each $j = 1, \ldots, k-1$, let $\hat{g}_{j,k}(\theta) = \hat{g}_{j,k}$ be such that

$$\frac{\hat{g}_{j,k}}{n\Phi_k} \left(\left(\sum_{j=1}^{k} \Phi_j \right)^n - \left(\sum_{j=1}^{k-1} \Phi_j \right)^n \right) = \hat{\chi}_{j,j}.$$

Let

$$\hat{g}_{k,k} = 1 - \sum_{j=1}^{k-1} \hat{g}_{j,k},$$

and define probability $\hat{\chi}_{k,k}$ that bid t_k generates offer t_k as

$$\hat{\chi}_{k,k} = \frac{\hat{g}_{k,k}}{n\Phi_k}\left(\left(\sum_{j=1}^{k}\Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right).$$

By construction, $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^{l}\}_{k=1}^{l}$, if feasible, satisfies conditions (iv) and (v) in Lemma 3. The following lemma verifies that it is feasible.

Lemma 7 *Fix any* $\{\Phi_k\}_{k=1}^{l}$ *such that* $\Phi_k > 0$, *and* $\sum_{k=1}^{l}\Phi_k = 1$. *The seller's auction* $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^{l}\}_{k=1}^{l}$ *satisfies conditions (iv) and (v) in Lemma 3.*

The proof of the above lemma is in the appendix. By construction, under the monotone auction $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^{l}\}_{k=1}^{l}$,

$$\sum_{j=1}^{k}\hat{\chi}_{j,j} = \frac{1}{n\Phi_k}\left(\left(\sum_{j=1}^{k}\Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right).$$

For any fixed $\{\Phi_k\}_{k=1}^{l}$, the auction $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^{l}\}_{k=1}^{l}$ constructed in the above lemma is not the unique monotone auction that satisfies conditions (iv) and (v) in Lemma 3, because we have made $g_{j,k}(\theta)$ for all k and $j \leq k$ independent of θ. However, for all such monotone auctions, the seller's revenue is the same. From (4) and (5), by Lemma 6 and Lemma 7, the seller's revenue in any equilibrium is given by

$$\pi = \sum_{k=1}^{l} R_k(p^*)\left(\left(\sum_{j=1}^{k}\Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right). \tag{7}$$

Compare (3) and (7). Since

$$\left(\sum_{j=1}^{k}\Phi_j\right)^n < \sum_{j=1}^{k}\Phi_j$$

for all $k = 1, \ldots, l-1$, the distribution of $\{R_k(p^*)\}_{k=1}^{l}$ in π according to (7) first order stochastic dominates the distribution of $\{R_k(p^*)\}_{k=1}^{l}$ in π^* according to (3). It follows that $\pi \geq \pi^*$ so long as $R_{k+1}(p^*) \geq R_k(p^*)$ for each $k = 1, \ldots, l-1$, with $\pi > \pi^*$ if at least one of the $l-1$ inequalities of the conditional revenue comparisons is strict.

3.5 Least and most profitable equilibria for given offers

Fix any offers $t_1 = p^* < t_2 < \ldots < t_l$. We construct an extension of the bidding strategy with piece-wise constant mixing $\{\hat{h}_k(w)\}_{k=1}^l$ for $w > t_1$ such that the equilibrium revenue is equal to π^*. For all $w \leq p^*$ and $k = 1, \ldots, l$, define

$$h_k^*(w) = \frac{\pi(t_k) - \pi(t_{k+1})}{\pi^*},$$

with $t_{l+1} = 1$. We have

$$\sum_{k=1}^{l} h_k^*(w) = \frac{\pi(t_1) - \pi(t_{l+1})}{\pi^*} = 1.$$

so $\{h_k^*(w)\}_{k=1}^l$ for $w \leq p^*$ is a feasible extension of $\{\hat{h}_k(w)\}_{k=1}^l$. Note that each $h_k^*(w)$ is independent of w. By (6), the ex ante probability that each bidder sending each message t_k, $k = 1, \ldots, l$, is given by

$$\Phi_k^* = \int_0^{p^*} h_k^*(w) f(w) \mathrm{d}w + \frac{\pi(t_k) - \pi(t_{k+1})}{p^*} = \frac{\pi(t_k) - \pi(t_{k+1})}{\pi^*}.$$

Proposition 2 *Fix any l offers $t_1 = p^* < t_2 < \ldots < t_l < 1$. The extension $\{h_k^*(w)\}_{k=1}^l$ to $\{\hat{h}_k(w)\}_{k=1}^l$ and the monotone auction $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$ form an equilibrium. Further, the equilibrium revenue is the lowest among all equilibria.*

Proof. By Lemma 7, to establish the first part of the proposition, it is sufficient to show that $\max_{p \leq t_1} R_k(p) = R_k(p^*)$ for each $k = 1, \ldots, l$, and $R_k(p^*)$ is weakly increasing in k.

For all $p \leq p^*$, we have

$$\begin{aligned} p \int_p^1 h_k^*(w) f(w) \mathrm{d}w &= p \left(\int_p^{p^*} h_k^*(w) f(w) \mathrm{d}w + \frac{\pi(t_k) - \pi(t_{k+1})}{p^*} \right) \\ &= p \left((F(p^*) - F(p)) \frac{\pi(t_k) - \pi(t_{k+1})}{\pi^*} + \frac{\pi(t_k) - \pi(t_{k+1})}{p^*} \right) \\ &= (\pi(t_k) - \pi(t_{k+1})) \frac{\pi(p)}{\pi^*}, \end{aligned}$$

where the first equality follows from Lemma 4. Thus,

$$\max_{p \le p^*} R_k(p) = R_k(p^*)$$

for each $k = 1, \ldots, l$. This implies

$$R_k(p^*) = \frac{p^* \int_{p^*}^1 \hat{h}_k(w) f(w) \mathrm{d}w}{\Phi_k^*} = \frac{\pi(t_k) - \pi(t_{k+1})}{\Phi_k^*} = \pi^*.$$

Thus, $R_k(p^*)$ is constant in k. It follows that the extension $\{h_k^*(w)\}_{k=1}^l$ to $\{\hat{h}_k(w)\}_{k=1}^l$ and the monotone auction $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$ form an equilibrium.

By (7), the equilibrium revenue $\pi = \pi^*$, because $R_k(p^*)$ is constant in k. ■

The idea of the construction of $\{h_k^*(w)\}_{k=1}^l$ is as follows. By Lemma 4, for each $k = 1, \ldots, l$ in any equilibrium

$$\int_{p^*}^1 h_k(w) f(w) = \frac{\pi(t_k) - \pi(t_{k+1})}{p^*}.$$

Thus, the contribution from values higher than p^* to each Φ_k, in terms of the share of the total measure $1 - F(p^*)$, is already pinned down. When the piece-wise constant mixing $\{\hat{h}_k(w)\}_{k=1}^l$ is adopted, by Lemma 5 we further have

$$\max_{p \ge p^*} R_k(p) = R_k(p^*).$$

The extension $\{h_k^*(w)\}_{k=1}^l$ assigns a share of the total measure $F(p^*)$ for values below p^* to each Φ_k in the same proportion as how the measure $1 - F(p^*)$ is assigned to Φ_k. This not only ensures that

$$\max_{p \le p^*} R_k(p) = R_k(p^*),$$

but also $R_k(p^*)$ is constant across k, because

$$R_k(p^*) = \frac{p^* \int_{p^*} h_k(w) f(w) \mathrm{d}w}{\int_0^1 h_k(w) f(w) \mathrm{d}w + \int_{p^*}^1 h_k(w) f(w) \mathrm{d}w}.$$

The same revenue of π^* can be obtained by a "babbling equilibrium," where bids are uninformative and the seller randomly selects a bidder to make an offer of p^*, or equivalently, with $l = 1$ and $t_1 = p^*$. Even though the equilibrium bidding strategy of bidders, given by the extension $\{h^*_k(w)\}_{k=1}^l$ to $\{\hat{h}_k(w)\}_{k=1}^l$, is informative of their values, the seller does not benefit from such informative communication.

Now we construct another extension of the bidding strategy with piece-wise constant mixing $\{\hat{h}_k(w)\}_{k=1}^l$ for $w > t_1$ that together with the monotone auction forms an equilibrium that yields an expected profit strictly greater than π^* to the seller. We will show that it is the most profitable equilibrium among all equilibria.

For each $k = 1, \ldots, l$, define y_k (we set $t_{l+1} = 1$) such that

$$y_k^2 f(y_k) = \pi^* - \pi(t_{k+1}).$$

Since

$$y^2 f(y) = \pi(y) - y\pi'(y)$$

the strict concavity of $\pi(\cdot)$ implies that $y^2 f(y)$ is strictly increasing in y, and thus y_k is uniquely defined, and is strictly increasing in k, with $y_l = p^*$.

Now, iteratively define an extension $\{\bar{h}_k(w)\}_{k=1}^l$ for $w \leq p^*$ to $\{\hat{h}_k(w)\}_{k=1}^l$ for $w > p^*$, starting from $k = 1$. For each $w \leq p^*$, let[6]

$$\bar{h}_1(w) = \begin{cases} (\pi^* - \pi(t_2))/(w^2 f(w)) & \text{if } w \in [y_1, p^*] \\ 1 & \text{if } w < y_1 \end{cases}$$

and for each $k = 2, \ldots, l$, let

$$\bar{h}_k(w) = \begin{cases} (\pi(t_k) - \pi(t_{k+1}))/(w^2 f(w)) & \text{if } w \in [y_k, p^*] \\ 1 - \sum_{j=1}^{k-1} \bar{h}_j(w) & \text{if } w \in [y_{k-1}, y_k) \\ 0 & \text{if } w < y_{k-1} \end{cases}$$

That is, for each $j = 1, \ldots, l$, bidders with values on the interval $[y_{j-1}, y_j)$ send each bid t_k,

[6] The construction of $\bar{h}_1(w)$ implies that upon selecting a bid t_k, $k = 1, \ldots, l$, buyer value w has a standard Pareto distribution over $[0, p^*]$. This ensures that the conditional revenue function is flat on the interval $[y_k, p^*]$. Pareto distributions have been used in constructions that arise from information design. See, e.g., Bergemann, Brooks and Morris (2015).

$k = 1, \ldots, j-1$, with probability $(\pi(t_k) - \pi(t_{k+1}))/(w^2 f(w))$, and send t_j with the remaining probability, with zero probability on all higher messages (we set $y_0 = 0$). In particular,

$$\sum_{k=1}^{l} \bar{h}_k(w) = \sum_{k=1}^{j} \bar{h}_k(w) = 1$$

for all $w \in [y_{j-1}, y_j)$, $j = 1, \ldots, l$.

By (6), we can calculate the ex ante mixing probabilities $\{\bar{\Phi}_k\}_{k=1}^{l}$ of bidders using messages $\{t_k\}_{k=1}^{l}$ with the extension $\{\bar{h}_k(w\}_{k=1}^{l}$. For each $k = 1, \ldots, l$, we have:

$$\begin{aligned}
\bar{\Phi}_k =& \int_{y_{k-1}}^{y_k} \left(1 - \frac{\pi^* - \pi(t_k)}{w^2 f(w)}\right) f(w)\mathrm{d}w + \int_{y_k}^{p^*} \frac{\pi(t_k) - \pi(t_{k+1})}{w^2 f(w)} f(w)\mathrm{d}w + \frac{\pi(t_k) - \pi(t_{k+1})}{p^*} \\
=& F(y_k) - F(y_{k-1}) - (\pi^* - \pi(t_k)) \left(\frac{1}{y_{k-1}} - \frac{1}{y_k}\right) + \frac{\pi(t_k) - \pi(t_{k+1})}{y_k} \\
=& F(y_k) + y_k f(y_k) - (F(y_{k-1}) + y_{k-1} f(y_{k-1})),
\end{aligned}$$

where the last equality uses the definitions of y_{k-1} and y_{k-1}, and we set $y_0 = 0$. Since

$$F(y) + yf(y) = 1 - \pi'(y),$$

we have $\bar{\Phi}_k > 0$ by $y_{k-1} < y_k \leq p^*$ and the strict concavity of $\pi(\cdot)$.

Proposition 3 *Fix any l offers $t_1 = p^* < t_2 < \ldots < t_l < 1$. The extension $\{\bar{h}_k(w)\}_{k=1}^{l}$ for $w \leq p^*$ to the bidding strategy with piece-wise constant mixing $\{\hat{h}_k(w)\}_{k=1}^{l}$ for $w > t_1$ and the monotone auction $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^{l}\}_{k=1}^{l}$ form an equilibrium. Further, the equilibrium revenue is the highest among all equilibria.*

Proof. By Lemma 7, to establish the first part of the proposition, it is sufficient to show that (i) $\max_{p \leq t_1} R_k(p) = R_k(p^*)$ for each $k = 1, \ldots, l$, and (ii) $R_k(p^*)$ is weakly increasing in k. Here we prove (i), and leave the proof of (ii) and the second part of the proposition to the appendix.

By construction, for all $p \leq [y_k, p^*]$, we have

$$
\begin{aligned}
p \int_p^1 \bar{h}_k(w) f(w) \mathrm{d}w &= p \left(\int_p^{p^*} \bar{h}_k(w) f(w) \mathrm{d}w + \frac{\pi(t_k) - \pi(t_{k+1})}{p^*} \right) \\
&= p \left(\int_{y_k}^{p^*} \frac{\pi(t_k) - \pi(t_{k+1})}{w^2 f(w)} f(w) \mathrm{d}w + \frac{\pi(t_k) - \pi(t_{k+1})}{p^*} \right) \\
&= p \left((\pi(t_k) - \pi(t_{k+1})) \left(\frac{1}{p} - \frac{1}{p^*} \right) + \frac{\pi(t_k) - \pi(t_{k+1})}{p^*} \right) \\
&= \pi(t_k) - \pi(t_{k+1}),
\end{aligned}
$$

where the first equality follows from Lemma 4. Thus, $R_k(p)$ is constant for all $p \in [y_k, p^*]$. For all $p \leq [y_{k-1}, y_k]$, we have

$$
\begin{aligned}
p \int_p^1 \bar{h}_k(w) f(w) \mathrm{d}w &= p \left(\int_p^{y_k} \bar{h}_k(w) f(w) \mathrm{d}w + \int_{y_k}^1 \bar{h}_k(w) f(w) \mathrm{d}w \right) \\
&= p \left(\int_p^{y_k} \left(1 - \frac{\pi^* - \pi(t_k)}{w^2 f(w)} \right) f(w) \mathrm{d}w + \frac{\pi(t_k) - \pi(t_{k+1})}{y_k} \right) \\
&= p \left(F(y_k) - F(p) - (\pi^* - \pi(t_k)) \left(\frac{1}{p} - \frac{1}{y_k} \right) + \frac{\pi(t_k) - \pi(t_{k+1})}{y_k} \right) \\
&= \pi(p) - p\pi'(y_k) - (\pi^* - \pi(t_k)),
\end{aligned}
$$

where the second equality uses the preceding result that $R_k(p)$ is constant for all $p \in [y_k, p^*]$. By the strict concavity of $\pi(\cdot)$, the above is maximized over $p \in [y_{k-1}, y_k]$ at $p = y_k$. Since $\bar{h}_k(w) = 0$ for $w < y_{k-1}$, we conclude that $R_k(p)$ is maximized at $p = p^*$ for $p \leq p^*$. ■

The idea of the construction of the extension $\{\bar{h}_k(w)\}_{k=1}^l$ is that it successively maximizes the ex ante mixing probabilities bidders use lower messages, starting from $k = 1$. For the lowest message t_1, $\bar{h}_1(w)$ for $w \leq p^*$ is such that the conditional revenue function $R_1(p)$ is flat for any $p \in [y_1, p^*]$, and $\bar{h}_1(w) = 1$ for $w < y_1$. Any additional mixing probability for $w \in [y_1, p^*]$ would violate the equilibrium condition (ii) in Lemma 3. Given that the ex ante mixing probabilities of bidders using the lowest message is maximized, $\bar{h}_2(w)$ for $w \leq p^*$ is constructed to maximize the ex ante mixing probability Φ_2 of bidders using the second lowest message subject to condition (ii) in Lemma 3, and so on.

More precisely, we construct $\bar{h}_1(w)$ for $w \leq t_1$ as the point-wise maximizer of Φ_1 subject to $\max_{p\leq p^*} R_1(p) = R_1(p^*)$, or equivalently,

$$p\left(\int_p^{p^*} h_1(w)f(w)dw + \frac{\pi(t_1)-\pi(t_2)}{p^*}\right) \leq \pi(t_1)-\pi(t_2)$$

for all $p \leq p^*$, where the left-hand side is $R_1(p)\Phi_1$ and the right-hand side is $R_1(p^*)\Phi_1$ from equation (4). Using induction, given $\{\bar{h}_j(w)\}_{j=1}^k$, we construct $\bar{h}_{k+1}(w)$ to be the point-wise maximizer of Φ_{k+1} subject to $\max_{p\leq p^*} R_{k+1}(p) = R_{k+1}(p^*)$. By construction, in any equilibrium with a feasible extension $\{h_k(w)\}_{k=1}^l$ and the corresponding sequence of mixing probabilities $\{\Phi_k\}_{k=1}^l$, we have

$$\sum_{j=1}^k \Phi_j \leq \sum_{j=1}^k \bar{\Phi}_j$$

for all $k = 1, \ldots, l$. Since

$$n\sum_{j=1}^k \chi_{j,j} = \frac{1}{\Phi_k}\left(\left(\sum_{j=1}^k \Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right) = \sum_{j=0}^{n-1}\left(\sum_{j=1}^k \Phi_j\right)^j \left(\sum_{j=1}^{k-1}\Phi_j\right)^{n-1-j},$$

the total probability $\sum_{j=1}^k \chi_{j,j}$ that any bid t_k generates offers $t_1, \ldots, t_k$ is maximized under the extension $\{\bar{h}_k(w)\}_{k=1}^l$ constructed for Proposition 3. In other words, $\{\bar{h}_k(w)\}_{k=1}^l$ successively minimizes the probability that no offer is made to any winning bid t_k. By (5), the seller's equilibrium revenue is maximized.

Since $\left(\sum_{j=1}^k \Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n$ in (7) is the probability that the highest bid is t_k, the revenue-maximizing extension $\{\bar{h}_k(w)\}_{k=1}^l$ induces the first-order stochastic dominated distribution of the highest bid. We have already observed from Lemma 4 and Lemma 5 that the seller's revenue in an equilibrium depends only on how bidders with values below p^* mix among the bids $t_1, \ldots, t_l$. These bidders do not accept any offer made to them. When they make high bids less often in the sense of first order stochastic dominance, the seller's equilibrium offers are more likely to be accepted. This increases the seller's equilibrium revenue.

3.6 Toward the most profitable equilibrium

Given the characterization of the most profitable equilibrium for given l offers $t_1 = p^* < t_2 < \ldots < t_l$, we can tackle the following question. For a fixed number $l \geq 2$ offers, what is the most profitable sequence of offers $\{t_k\}_{k=1}^l$? The answer to this question generally depends on the number of bidders n and the underlying underlying value distribution F. We show that regardless of n and F, in the most profitable equilibrium, the offers $t_2, \ldots, t_l$ are all distinct and interior to $[p^*, 1]$.

For any sequence of l offers, $t_1 = p^* < t_2 \ldots < t_l \leq 1$, by (7) and Proposition 3, we can write the seller's revenue in the most profitable equilibrium, jointly given the extension $\{\bar{h}_k(w)\}_{k=1}^l$ to $\{\hat{h}_k(w)\}_{k=1}^l$ and the monotone auction $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$, as

$$\pi = \sum_{k=1}^{l} (\pi(t_k) - \pi(t_{k+1})) \frac{(1 - \pi'(y_k))^n - (1 - \pi'(y_{k-1}))^n}{\pi'(y_{k-1}) - \pi'(y_k)}. \tag{8}$$

For any fixed $k = 2, \ldots, l$, we consider how an infinitesimal increase in t_k when $t_k = t_{k-1}$ and an infinitesimal decrease in t_k when $t_k = t_{k+1}$ affect π. The proof of the following result is in the appendix.

Proposition 4 *Fix any $l \geq 2$. In the most profitable equilibrium, the sequence of offers $\{t_k\}_{k=1}^l$ is strictly increasing, with $t_1 = p^*$ and $t_l < 1$.*

The argument in the proof of Proposition 4 is that any offer t_k should be distinct from the two adjacent offers t_{k-1} and t_{k+1}, so long as we assign the mixing probabilities for the three offers, t_{k-1}, t_k and t_{k+1}, according to the most profitable equilibrium as characterized in Proposition 3. An implication of Proposition 4 is then the following: for any equilibrium with a finite number l of offers, there is another equilibrium with a strictly higher revenue for the seller with $l+1$ offers. This is because the most profitable equilibrium with l offers, given by some strictly increasing sequence $t_1 = p^* < t_2 < \ldots < t_l < 1$ as characterized by Proposition 4, is identical to an equilibrium with $l+1$ offers, given by $t_1 = p^* < t_2 < \ldots < t_l < t_{l+1} = 1$. By Proposition 4, the seller's revenue can be increased by lowering t_{l+2} from 1.

4 A Continuum of Offers

Proposition 4 implies that any offer $t \in [p^*, 1]$ is made in the equilibrium that maximizes the seller's revenue. In this section, we construct equilibria with continuum of offers and monotone auctions. Although it is possible to construct equilibria where only offers in any given strict subinterval of $[p^*, 1]$ are made, as suggested by Proposition 4 we will focus on the equilibrium that offers span the entire interval $[p^*, 1]$. Correspondingly, bidders make all bids on the same interval. Each offer $t \in [p^*, 1]$ is associated with a distinct bid t.

4.1 Equilibrium bidding with a continuum of bids

In any equilibrium with a finite number l of offers, the mixing probabilities $\{h_k(w)\}_{k=1}^{l}$ of bidders with values $w > p^*$ are not pinned down, as long as they satisfy Lemma 4. With a continuum of offers, the mixing probabilities are pinned down as follows. For any $w > p^*$, let $\hat{H}(t, w)$ be the probability that a bidder with value w makes a bid t or lower. Define

$$\hat{H}(t, w) = \begin{cases} 0 & \text{if } t < w \\ -\pi'(w)/(wf(w)) & \text{if } t = w \\ 1 - \pi(t)/(w^2 f(w)) & \text{if } t \in (w, 1] \end{cases}$$

Note that

$$1 - \frac{\pi(w)}{w^2 f(w)} = \frac{-\pi'(w)}{wf(w)}.$$

A bidder with value w puts a probability mass of $-\pi'(w)/(wf(w))$ on the bid w, and a continuous positive density of $-\pi'(t)/(w^2 f(w))$ on all bids strictly higher than w. The bidder never bids strictly below the value w.

We say that a bidding strategy $H(\cdot, w)$ is a feasible extension of $\hat{H}(\cdot, w)$ if for all $w > p^*$, $H(t, w) = \hat{H}(t, w)$ for all $t \in [p^*, 1]$, and for all $w \leq p^*$, $H(p^*, w) = 0$ and $H(1, w) = 1$, with $H(t, w)$ weakly increasing. Define the ex ante probability $\Phi(t)$ that a bidder bids any $t \in [p^*, 1]$ as

$$\Phi(t) = \int_0^1 h(t, w) f(w) \mathrm{d}w.$$

The seller's revenue function $R(p,t)$ after selecting any fixed bid t is

$$R(p,t) = \frac{p}{\Phi(t)} \int_p^1 h(t,w) f(w) \mathrm{d}w.$$

We now show that under any feasible extension $H(\cdot,w)$, for any bid $t \in [p^*,1]$, each offer $p \in [p^*,t]$ is optimal over $p \geq p^*$ for the seller after selecting bid t. This is the counterpart of Lemma 5.

Lemma 8 *For any feasible extension $H(\cdot,w)$ of $\hat{H}(\cdot,w)$, any $p \in [p^*,t]$ is a solution to $\max_{p \geq p^*} R(p,t)$ for any $t \in [p^*,1]$. Further,*

$$\int_{p^*}^1 \hat{h}(t,w) f(w) \mathrm{d}w = -\frac{\pi'(t)}{p^*}. \tag{9}$$

Proof. By construction, for each $w \in [p^*,1]$, the derivative of $\hat{H}(t,w)$ with respect to the first argument, denoted as $h(t,w)$, is well-defined except at $t = w$, and so for all $p \in [p^*,t]$ we have

$$\begin{aligned}
\int_p^1 \hat{h}(t,w) f(w) \mathrm{d}w &= \int_p^t \hat{h}(t,w) f(w) \mathrm{d}w + \left(-\frac{\pi'(t)}{t f(t)}\right) f(t) \\
&= \int_p^t \left(-\frac{\pi'(t)}{w^2 f(w)}\right) f(w) \mathrm{d}w - \frac{\pi'(t)}{t} \\
&= -\frac{\pi'(t)}{p}.
\end{aligned}$$

The lemma and (9) follow immediately. ■

The above result is not surprising, because the construction of the bidding strategy $\hat{H}(t,w)$ for $w > p^*$ is the limit of the piece-wise constant mixing strategy $\{\hat{h}_k(w)\}_{k=1}^l$ for $w > p^*$ constructed for Lemma 5 as l goes to infinity. In the latter construction, for each offer $t_j = 1, \ldots, l$, a bidder with value w in the union of the interval $(t_j, z_j]$ and $(z_j, t_{j+1}]$ bids only t_j and higher. The ex ante probability that the bid is t_k and lower, $k = j, \ldots, l$, is given by

$$
\begin{aligned}
&\frac{F(t_{j+1})-F(z_j)}{F(t_{j+1})-F(t_j)}\cdot 1+\frac{F(z_j)-F(t_j)}{F(t_{j+1})-F(t_j)}\cdot\frac{\pi(t_j)-\pi(t_{k+1})}{\pi(t_j)}\\
=&\frac{\pi(t_j)-\pi(t_{j+1})}{(F(t_{j+1})-F(t_j))t_{j+1}}+\frac{(1-F(t_j))(t_{j+1}-t_j)}{(F(t_{j+1})-F(t_j))t_{j+1}}\cdot\frac{\pi(t_j)-\pi(t_{k+1})}{\pi(t_j)}\\
=&1-\frac{t_{j+1}-t_j}{F(t_{j+1})-F(t_j)}\cdot\frac{\pi(t_{k+1})}{t_jt_{j+1}}.
\end{aligned}
$$

The limit of the above expression as t_{j+1} goes to $t_j = w$, for any $t = t_{k+1}$, is precisely $\hat{H}(t, w)$.

4.2 Monotone auction with a continuum of offers

Next, we construct the monotone auction that supports any equilibrium with a continuum of offers. Let $H(\cdot, w)$ be the equilibrium bidding strategy of a bidder with value w. For any profile of bids $b = (b_1, \ldots, b_n)$, let $\hat{q}_i(b) = 1$ for any i such that $b_i = \max_{j \neq i} b_j$. That is, the highest bid is selected with probability one; tie-breaking rule is irrelevant as there is a continuum of bids. Denote the highest bid as t. Let the offer strategy of the seller $\hat{G}(\cdot, t)$ conditional on selecting bid t, which represents the probability the seller makes an offer less than or equal to $\hat{G}(p, t)$ for any p, be given by

$$
\hat{G}(p,t)=\left(\frac{\int_0^1 H(p,w)f(w)\mathrm{d}w}{\int_0^1 H(t,w)f(w)\mathrm{d}w}\right)^{n-1}.
$$

The following is the counterpart of Lemma 7.

Lemma 9 *For any feasible extension $H(\cdot, w)$, under the monotone auction $(\hat{q}(b), \hat{G}(p, t))$, a bidder with any given value $w \in [p^*, 1]$ is indifferent among all bids $t \in [w, 1]$.*

Proof. For any feasible extension $H(\cdot, w)$, we have $H(p, w) = 0$ for all w and all $p < p^*$, and thus $\hat{G}(p, t) = 0$ for all $p < p^*$; that is, all offers are greater than or equal to p^*. By construction, $\hat{G}(t, t) = 1$, so that all offers to bid t are smaller than or equal to t. Under $\hat{G}(\cdot, t)$, the probability that bid t generates an offer that is smaller than or equal to any

$p \in [p^*, t]$ is given by

$$\hat{X}(p) = \left(\int_0^1 H(t,w)f(w)\mathrm{d}w\right)^{n-1} \cdot \hat{G}(p,t) = \left(\int_0^1 H(p,w)f(w)\mathrm{d}w\right)^{n-1},$$

which is independent of t. Thus, any two bids $t < \tilde{t}$ generate all offers p or lower from the interval $[p^*, t]$ with the same probability $\hat{X}(p)$. The lemma follows immediately. ■

To see the above monotone auction $(\hat{q}(b), \hat{G}(p,t))$ as the limit of $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$, recall that in the latter construction, the ex ante probability that any fixed bid t_k or higher, $k = 1, \ldots, l$, generates offers $t_1, \ldots, t_k$, is given by

$$\sum_{j=1}^k \hat{\chi}_{j,j} = \frac{1}{n\Phi_k}\left(\left(\sum_{j=1}^k \Phi_j\right)^n - \left(\sum_{j=1}^{k-1} \Phi_j\right)^n\right).$$

The limit of the above expression, as t_k goes to t_{k-1}, and thus Φ_k goes to 0, for any $t_{k-1} = p$, is precisely $\hat{X}(p)$.

Under the monotone auction, we can compute the seller's expected revenue π as

$$\begin{aligned}
\pi &= \int_{p^*}^1 R(p^*,t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 H(t,w)f(w)\mathrm{d}w\right)^n \mathrm{d}t \\
&= \int_{p^*}^1 nR(p^*,t)\Phi(t)\left(\int_0^1 H(t,w)f(w)\mathrm{d}w\right)^{n-1} \mathrm{d}t \\
&= -\int_{p^*}^1 n\pi'(t)\left(\int_0^1 H(t,w)f(w)\mathrm{d}w\right)^{n-1} \mathrm{d}t,
\end{aligned} \tag{10}$$

where the second inequality follows from the definition of $\Phi(t)$.

4.3 Least profitable equilibrium

In any feasible extension $H(\cdot, w)$, from (9) we have

$$\Phi(t) = \int_0^{p^*} h(t,w)f(w)\mathrm{d}w - \frac{\pi'(t)}{p^*}.$$

The maximum revenue from selecting any bid $t \in [p^*, 1]$ satisfies

$$R(p^*, t)\Phi(t) = -\pi'(t).$$

Integrating the above over t from p^* to 1, we have

$$\pi^* = \int_{p^*}^{1} R(p^*, t)\Phi(t)\mathrm{d}t = \int_{p^*}^{1} R(p^*, t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 H(t, w)f(w)\mathrm{d}w\right)\mathrm{d}t,$$

where the second equality follows from the definition of $\Phi(t)$. In any equilibrium, the seller's revenue π is given by

$$\pi = \int_{p^*}^{1} R(p^*, t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 H(t, w)f(w)\mathrm{d}w\right)^n \mathrm{d}t.$$

The distribution of $R(p^*, t)$ over t in the expression for π first-order stochastically dominates that in the expression for π^*, because for all t,

$$\left(\int_0^1 H(t, w)f(w)\mathrm{d}w\right)^n \leq \int_0^1 H(t, w)f(w)\mathrm{d}w.$$

It follows that $\pi \geq \pi^*$ in any equilibrium, with a strict inequality if $R(p^*, t)$ is strictly increasing over any positive measure of t.

As in Proposition 2, we can construct an extension $H^*(\cdot, w)$ for $w \leq p^*$ such that the seller's revenue is π^*. For each $w \in [0, p^*]$, define

$$H^*(t, w) = 1 - \frac{\pi(t)}{\pi^*}.$$

This is the probability that a bidder with value $w \in [0, p^*]$ places a bid that is less than or equal to a fixed $t \in [p^*, 1]$. Note that $H^*(\cdot, w)$ is independent of w. We have the following counterpart of Proposition 2 with a continuum of offers.

Proposition 5 *The extension $H^*(\cdot, w)$ to $\hat{H}(\cdot, w)$ and the monotone auction $(\hat{q}(b), \hat{G}(p, t))$ form an equilibrium of the unobserved auction game, with the seller's revenue equal to π^*.*

Proof. For any $p \in [0, p^*]$, and for any $t \in [p^*, 1]$, we have

$$\begin{aligned}
&\int_p^{p^*} h^*(t,w)f(w)\mathrm{d}w + \int_{p^*}^1 \hat{h}(t,w)f(w)\mathrm{d}w \\
=& -(F(p^*) - F(p))\frac{\pi'(t)}{\pi^*} - \frac{\pi'(t)}{p^*} \\
=& -\frac{\pi'(t)}{\pi^*}(1 - F(p)),
\end{aligned}$$

where the first equality follows from (9). Thus, the seller's revenue from the offer p upon selecting a bid $t \in [p^*, 1]$ is given by

$$R(p,t) = p \cdot \frac{1 - F(p)}{1 - F(0)} = \pi(p).$$

This implies that for any $t \in [p^*, 1]$,

$$\max_{p \le p^*} R(p,t) = R(p^*, t) = \pi^*.$$

The proposition follows immediately. ■

The construction of $H^*(\cdot, w)$ is the limit of $\{h_k^*(w)\}_{k=1}^l$ constructed for Proposition 2 as l goes to infinity. In the latter construction, a bidder bids any t_k, $k = 1, \ldots, l$, with probability $(\pi(t_k) - \pi(t_{k+1}))/\pi^*$, independent of his value w for $w \le p^*$. The probability that the bidder bids t_k or lower is therefore

$$1 - \frac{\pi(t_{k+1})}{\pi^*}.$$

This converges to $H^*(t, w)$ as t_k goes to $t_{k+1} = t$.

4.4 Most profitable equilibrium

Now we construct the continuous-offer limit of the extension $\{\bar{h}_k(w)\}_{k=1}^l$ constructed for Proposition 3. For each $t \in [p^*, 1]$, let $y(t) \in [0, p^*]$ be uniquely given by

$$y^2(t)f(y(t)) = \pi^* - \pi(t).$$

The strict concavity of $\pi(\cdot)$ implies that $y(\cdot)$ is a strictly increasing function, with $y(p^*) = 0$ and $y(1) = p^*$. For each $w \in [0, p^*]$, denote as $\bar{H}(t, w)$ the probability that a bidder with value w makes a bid t or lower. Define

$$\bar{H}(t,w) = \begin{cases} 0 & \text{if } t < p^* \\ (\pi^* - \pi(t))/(w^2 f(w)) & \text{if } 0 \le y(t) \le w \\ 1 & \text{if } y(t) > w \end{cases}$$

By construction, for any $w \in [0, p^*]$, we have $\bar{H}(p^*, w) = 0$ and $\bar{H}(y^{-1}(w), w) = 1$. That is, a bidder with value any $w \in [0, p^*]$ mixes over bids on the interval $[p^*, y^{-1}(w)]$.

Below we directly verify that the constructed bidding strategy from combining $\hat{H}(p, w)$ for $w > p^*$ with $\bar{H}(p, w)$ for $w \le p^*$, and the monotone auction $(\hat{q}(b), \hat{G}(p, t))$ together satisfy that the equilibrium conditions (1) and (2).

Proposition 6 *The bidding strategy of $\hat{H}(p, w)$ for $w > p^*$ and $\bar{H}(p, w)$ for $w \le p^*$, and the monotone auction $(\hat{q}(b), \hat{G}(p, t))$ form an equilibrium of the unobserved auction game. The equilibrium revenue of the seller, given by*

$$\pi = -\int_{p^*}^{1} n\pi'(t)(1 - \pi'(y(t)))^{n-1} \mathrm{d}t, \tag{11}$$

achieves the highest revenue among all equilibria.

Proof. In Lemma 9, we have already verified that under the seller's auction $(\hat{q}(b), \hat{G}(p, t))$, any two bids $t < \tilde{t}$ generate all offers p or lower from the interval $[p^*, t]$ with the same probability $\hat{X}(p)$. Thus, for a bidder with value $w \in (p^*, 1]$, all offers lower than w lead to a strictly positive payoff. It follows that any mixing over bids from the interval $[w, 1]$ is a best response for the bidder. Further, since all equilibrium offers are higher than p^*, for a bidder with value $w \in [0, p^*]$, any mixing over all bids is optimal. Thus, the constructed bidding strategy of $\hat{H}(p, w)$ for $w > p^*$ and $\bar{H}(p, w)$ for $w \le p^*$ satisfies equilibrium condition (1) for all $w \in [0, 1]$.

For equilibrium condition (2), define

$$\bar{\Phi}(t) = \int_0^{p^*} \bar{h}(t,w)f(w)\mathrm{d}w + \int_{p^*}^1 \hat{h}(t,w)f(w)\mathrm{d}w$$

as the ex ante probability that a bidder bids any $t \in [p^*, 1]$. By (9), we have

$$\bar{\Phi}(t) = \int_{y(t)}^{p^*} \left(-\frac{\pi'(t)}{w^2 f(w)}\right) f(w)\mathrm{d}w - \frac{\pi'(t)}{p^*} = -\frac{\pi'(t)}{y(t)}.$$

Denote the conditional revenue from selecting bid t, as a function of the offer p, as

$$R(p,t) = \frac{1}{\bar{\Phi}(t)} \left(\int_{\min\{p,p^*\}}^{p^*} \bar{h}(t,w)f(w)\mathrm{d}w + \int_{\min\{p,p^*\}}^1 \hat{h}(t,w)f(w)\mathrm{d}w\right).$$

We can verify that

$$R(p,t) = \begin{cases} p & \text{if } p < y(t) \\ y(t) & \text{if } p \in [y(t), t] \\ 0 & \text{if } p \in (t, 1] \end{cases}$$

For any $t \in [p^*, 1]$, the conditional revenue function is strictly increasing in the offer p for $p < y(t)$, and is equal to a constant $y(t)$ for $p \in [y(t), t]$, and 0 for $p > t$. Since the constant $y(t)$ is strictly increasing in t, equilibrium condition (2) is satisfied. The first part of the proposition follows immediately.

For the second part of the proposition, by construction we have

$$\begin{aligned}
\int_0^1 H(t,w)f(w)\mathrm{d}w =& F(y(t)) + \int_{y(t)}^{p^*} \frac{\pi^* - \pi(t)}{w^2 f(w)} f(w)\mathrm{d}w + \int_{p^*}^{t} \left(1 - \frac{\pi(t)}{w^2 f(w)}\right) f(w)\mathrm{d}w \\
=& F(y(t)) + (\pi^* - \pi(t))\left(\frac{1}{y(t)} - \frac{1}{p^*}\right) + \left(F(t) - F(p^*) - \pi(t)\left(\frac{1}{p^*} - \frac{1}{t}\right)\right) \\
=& F(y(t)) + \frac{\pi^* - \pi(t)}{y(t)} \\
=& 1 - \pi'(y(t)).
\end{aligned}$$

Equation (11) follows immediately from (10).

Let π_l be the revenue for the seller in the most profitable equilibrium with l offers, with the strictly increasing sequence $t_1 = p^* < t_2 < \ldots < t_l < 1$ characterized by Proposition 4, and the corresponding $y_k = y(t_{k+1})$ for each $k = 0, \ldots, l$. By (8), we have

$$\pi_l = \sum_{k=1}^{l} (\pi(t_k) - \pi(t_{k+1})) \frac{(1 - \pi'(y(t_{k+1})))^n - (1 - \pi'(y(t_k)))^n}{\pi'(y(t_k)) - \pi'(y(t_{k+1}))}.$$

Since $y(\cdot)$ is strictly increasing,

$$n(1 - \pi'(y(t_k)))^{n-1} < \frac{(1 - \pi'(y(t_{k+1})))^n - (1 - \pi'(y(t_k)))^n}{\pi'(y(t_k)) - \pi'(y(t_{k+1}))} < n(1 - \pi'(y(t_{k+1})))^{n-1}.$$

By Proposition 4, as l goes to infinity, the sequence of offers $t_1 = p^* < t_2 < \ldots < t_l < 1$ defines an arbitrarily fine partition of $[p^*, 1]$. This implies that

$$\lim_{l \to \infty} \pi_l = -\int_{p^*}^{1} n\pi'(t)(1 - \pi'(y(t)))^{n-1} \mathrm{d}t.$$

The above is precisely the seller's equilibrium revenue π given by (11). The second part of the proposition follows immediately from Proposition 4, because any equilibrium can be approximated arbitrarily closely by an equilibrium with a finite number of offers. ■

In the equilibrium given by Proposition 6, the bidding strategy $\bar{H}(\cdot, w)$ for $w \leq p^*$ extends $\hat{H}(\cdot, w)$ for $w > p^*$ by making each conditional revenue function $R(p, t)$ flat over the interval $[y(t), p^*]$. Since $y(t)$ is strictly increasing, the support of $\bar{H}(\cdot, w)$ for $w \leq p^*$, given by $[p^*, y^{-1}(w)]$, shifts to the right as w increases. Thus, as in the construction for Proposition 3, given $\hat{H}(\cdot, w)$ for $w > p^*$, the extension $\bar{H}(\cdot, w)$ for $w \leq p^*$ maximizes the probability that the highest bid is each t or lower, given by

$$\int_{p^*}^{t} \Phi(\tau) \mathrm{d}\tau = \int_{0}^{1} H(t, w) f(w) \mathrm{d}w,$$

subject to the constraint that $\max_{p \leq p^*} R(p, \tau) = R(p^*, \tau)$ for all $\tau \in [p^*, t]$. By (10), the extension $\bar{H}(\cdot, w)$ to $\hat{H}(\cdot, w)$ maximizes the seller's revenue among all equilibria where all offers on the interval $[p^*, 1]$ are made.

Indeed, we can show that $\bar{H}(t, w)$ is the limit of $\{\bar{h}_k(w)\}_{k=1}^{l}$ constructed for Proposition 3 as l goes to infinity. In the latter construction, for each $j = 1, \ldots, l$, bidders with values in $[y_{j-1}, y_j)$ bid each t_k, $k = 1, \ldots, j-1$ with probability $(\pi(t_k) - \pi(t_{k+1})/(w^2 f(w))$, and bid t_j

with the remaining probability. The probability of bidders with value $w \in [y_{j-1}, y_j)$ bidding t_k and lower, $k = 1, \ldots, j-1$, is given by

$$\frac{\pi^* - \pi(t_{k+1})}{w^2 f(w)},$$

which converges to $\bar{H}(t, w)$ for y_{j-1} converging to $y_j = w$, and any $t_k = t \leq y^{-1}(w)$.

4.5 A uniform example

Consider the uniform value distribution, with distribution function $F(w) = w$ for $w \in [0, 1]$. We have

$$\pi(p) = p(1-p),$$

with $p^* = \frac{1}{2}$ and $\pi^* = \frac{1}{4}$. In the optimal auction (Myerson, 1981), the optimal reserve price is p^*, regardless of the number of bidders n. The optimal revenue is given by

$$n \int_{\frac{1}{2}}^{1} w^{n-1}(2w-1)dw = \frac{(n-1)2^n + 1}{(n+1)2^n}.$$

For $n = 2$, the optimal revenue is $\frac{5}{12}$.

For any given sequence of l offers $t_1 = p^* < t_2 < \ldots < t_l < 1$, the revenue for the seller in the most profitable equilibrium as characterized by Proposition 3 is given by (8). For each $k = 1, \ldots, l$, by definition we have

$$y_k = t_{k+1} - \frac{1}{2},$$

where we set $t_{l+1} = 1$. By (8), the maximum revenue is

$$\pi = \frac{1}{2} \sum_{k=1}^{l} (t_{k+1} + t_k - 1) \left((2t_{k+1} - 1)^n - (2t_k - 1)^n \right).$$

By Proposition 4, for any fixed number of offers l, the most profitable equilibrium involves a strictly increasing and interior sequence $t_1 = p^* < t_2 < \ldots < t_l < 1$. This sequence can be found taking derivatives of π with respect to $t_2, \ldots, t_l$ and setting them to zero. The first order condition with respect to t_k, $k = 2, \ldots, l$, gives a second-order difference equation in

$(t_1, t_2, \ldots, t_l)$

$$n(2t_k - 1)^{n-1} = \frac{(2t_{k+1} - 1)^n - (2t_{k-1} - 1)^n}{2(t_{k+1} - t_{k-1})},$$

with two boundary conditions of $t_1 = \frac{1}{2}$ and $t_{l+1} = 1$. For $n = 2$, the difference equation can be solved explicitly. The revenue-maximizing sequence is evenly spaced on $\left[\frac{1}{2}, 1\right]$, with

$$t_k = \frac{k + l - 1}{2l}$$

for each $k = 1, \ldots, l$. The maximized revenue with l optimally placed offers is

$$\frac{1}{4l^3} \sum_{k=1}^{l} (2k - 1)^2.$$

This is an increasing sequence in l, and converges to $\frac{1}{3}$ as l goes to infinity.

For the equilibrium with a continuum of offers characterized in Proposition 6, note that by definition

$$y(t) = t - \frac{1}{2}.$$

By (11), the seller's revenue is equal to

$$\int_{\frac{1}{2}}^{1} n(2t - 1)^n \mathrm{d}t = \frac{n}{2(n + 1)}.$$

For $n = 2$, the seller's revenue is $\frac{1}{3}$, which is the limit of the seller's revenue in the most profitable equilibrium with l offers as l goes to infinity.

5 Concluding Remarks

We have used Google Ads auctions to motivate our unobserved auction game, but we have abstracted away from two essential features of the actual auctions. One is that multiple goods are sold; the other is that there is a quality or relevance dimension to each bid. In future research, we want to investigate how to incorporate one or both features into the analysis.

The present paper is a continuation of our research on mechanism design problems where agents may not observe the committed mechanism. In our earlier papers, we have studied the same independent private auction setting but assumed that each buyer has an independent and identical probability of not observing the seller's auction. The outcome depends on the selection of equilibrium behavior of unobservant buyers. In our first paper (Li and Peters, 2022a), unobservant buyers babble, so the offer they receive from the seller does not dependent on their bid. The result we obtain is that in equilibrium the seller holds an "equal priority auction," where buyers with values on a strict subinterval in the value support have the same allocation priority as unobservant buyers. In our second paper (Li and Peters, 2022b), unobservant buyers with values above a threshold make a high bid to indicate they are "interested," while those with values below the threshold make a low bid to indicate that they are uninterested. The seller's equilibrium auction targets interested buyers, and the commitment to not making an offer to uninterested buyers is made credible by observant buyers with low values. We plan to make use of the construction in the present paper to enrich the analysis of unobserved mechanisms by considering a broader class of equilibrium behavior of unobservant buyers.

Appendix: Omitted Proofs

Proof of Lemma 6

For any $t_1 < \ldots < t_l$, fix an equilibrium with buyers' bidding strategy $\{h_k(w)\}_{k=1}^{l}$ and seller's auction $\{q_k(\theta), \{g_{j,k}(\theta)\}_{j=1}^{l}\}_{k=1}^{l}$. We prove the lemma with a series of claims. First, we provide a useful lower bound on the weighted sum of the partial sums of offer generating probabilities.

Claim 1 *For any $1 \leq k_1 \leq \cdots \leq k_J \leq l$,*

$$\sum_{j=1}^{J} n\Phi_{k_j}(\chi_{1,1} + \cdots + \chi_{k_j,k_j}) \geq \left(\sum_{k_j=k_1}^{k_J} \Phi_{k_j} \right)^n,$$

with equality if, for any $j = 1, \ldots, J$, $q_{k_j}(\theta) > 0$ implies $\theta_{k_1} + \cdots + \theta_{k_J} = n$.

Proof. For any $j = 1, \ldots, J$, by condition (iv) of Lemma 3 we have

$$\begin{aligned}
&\chi_{1,k_j} + \cdots + \chi_{k_j,k_j} \\
\geq &\sum_{\{\theta|\theta_{k_1}+\cdots+\theta_{k_j}+1+\cdots+\theta_{k_J}=n\}} \frac{(n-1)! \cdot q_{k_j}(\theta)}{\theta_{k_1}! \times \cdots \times \theta_{k_j}! \times \cdots \times \theta_{k_J}!} \Phi_{k_1}^{\theta_{k_1}} \times \cdots \times \Phi_{k_j}^{\theta_{k_j}} \times \cdots \times \Phi_{k_J}^{\theta_{k_J}} \\
= &\frac{1}{n\Phi_j} \sum_{\{\theta|\theta_{k_1}+\cdots+\theta_{k_j}+\cdots+\theta_{k_J}=n\}} \frac{n! \cdot \theta_{k_j} q_{k_j}(\theta)}{\theta_{k_1}! \times \cdots \times \theta_{k_j}! \times \cdots \times \theta_{k_J}!} \Phi_{k_1}^{\theta_{k_1}} \times \cdots \times \Phi_{k_j}^{\theta_{k_j}} \times \cdots \times \Phi_{k_J}^{\theta_{k_J}},
\end{aligned}$$

where the inequality holds as an equality if t_{k_j} is selected only when $\theta_{k_1} + \cdots + \theta_{k_J} = n$, and the equality follows from a change of variable (setting $\theta_{k_j}+1$ to θ_{k_j}). Since $\sum_{j=1}^{J} \theta_{k_j} q_{k_j}(\theta) = 1$ for any θ such that $\theta_{k_1} + \cdots + \theta_{k_J} = n$, the claim follows from condition (v) of Lemma 3 by summing over $j = 1, \ldots, J$. ■

Next, we derive a monotonicity result on the weighted average of the partial sums of offer generating probabilities.

Claim 2 *For any* $1 \leq \underline{k}_1 \leq \overline{k}_1 \leq \underline{k}_2 \leq \overline{k}_2 \leq l$,

$$\frac{\sum_{j=\underline{k}_1}^{\overline{k}_1} \Phi_j(\chi_{1,1} + \cdots + \chi_{j,j})}{\sum_{j=\underline{k}_1}^{\overline{k}_1} \Phi_j} \leq \frac{\sum_{j=\underline{k}_2}^{\overline{k}_2} \Phi_j(\chi_{1,1} + \cdots + \chi_{j,j})}{\sum_{j=k_3}^{k_4} \Phi_j}.$$

Proof. We verify the claim by comparing the coefficient of $\chi_{j,j}$ for each $j = 1, \ldots, l$. For any $j \leq \underline{k}_1$, the coefficients of $\chi_{j,j}$ on the left-hand side and the right-hand side are both equal to 1. For any j such that $\underline{k}_1 < j \leq \overline{k}_1$, we have $j \leq \underline{k}_2$, and the coefficient of $\chi_{j,j}$ on the left-hand side is less than 1, which is the coefficient on the right-hand side. Finally, for any $j > \overline{k}_1$, the coefficient $\chi_{j,j}$ on the left-hand side is equal to 0. ■

Lastly, we establish a useful inequality.

Claim 3 *For any* $\sigma_1, \sigma_2, \sigma_3 > 0$.

$$\frac{1}{\sigma_3}\left((\sigma_1 + \sigma_3)^n - \sigma_1^n\right) > \frac{1}{\sigma_2}\left((\sigma_1 + \sigma_2 + \sigma_3)^n - (\sigma_1 + \sigma_3)^n\right).$$

Proof. For fixed σ_1 and σ_3, define

$$\Delta(\sigma_2) = \sigma_2((\sigma_1+\sigma_3)^n - \sigma_1^n) - \sigma_3((\sigma_1+\sigma_2+\sigma_3)^n - (\sigma_1+\sigma_3)^n).$$

Note that $\Delta(0) = 0$,

$$\Delta'(\sigma) = (\sigma_1+\sigma_3)^n - \sigma_1^n - n\sigma_3(\sigma_1+\sigma_2+\sigma_3)^{n-1}$$

and $\Delta''(\sigma_2) \leq 0$ for all σ_2. To establish the claim, it suffices to show that $\Delta'(0) < 0$. We accomplish this by induction on n. For $n = 2$, we have $\Delta'(0) = -2\sigma_3^2 < 0$. Suppose that $\Delta'(0) < 0$ holds for n. Then,

$$\begin{aligned}
&(\sigma_1+\sigma_3)^{n+1} - \sigma_1^{n+1} - (n+1)\sigma_3(\sigma_1+\sigma_3)^n \\
<&(\sigma_1+\sigma_3)^{n+1} - \sigma_1^{n+1} - \frac{n+1}{n}(\sigma_1+\sigma_3)\left((\sigma_1+\sigma_3)^n - \sigma_1^n\right) \\
=&\frac{1}{n}\left(\sigma_1^{n+1} + (n+1)\sigma_1^n\sigma_3 - (\sigma_1+\sigma_3)^{n+1}\right),
\end{aligned}$$

which is strictly negative. The claim follows immediately. ■

Now, we use the above three claims to show that in any equilibrium the set of minimizers of $R_j(p^*)$ over $j = 1, \ldots, l$ takes the form of $\{1, \ldots, k\}$ for some k. Suppose this is false. Then, there exist J pairs of numbers $\underline{k}_j$ and $\overline{k}_j$, $j = 1, \ldots, J$, satisfying $\underline{k}_1 \leq \overline{k}_1 < \underline{k}_2 \leq \overline{k}_2 < \cdots < \underline{k}_J \leq \overline{k}_J$, such that any κ satisfying $\underline{k}_j \leq \kappa \leq \overline{k}_j$ for some k_j is a minimizer. By condition (iii) of Lemma 3 and Claim 1,

$$\sum_{j=1}^{J}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j} n\Phi_\kappa(\chi_{1,1} + \cdots + \chi_{\kappa,\kappa}) = \left(\sum_{j=1}^{J}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n.$$

We use the above expression to derive two inequalities that contradict that Claim 3. First, by Claim 1, we have

$$\sum_{j=1}^{J-1}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j} n\Phi_\kappa(\chi_{1,1} + \cdots + \chi_{\kappa,\kappa}) \geq \left(\sum_{j=1}^{J-1}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n,$$

and therefore

$$\sum_{\kappa=\underline{k}_J}^{\overline{k}_J} \Phi_\kappa(\chi_{1,1}+\cdots+\chi_{\kappa,\kappa}) \le \left(\sum_{j=1}^{J}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n - \left(\sum_{j=1}^{J-1}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n.$$

This the first inequality. Next, denoting $\overline{k}_0 = 0$, and using the above expression and Claim 1 again, we have

$$\sum_{j=1}^{J}\sum_{\kappa=\overline{k}_{j-1}+1}^{\underline{k}_j-1} n\Phi_\kappa(\chi_{1,1}+\cdots+\chi_{\kappa,\kappa}) \ge \left(\sum_{\kappa=1}^{k_J}\Phi_\kappa\right)^n - \left(\sum_{j=1}^{J}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n.$$

By Claim 2, for each $j = 1,\dots,J$,

$$\sum_{\kappa=\overline{k}_{j-1}+1}^{\underline{k}_j-1} \Phi_\kappa(\chi_{1,1}+\cdots+\chi_{\kappa,\kappa}) \le \frac{\sum_{\kappa=\overline{k}_{j-1}+1}^{\underline{k}_j-1}\Phi_\kappa}{\sum_{\kappa=\underline{k}_J}^{\overline{k}_J}\Phi_\kappa}\left(\sum_{\kappa=\underline{k}_J}^{\overline{k}_J}\Phi_\kappa(\chi_{1,1}+\cdots+\chi_{\kappa,\kappa})\right)$$

Summing over $j = 1,\dots,J$, we have

$$\frac{\sum_{\kappa=\underline{k}_J}^{\overline{k}_J} n\Phi_\kappa(\chi_{1,1}+\cdots+\chi_{\kappa,\kappa})}{\sum_{\kappa=\underline{k}_J}^{\overline{k}_J}\Phi_\kappa} \ge \frac{\left(\sum_{\kappa=1}^{k_J}\Phi_\kappa\right)^n - \left(\sum_{j=1}^{J}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n}{\sum_{j=1}^{J}\sum_{\kappa=\overline{k}_{j-1}+1}^{\underline{k}_j-1}\Phi_\kappa}.$$

This is the second inequality. Combining the two inequalities, we have a contradiction to Claim 2:

$$\frac{\left(\sum_{j=1}^{J}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n - \left(\sum_{j=1}^{J-1}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n}{\sum_{\kappa=\underline{k}_J}^{\overline{k}_J}\Phi_\kappa} \ge \frac{\left(\sum_{\kappa=1}^{k_J}\Phi_\kappa\right)^n - \left(\sum_{j=1}^{J}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa\right)^n}{\sum_{j=1}^{J}\sum_{\kappa=\overline{k}_{j-1}+1}^{\underline{k}_j-1}\Phi_\kappa},$$

where $\sigma_1 = \sum_{j=1}^{J-1}\sum_{\kappa=\underline{k}_j}^{\overline{k}_j}\Phi_\kappa$, $\sigma_2 = \sum_{j=1}^{J}\sum_{\kappa=\overline{k}_{j-1}+1}^{\underline{k}_j-1}\Phi_\kappa$ and $\sigma_3 = \sum_{\kappa=\underline{k}_J}^{\overline{k}_J}\Phi_\kappa$.

Now that we have shown that the set of minimizers of $R_j(p^*)$ over $j = 1,\dots,l$ is $\{1,\dots,k_1\}$ for some k_1, the next step is to show that, if $k_1 < l$, the set of minimizers of $R_j(p^*)$ over $j = k_1+1,\dots,l$ takes the form of $\{k_1+1,\dots,k_2\}$ for some $k_2 > k_1$. The argument is the same as the first step. The lemma follows immediately.

Proof of Lemma 7

It suffices to show that $\{\hat{q}_k(\theta), \{\hat{g}_{j,k}(\theta)\}_{j=1}^l\}_{k=1}^l$ as constructed is feasible. That is, for all $k = 2, \ldots, l$, if the auction already constructed in previous steps specifying $\{\hat{q}_j(\theta), \{\hat{g}_{\tilde{j},j}(\theta)\}_{\tilde{j}=1}^l\}_{j=1}^{k-1}$ is feasible, then the specification for $\{\hat{q}_k(\theta), \{\hat{g}_{\tilde{j},k}(\theta)\}_{\tilde{j}=1}^k\}$ is feasible, meaning $\sum_{j=1}^{k-1} \hat{g}_{j,k} < 1$. This is equivalent to

$$\frac{1}{n\Phi_k}\left(\left(\sum_{j=1}^{k}\Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right) > \frac{1}{n\Phi_{k-1}}\left(\left(\sum_{j=1}^{k-1}\Phi_j\right)^n - \left(\sum_{j=1}^{k-2}\Phi_j\right)^n\right).$$

For fixed $\Phi_1, \ldots, \Phi_{k-1}$, define

$$\Delta(\Phi_k) = \Phi_{k-1}\left(\left(\sum_{j=1}^{k}\Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right) - \Phi_k\left(\left(\sum_{j=1}^{k-1}\Phi_j\right)^n - \left(\sum_{j=1}^{k-2}\Phi_j\right)^n\right)$$

We claim that $\Delta(\Phi_k) > 0$ for all $\Phi_k > 0$. Observe that $\Delta(0) = 0$, and

$$\Delta'(\Phi_k) = n\Phi_{k-1}\left(\sum_{j=1}^{k}\Phi_j\right)^{n-1} - \left(\left(\sum_{j=1}^{k-1}\Phi_j\right)^n - \left(\sum_{j=1}^{k-2}\Phi_j\right)^n\right).$$

Thus, $\Delta'(\Phi_k)$ is strictly increasing in Φ_k. To show that $\Delta(\Phi_k) > 0$ for all $\Phi_k > 0$, we only need to show that $\Delta'(0) > 0$. We establish this result by induction on n. For $n = 2$, we have

$$\Delta_2'(0) = 2\Phi_{k-1}\sum_{j=1}^{k-1}\Phi_j - \Phi_{k-1}\left(2\sum_{j=1}^{k-2}\Phi_j + \Phi_{k-1}\right) = \Phi_{k-1}^2 > 0.$$

Suppose that $\Delta_n'(0) > 0$. Then,

$$\begin{aligned}
\Delta_{n+1}'(0) =& (n+1)\Phi_{k-1}\left(\sum_{j=1}^{k-1}\Phi_j\right)^n - \left(\left(\sum_{j=1}^{k-1}\Phi_j\right)^{n+1} - \left(\sum_{j=1}^{k-2}\Phi_j\right)^{n+1}\right) \\
>& \frac{n+1}{n}\left(\sum_{j=1}^{k-1}\Phi_j\right)\left(\left(\sum_{j=1}^{k-1}\Phi_j\right)^n - \left(\sum_{j=1}^{k-2}\Phi_j\right)^n\right) - \left(\left(\sum_{j=1}^{k-1}\Phi_j\right)^{n+1} - \left(\sum_{j=1}^{k-2}\Phi_j\right)^{n+1}\right) \\
=& \frac{1}{n}\left(\left(\sum_{j=1}^{k-1}\Phi_j\right)^{n+1} - (n+1)\Phi_{k-1}\left(\sum_{j=1}^{k-2}\Phi_j\right)^n - \left(\sum_{j=1}^{k-2}\Phi_j\right)^{n+1}\right),
\end{aligned}$$

which is strictly positive. It follows that $\Delta(\Phi_k) > 0$ for all $\Phi_k > 0$, and thus the specification for $\{\hat{q}_k(\theta), \{\hat{g}_{\tilde{j},k}(\theta)\}_{\tilde{j}=1}^{k}\}$ is feasible.

Omitted proof of Proposition 3

We show that under the extension $\{\bar{h}_k(w)\}_{k=1}^{l}$ to $\{\hat{h}_k(w)\}_{k=1}^{l}$, the maximum conditional revenue $R_k(p^*)$ is strictly increasing in k. For all t and $\tilde{t}$ such that $p^* < t < \tilde{t} < 1$, define

$$\rho(t, \tilde{t}) = \frac{\pi(t) - \pi(\tilde{t})}{\pi'(y(t)) - \pi'(y(\tilde{t}))},$$

where the function $y(\cdot)$ is given by

$$y^2(t) f(y(t)) = \pi^* - \pi(t).$$

We claim that $\rho(t, \tilde{t})$ is increasing in t for $t < \tilde{t}$ and in $\tilde{t}$ for $\tilde{t} > t$. To see this, note that the sign of $\partial\rho(t, \tilde{t})/\partial t$ is the same as

$$\begin{aligned}
&\pi'(t)(\pi'(y(t)) - \pi'(y(\tilde{t}))) - \pi''(y(t))y'(t)(\pi(t) - \pi(\tilde{t})) \\
=&\pi'(t)(\pi'(y(t)) - \pi'(y(\tilde{t}))) - \frac{\pi'(t)}{y(t)}(\pi(t) - \pi(\tilde{t})) \\
=& - \pi'(t)\left(\frac{\pi(t) - \pi(\tilde{t})}{y(t)} - (\pi'(y(t)) - \pi'(y(\tilde{t})))\right).
\end{aligned}$$

The derivative of the above expression with respect to $\tilde{t}$ has the same sign as

$$-\frac{\pi'(\tilde{t})}{y(t)} + \pi''(y(\tilde{t}))y'(\tilde{t}) = -\frac{\pi'(\tilde{t})}{y(t)} + \pi''(y(\tilde{t}))\frac{\pi'(\tilde{t})}{\pi''(y(\tilde{t}))y(\tilde{t})} > 0,$$

because $\tilde{t} > t$. At $\tilde{t} = t$, we have $\partial\rho(t, \tilde{t})/\partial t = 0$. This implies $\partial\rho(t, \tilde{t})/\partial t > 0$. By a symmetric argument, we have $\partial\rho(t, \tilde{t})/\partial\tilde{t} > 0$. It follows that for all $k = 2, \ldots, l$,

$$R_{k-1}(p^*) = \frac{\pi(t_{k-1}) - \pi(t_k)}{\bar{\Phi}_{k-1}} = \rho(t_{k-1}, t_k) < \rho(t_k, t_k) < \rho(t_k, t_{k+1}) = \frac{\pi(t_k) - \pi(t_{k+1})}{\bar{\Phi}_k} = R_k(p^*).$$

For the second part of the proposition, we use an induction argument. First, we show that if $h_1(w) < \bar{h}_1(w)$ for a positive measure of $w \le p^*$ in an equilibrium, then there is another equilibrium with a higher revenue for the seller. Consider marginally increasing $h_1(w)$ for a positive measure of $w \le p^*$ such that $h_1(w) + \mathrm{d}h_1(w) \le \bar{h}_1(w)$ for all $w \le p^*$, with $\mathrm{d}\Phi_1 > 0$. Suppose that

$$R_1(p^*) \le R_2(p^*) = \ldots = R_{\hat{j}}(p^*) < R_{\hat{j}+1}(p^*) \le \ldots \le R_l(p^*).$$

We marginally decrease each $h_k(w)$, $k = 2, \ldots, \hat{j}$, for a positive measure of $w \le p^*$, such that

$$\mathrm{d}\Phi_k = -\mathrm{d}\Phi_1 \left(\sum_{j=2}^{\hat{j}} \frac{\Phi_j^2}{\pi(t_j) - \pi(t_{j+1})} \right)^{-1} \frac{\Phi_k^2}{\pi(t_k) - \pi(t_{k+1})} < 0.$$

By construction,

$$\sum_{k=1}^{\hat{j}} \mathrm{d}\Phi_j = 0.$$

Since $h_1(w) + \mathrm{d}h_1(w) \le \bar{h}_1(w)$ for all $w \le p^*$, for any $p \le p^*$, we have

$$\begin{aligned} p \int_p^1 (h_1(w) + \mathrm{d}h_1(w)) f(w) \mathrm{d}w &\le p^* \int_{p^*}^1 \bar{h}_1(w) f(w) \mathrm{d}w \\ &= p^* \int_{p^*}^1 (h_1(w) + \mathrm{d}h_1(w)) f(w) \mathrm{d}w, \end{aligned}$$

and therefore it remains true that $p = p^*$ maximizes $R_1(p)$ for all $p \le p^*$. For each $k = 2, \ldots, \hat{j}$, since $\mathrm{d}\Phi_k < 0$, it also remains true that $p = p^*$ maximizes $R_1(p)$ for all $p \le p^*$. Further,

$$\frac{\mathrm{d}R_1(p^*)}{\mathrm{d}\Phi_1} = -\frac{\pi^* - \pi(t_2)}{\Phi_1^2} < 0,$$

and for each $k = 2, \ldots, \hat{j}$,

$$\frac{\mathrm{d}R_k(p^*)}{\mathrm{d}\Phi_1} = -\frac{\pi(t_k) - \pi(t_{k+1})}{\Phi_k^2} \frac{\mathrm{d}\Phi_k}{\mathrm{d}\Phi_1} = \left(\frac{\Phi_j^2}{\pi(t_j) - \pi(t_{j+1})} \right)^{-1},$$

which is strictly positive and independent of k. We have thus constructed another equilibrium. By (7), the change in the seller's equilibrium revenue is given by

$$
\begin{aligned}
\frac{\mathrm{d}\pi}{\mathrm{d}\Phi_1} = & \frac{\mathrm{d}}{\mathrm{d}\Phi_1}\left(\sum_{k=1}^{\hat{j}} R_k(p^*)\left(\left(\sum_{j=1}^{k}\Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right)\right) \\
= & \sum_{k=1}^{\hat{j}} \frac{\mathrm{d}R_k(p^*)}{\mathrm{d}\Phi_1}\left(\left(\sum_{j=1}^{k}\Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right) + \sum_{k=1}^{\hat{j}} R_k(p^*)\frac{\mathrm{d}}{\mathrm{d}\Phi_1}\left(\left(\sum_{j=1}^{k}\Phi_j\right)^n - \left(\sum_{j=1}^{k-1}\Phi_j\right)^n\right) \\
= & \left(\frac{\mathrm{d}R_1(p^*)}{\mathrm{d}\Phi_1} - \frac{\mathrm{d}R_2(p^*)}{\mathrm{d}\Phi_1}\right)\Phi_1^n + \frac{\mathrm{d}R_{\hat{j}}(p^*)}{\mathrm{d}\Phi_1}\left(\sum_{j=1}^{\hat{j}}\Phi_j\right)^n + (R_1(p^*) - R_2(p^*))n\Phi_1^{n-1} \\
> & R_2(p^*)\left(\frac{\Phi_2}{\pi(t_2) - \pi(t_3)}\left(\sum_{j=2}^{\hat{j}}\frac{\Phi_j^2}{\pi(t_j) - \pi(t_{j+1})}\right)^{-1}\left(\left(\sum_{j=1}^{\hat{j}}\Phi_j\right)^n - \Phi_1^n\right) - n\Phi_1^{n-1}\right) \\
= & R_2(p^*)\left(\frac{\left(\sum_{j=1}^{\hat{j}}\Phi_j\right)^n - \Phi_1^n}{\sum_{j=2}^{\hat{j}}\Phi_j} - n\Phi_1^{n-1}\right),
\end{aligned}
$$

where the first equality follows because there is no change to any $h_k(w)$ for $w \le p^*$ and $k = \hat{j}+1, \ldots, l$, with Φ_k and hence $R_k(p^*)$ staying the same; the third equality follows because by construction $\mathrm{d}R_k(p^*)/\mathrm{d}\Phi_1 = \mathrm{d}R_{k+1}(p^*)/\mathrm{d}\Phi_1$ and $R_k(p^*) = R_{k+1}(p^*)$ for all $k = 2, \ldots, \hat{j}-1$; the inequality follows from dropping a positive term involving $R_1(p^*)$, and the last equality follows again from $R_k(p^*) = R_{k+1}(p^*)$ for all $k = 2, \ldots, \hat{j}-1$. The above is strictly positive, establishing that in the most profitable equilibrium, we have $h_1(w) = \bar{h}_1(w)$ for all $w \le p^*$.

Suppose that for all $j = 1, \ldots, l-2$, we have shown that $h_k(w) = \bar{h}_k(w)$ for $w \le p^*$ and all $k = 1, \ldots, j$, but $h_{j+1}(w) < \bar{h}_{j+1}(w)$ for a positive measure of $w \le p^*$. We proceed in the same way as above, by increasing $h_{j+1}(w)$ for a positive measure of $w \le p^*$, and hence Φ_{j+1} by $\mathrm{d}\Phi_{j+1}$. Corresponding, if

$$R_{j+1}(p^*) \le R_{j+2}(p^*) = \ldots = R_{\hat{j}}(p^*) < R_{\hat{j}+1}(p^*) \le \ldots \le R_l(p^*),$$

we decrease each Φ_k, $k = j+2, \ldots, \hat{j}$, such that

$$\sum_{k=j+1}^{\hat{j}} \mathrm{d}\Phi_j = 0,$$

and

$$\mathrm{d}\Phi_k = -\mathrm{d}\Phi_{j+1}\left(\sum_{\tilde{j}=2}^{\hat{j}} \frac{\Phi_{\tilde{j}}^2}{\pi(t_{\tilde{j}}) - \pi(t_{\tilde{j}+1})}\right)^{-1} \frac{\Phi_k^2}{\pi(t_k) - \pi(t_{k+1})}.$$

The same argument as above establishes that $\mathrm{d}\pi/\mathrm{d}\Phi_{j+1} > 0$.

Proof of Proposition 4

By construction, each y_k depends on t_{k+1} only through $\pi(t_{k+1})$, with

$$\frac{\mathrm{d}y_k}{\mathrm{d}\pi_{k+1}} = \frac{1}{y_k \pi''(y_k)},$$

where for simplicity we write π_{k+1} for $\pi(t_{k+1})$ for each $k = 1, \ldots, l-1$. It follows that changes in t_k affect π only in two terms in the summation of π, and only through π_k.

Consider the derivative of π with respect π_k, $k = 2, \ldots, l$, for $\pi_k \in [\pi_{k-1}, \pi_{k+1}]$. Evaluating the derivative at $\pi_k = \pi_{k-1}$, and using L'Hôptial's rule, we have

$$\begin{aligned}\left.\frac{\mathrm{d}\pi}{\mathrm{d}\pi_k}\right|_{\pi_k=\pi_{k-1}} = &\left(\frac{(1-\pi'(y_k))^n - (1-\pi'(y_{k-2}))^n}{\pi'(y_{k-2}) - \pi'(y_k)} - n(1-\pi'(y_{k-2}))^{n-1}\right) \\ &\cdot \left(1 - \frac{\pi_{k-1} - \pi_{k+1}}{y_{k-2}(\pi'(y_{k-2}) - \pi'(y_k))}\right).\end{aligned}$$

We have

$$\begin{aligned}(1-\pi'(y_k))^n - (1-\pi'(y_{k-2}))^n &= (1-\pi'(y_{k-2}) + \pi'(y_{k-2}) - \pi'(y_k))^n - (1-\pi'(y_{k-2}))^n \\ &> n(1-\pi'(y_{k-2}))^{n-1}(\pi'(y_{k-2}) - \pi'(y_k)),\end{aligned}$$

so the term in the first bracket is strictly positive. By the definitions of y_{k-2} and y_{k-1}, the term in the second bracket has the same sign as

$$y_{k-2}(F(y_k) - F(y_{k-2})) - y_k f(y_k)(y_k - y_{k-2}).$$

For fixed y_k, the above is equal to 0 if $y_{k-2} = y_k$. As the function of y_{k-2}, the above is increasing for all $y_{k-2} < y_k$, because the derivative with respect to y_{k-2} is $\pi'(y_{k-2}) - \pi'(y_k)$, which is positive by the strict concavity of $\pi(\cdot)$, because $y_{k-2} < y_k \leq p^*$. We conclude that the derivative of π with respect to π_k at $\pi_k = \pi_{k-1}$ is strictly negative. As a result, π is strictly increasing in t_k at $t_k = t_{k-1}$. A symmetric argument establishes that the derivative of π with respect to π_k at $\pi_k = \pi_{k+1}$ is strictly positive, imply that π is strictly decreasing in t_k at $t_k = t_{k+1}$. The proposition follows immediately.

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