Econ 421
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Problem Set 8

1. Exercise 296.1 (Auctions with risk-averse bidders)

2. Exercise 299.2 (First-price sealed-bid auction with common valuations)

3. (*Must love dogs*) MLD is an online platform that randomly matches dog-owners who are willing to pay someone to take their dogs for a walk and dog-walkers who are potentially interested in providing a one-time service. A dog-owner of course knows her valuation V for the service, but not the cost C of providing the service for any dog-walker she is matched with. Likewise, a dog-walker knows his cost C, but not the valuation V of any dog-owner he is matched with. After they are randomly matched by MLD, the dog-owner has to enter online how much she is bidding for the service and simultaneously the dog-walker enters how much he is asking for it. If the bid B is less than or equal to the ask A, MLD informs both parties that there is no deal. In this case, both the owner and the walker receive a payoff of 0. If instead $B > A$, MLD informs the two parties that there is a deal. MLD charges the dog-owner A and pays the dog-walker B. The payoff is $V - A$ to the dog-owner, and $B - C$ to the dog-walker.

(a) Argue that it is weakly dominant for the dog-owner to enter her valuation V, and for the dog-walker to enter his cost C.

Now suppose that MLD adopts a new platform. If $B \leq A$, there is still no deal. But if $B > A$, MLD charges the dog-owner $\frac{1}{2}(B + A)$ and pays the dog-walker $\frac{1}{2}(B + A)$.

To answer question (b) below, you should assume that the dog-owner knows that C is uniformly distributed between 0 and 1, and the dog-walker knows that V is uniformly distributed between 0 and 1.

(b) Show that it is a Bayesian Nash equilibrium for the dog-owner to bid $\frac{2}{3}V + \frac{1}{12}$, and for the dog-walker to ask $\frac{2}{3}C + \frac{1}{4}$.

4. (*First price auction with many potential buyers*) A seller of an indivisible object faces n potential buyers with independent valuations. The seller's own valuation is 0. The seller knows that each buyer's valuation is v_H with probability p_H and $v_L < v_H$ with probability $p_L = 1 - p_H$. Consider the following sealed bid first price auction: Bidders submit bids independently; if there is a single bidder with the highest bid, then he gets the object and pays his bid; if $m \leq n$ bidders bid the same highest bid, then each of the m bidders gets the object with probability $1/m$ and pays the bid.

(a) Set up the auction as a static Bayesian game among the n potential buyers.

(b) Show that there is no symmetric pure strategy Bayesian Nash equilibrium in this game. (Hint: By way of contradiction, suppose that $(b^*(v_H), b^*(v_L))$ is a symmetric Bayesian Nash equilibrium. First show that $b^*(v_L) \leq v_L$. Then show that $b^*(v_H) < v_H$. Then show that type v_H can gain by bidding instead just above $b^*(v_H)$.)

(c) Show the following strategy profile is a mixed strategy symmetric Bayesian Nash equilibrium: Type v_L bids v_L; type v_H randomizes over $(v_L, \bar{b}]$, where $\bar{b} = v_H - p_L^{n-1}(v_H - v_L)$, in such a way that $F(b)$, the probability that he bids below any $b \in (v_L, \bar{b}]$ satisfies

$$(v_H - b)(p_L + p_H F(b))^{n-1} = p_L^{n-1}(v_H - v_L).$$

(Hint: You have to show that for type v_H bidding any $b \notin (v_L, \bar{b}]$ is not optimal, and that he is indifferent among all bids $b \in (v_L, \bar{b}]$. Use the fact that bidders have independent evaluations, and the fact that the bidders bid independently with their proposed mixed strategies. You may also have to use the following binomial formula: $(a+b)^k = \sum_{r=0}^{k} k!/(r!(k-r)!)a^{k-r}b^r$.)

Note that the result here applies to any $n \geq 2$. Now consider how the mixed strategy Bayesian equilibrium in (c) changes with n.

(d) Show that as n increases, type v_H bids more aggressively. That is, type v_H randomizes over a bigger interval as n increases (i.e. $\bar{b}$ increases with n), and for any $b \in (v_L, v_H)$, the probability that type v_H bids above b increases with n. Give an intuitive explanation to this result.

(e) Show that the expected total surplus (including n buyers' and the seller's) is $p_L^n v_L + (1 - p_L^n)v_H$. Show that as n increases, the total expected surplus of n potential buyers decreases and the seller's surplus from the auction increases. Give an intuitive explanation to this result.

As in the case $n = 2$, first price auction is far from the optimal mechanism for the seller. Consider the following direct mechanism: Potential buyers are asked to report their valuations independently; if all report v_L, then with probability $1/n$ each buyer pays v_L and gets the object; if there is only one potential buyer who reports v_H, then he pays $v_H - (v_H - v_L)/n$ and gets the object; if $m \geq 1$ potential buyers report v_H, then with probability $1/m$ each of the m buyers pays v_H and gets the object.

(f) Show that truth-telling is a Bayesian Nash equilibrium in the above direct mechanism.

(g) Compute the seller's expected surplus and show that this surplus is always greater than his surplus from first price auction that you have found in (e). Would the seller use the above direct mechanism if $v_L < p_H v_H$?

5. (*Email game*) Maria of UBC and her boyfriend James of SFU plan a meeting at 6 this coming Friday afternoon, at either Granville Island or Metrotown. Depending on Maria's schedule in the early afternoon, one of the two places is more convenient for them than the other. Maria and James choose independently to go to Granville Island or Metrotown (neither of them has a cellular phone). Each gets a payoff of 2 if both go to the more convenient place, and 0 if they meet at the inconvenient place. If the two go to different places, the one that goes to Granville Island gets a payoff of 1 and the other gets a payoff of -3 (because the agreement is that if they don't see each other at 6, the one who has gone to Metrotown has to go to Granville Island to meet his/her partner.) For example, if the more convenient place is Metrotown and Maria goes to Metrotown but James shows up at Granville Island, then Maria gets a payoff of -3 and James gets 1.

(a) Show that if it is common knowledge between Maria and James that Granville Island is the more convenient place to meet (that is, both Maria and James know that Granville Island is more convenient, both know that each other knows this, and both know that each other knows that both know this, etc.), then there is a unique Nash equilibrium in this game of complete information and both go to Granville Island. (Hint: use strict dominance.) Show that if it is common knowledge that Metrotown is more convenient, there is a Nash equilibrium where both go to Metrotown.

Now suppose that initially only Maria knows which place is more convenient, and James knows only that with probability $p \leq .5$ Metrotown is more convenient and with probability $1 - p$ Granville Island is more convenient. Furthermore, suppose that it is impossible for

Maria to tell James which is more convenient (she learns which is more convenient on Friday morning but she is busy before 6 and neither has cellular phones.)

(b) Set up the above game as a static Bayesian game. Show that there is a unique Bayesian Nash equilibrium and both go to Granville Island in equilibrium. (Hint: First show that for Maria going to Granville Island strictly dominates going to Metrotown when Granville Island is more convenient. Given that Maria goes to Granville Island when it is more convenient, show that James gets an expected utility of at least $2(1-p)$ by going to Granville Island, and gets at most $-3(1-p)+2p$ by going to Metrotown, which is smaller. Finally, show that given that it is optimal for James to go to Granville Island, Maria goes to Granville Island even when Metrotown is more convenient.)

For the rest of this problem, suppose that Maria and James can communicate to each other via the following *automatic* email system. If Maria learns that Metrotown is more convenient, her computer *automatically* sends a message to James' computer. If Maria learns that Granville Island is more convenient, no email message is sent. If a computer receives a message then it *automatically* sends back a confirmation message; this is so not only for the original message but also for the confirmation, the confirmation of confirmation, and so on. The email system is designed to send confirmations because the technology has the property that there is a small probability $\epsilon > 0$ that any given message does not arrive at its intended destination. If a message does not arrive then the communication stops. At the end of the communication phase each player's screen displays the number of messages that his/her computer has sent. (Each player observes the number on his/her computer screen only.)

If the number displayed in James' computer screen is 1, he knows that his computer has sent 1 message, but he does not know whether the number on Maria's screen is 1 (because his message did not reach Maria's computer) or 2 (because Maria's second message did not reach his computer). Therefore, James knows that Metrotown is more convenient, but he does not know whether Maria knows that he knows it. If the number is instead 2, James knows that the number on Maria's computer screen is either 2 or 3. In this case, James knows that Metrotown is more convenient, knows that Maria knows that he knows it, but he does not know whether Maria knows that he knows that she knows that he knows it. Since ϵ is very small, if Maria learns that Metrotown is more convenient, the confirmation process can go on for a long time before it stops. A very big number on the screen indicates to the player that it is "almost" common knowledge that Metrotown is the more convenient meeting place. From question (a) we know that there is a Nash equilibrium where both go to Metrotown if it is common knowledge that Metrotown is more convenient, and so we may conjecture that with this email system of confirmation, if the numbers displayed on the computer screens are very big, both players should go to Metrotown. However, we will show below that this conjecture is false. No matter how big the numbers are, Maria and James will go to Granville Island, which from question (b) is what they would do in the absence of any communication technology. Thus, the above email system is useless as a means of communication.

(c) Set up the game as a static Bayesian game. (Hint: The type space of each player is set of all natural numbers, $\{0, 1, \ldots, \}$, corresponding to the number of messages sent by his/her computer. The distributions of types are not independent. If Maria sees a number q_1 on her computer screen, then, the number on James' computer screen is either $q_1 - 1$ or q_1.)

(d) Show that in any Bayesian Nash equilibrium Maria chooses Granville Island if the number on her computer screen is 0. (Hint: Show that Granville Island strictly dominates Metrotown for Maria when she sees 0 on her computer screen.)

(e) Show that given (d), in any equilibrium James chooses Granville Island if he sees 0 on his computer screen. (Hint: Show that conditional on seeing the number 0 on his computer screen, James assesses a probability $(1-p)/(1-p+p\epsilon)$ to the event that Maria did not send the message, and probability $p\epsilon/(1-p+p\epsilon)$ to the event that Maria sent the message but he did not get it. Show that given these probabilities, it is optimal for James to choose Granville Island.)

Questions (d) and (e) have completed the first step in mathematical induction in proving that both will go to Granville Island no matter how greater the numbers they see on their computer screens. Assume now that we have shown that for all (q_1, q_2) with $q_1 + q_2 < 2Q$ both Maria and James choose Granville Island, where q_1 is the number Maria sees on her screen and q_2 is the number that James sees on his screen, and Q is the induction index number. The following two questions complete the proof by induction that Maria and James choose Granville Island no matter how great the numbers on their computer screens are.

(f) Show that Maria chooses Granville Island when the number on her computer screen is Q. (Hint: Maria knows that the number on James' computer screen is either $Q-1$ or Q. Show that Maria assigns probability $\epsilon/(\epsilon+(1-\epsilon)\epsilon) > \frac{1}{2}$ and $(1-\epsilon)\epsilon/(\epsilon+(1-\epsilon)\epsilon) < \frac{1}{2}$ to the two events respectively. Given the induction assumption James chooses Granville Island in the first event, show that Maria's optimal choice is Granville Island, regardless of James' choice in the second event.)

(g) Assume that we have show that for all (q_1, q_2) with $q_1 + q_2 < 2Q + 1$ both Maria and James choose Granville Island. Show that James chooses Granville Island when the number on his computer screen is Q.