

Midterm Practice Questions

1. (*A trading game*) A buyer and a seller bargain over an object. The object is worth 0 to the seller and 3 to the buyer. After failing to agree on a price, the buyer and the seller decide to adopt the following trading rule. Both the buyer and the seller simultaneously name one of four possible prices 0,1,2 or 3. If the price named by the buyer (bid price) is at least as large as the price named by the seller (ask price) they will trade at a price that is the average of the bid and the ask price. The buyer's payoff is equal to his valuation, which equals 3, minus the price paid. The seller's payoff is just the price. (For example, if the ask price is 2 and the bid price is 3, the buyer acquires the object at a price equal to $(2 + 3)/2 = 2.5$, with the buyer's payoff equal to 0.5 and the seller payoff equal to 2.5.) If the bid price is strictly smaller than the ask price, there is no trade. Both the buyer and the seller receive a payoff of 0.

(a) Find all pure strategy Nash equilibria of the game.

(b) Find a mixed strategy Nash equilibrium in which the buyer and the seller name the price 1 and the price 2 with strictly positive probability.

2. (*Wage competition*) Firm 1 and Firm 2 compete for workers by simultaneously announcing their wages w_1 and w_2 respectively. Output per worker is 1 for each firm, and wages must be non-negative. Suppose a fraction t of these workers are informed about this industry and these workers will only work for the firm with a higher wage. If the wages are the same, the informed workers will work for either firm with equal probability. The

remaining $(1 - t)$ fraction of the workers are uninformed about this industry and they will work for either firm with equal probability. Thus, if $w_1 < w_2$, the payoff to Firm 1 is $(1 - w_1)(1 - t)/2$ and the payoff to Firm 2 is $(1 - w_2)(t + (1 - t)/2)$, and if $w_1 = w_2 = w$, the payoff to each firm is $(1 - w)(t/2 + (1 - t)/2) = (1 - w)/2$.

(a) Suppose that $t = 0$. Find the Nash equilibrium of this game.

(b) Suppose that $t = 1$. Find the Nash equilibrium of this game.

(c) Suppose that $0 < t < 1$. Find all the pure strategy Nash equilibrium if they exist and show your work. Otherwise show that there does not exist a pure strategy Nash equilibrium.

3. (*Location of Cash Machines*) Two banks, CIBC and RBC, each choose a location for its cash machine on a street block. Imagine that street addresses are represented by a real number on the interval $[0, 1]$. For example, the two end-points of the street have addresses of 0 and 1, and the mid-point has an address of $\frac{1}{2}$. Given the locational choices of CIBC and RBC, say a_C and a_R , each resident of the street chooses the bank with the closer cash machine for his banking needs. For example, if $a_C < a_R$, then the mid-point between the two cash machines is $\frac{1}{2}(a_C + a_R)$, so residents with addresses between 0 and $\frac{1}{2}(a_C + a_R)$ find CIBC's cash machine closer and bank with CIBC, and those with addresses between $\frac{1}{2}(a_C + a_R)$ and 1 bank with RBC. Imagine that residents are evenly located on the street block. Then, if $a_C < a_R$, the share of CIBC's banking business is $\frac{1}{2}(a_C + a_R)$, and the share of RBC is $1 - \frac{1}{2}(a_C + a_R)$. We assume that if $a_C = a_R$ (i.e. the two banks choose the same address for their cash machines), half of the residents choose CIBC and the other half choose RBC so each bank's share of business is $\frac{1}{2}$. Each bank chooses cash machine location to maximize its banking business share from the street.

- (a) Suppose that due to city zoning regulations, CIBC's only two locational choices are $\frac{1}{6}$ and $\frac{5}{6}$, and RBC's only two choices are $\frac{1}{3}$ and $\frac{2}{3}$. Suppose that the locational choices of the two banks are made simultaneously. Model the situation as a strategic game with two pure strategies for each bank. Is there a pure-strategy Nash equilibrium? Is there a mixed-strategy Nash equilibrium? Explain your answers.
- (b) Suppose that CIBC and RBC can choose any location on $[0, 1]$ for their cash machines and that the two banks make their choices simultaneously. Is the choice of either end-point of the street block a strictly dominated strategy for each bank? If yes, give a location that strictly dominates the choice of an end-point. If no, explain why.
- (c) Suppose that CIBC and RBC can choose any location on $[0, 1]$ for their cash machines and that the two banks make their choices simultaneously. Show that $a_C = a_R = \frac{1}{2}$ is a Nash equilibrium. Is there another pure-strategy Nash equilibrium? Explain.
- (d) Suppose first CIBC chooses a location for its cash machine and then RBC chooses a location after observing CIBC's choice. Suppose further that only 6 choices are available to the two banks, $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ and 1 . Use backward induction to find all subgame perfect equilibrium outcomes.
- (e) Suppose that three banks, CIBC, RBC and BOM, can choose any location on $[0, 1]$ for their cash machines and that the three banks make their choices simultaneously. As before, each resident chooses the bank with the closest cash machine; residents are evenly located on the street block; banks split the businesses evenly if more than one bank chooses the same address; each bank chooses cash machine location to maximize its banking business share. Show that there is no pure-strategy Nash equilibrium.
4. (*Majority voting*) There are three players, 1, 2, and 3, and three alternatives, A , B , and C .

For the first six questions, players vote simultaneously for an alternative. Abstaining is not allowed. The alternative with the most votes wins. For example, if the three votes are A , B , and B , then B is chosen. If no alternative receives a majority, which happens when the three votes are A , B , and C , then A is selected. A is called “status quo,” and B and C are “challengers.” Player 1 prefers A to B , and B to C . That is, A is player 1’s first choice, B the second, and C the last. Player 2 prefers B to C , and C to A ; player 3 prefers C to A , and A to B .

Given a strategy profile, a player is called a “swing voter” if he can change the outcome of the game by changing his vote. For example, if the profile is (A, B, B) (meaning player 1 votes A , player 2 and 3 vote B), then player 2 and 3 are swing voters but player 1 is not; if the profile is (A, A, A) , then nobody is a swing voter; if the profile is (A, B, C) , then everyone is a swing voter.

- (a) Is it a Nash equilibrium where everyone votes for his first choice? How about everyone voting for his second choice?
- (b) Is there a Nash equilibrium where everyone is a swing voter?
- (c) Argue that for each player, voting for his last choice is weakly dominated by voting for his first choice. (Hint: discuss two cases: the player is not a swing voter, and the player is a swing voter.)
- (d) Show that for each alternative, say A , there is a Nash equilibrium where the alternative A is chosen and nobody is a swing voter.
- (e) Find another equilibrium where alternative A is chosen and only player 2 is not a swing voter.
- (f) Does there exist any other equilibrium where A is chosen? Explain.

For the next four questions, consider a two-stage game among the three players, with the same preferences. In the first stage, players choose a challenger by casting votes for either B or C , with the majority choice being the winner and abstention not allowed. At the second stage, players choose between the status quo A and whichever alternative was victorious in the first stage, with majority rule again determining the winner. Players vote simultaneously in each round. They care only about the alternative that is finally selected, and are indifferent as to the sequence of votes that leads to a given selection.

(g) Show that the status quo A will be chosen under “sincere voting,” that is, if at each stage the players voted for the alternative they would prefer as the final outcome. (Hint: sincere voting means that player 2 chooses B over C in the first stage since he prefers B to C as the final outcome.)

(h) Show that sincere voting is not a Nash equilibrium. (Hint: show that player 2 will not vote sincerely when others vote sincerely.)

(i) Construct a subgame perfect equilibrium where A is chosen as the final outcome.

(j) Find the subgame perfect equilibrium where C is chosen as the final outcome.

5. (*Shark, fish and shrimp*) Three firms play a quantity game in a market of a homogenous product. The demand function is given by $p = a - \sum_{i=1}^3 q_i$, where p is the price and q_i is the quantity produced by firm i , with $a > 0$. Marginal cost for each firm is $c < a$; there is no fixed cost. The game has three stages. Firm 1 chooses a quantity q_1 in the first stage, firm 2 observes the choice q_1 and then chooses q_2 in the second stage, and firm 3 observes both q_1 and q_2 before it chooses q_3 in the third stage. The game then ends with each firm receiving profits as its payoff.

(a) Describe all subgames.

(b) Show that the following strategy profile is a Nash equilibrium: Firm 1 produces 0; firm 2 produces $(a - c)/2$ if firm 1 produces 0 and $(a - c)$ otherwise; firm 3 produces $(a - c - q_1 - q_2)/2$ if $q_1 + q_2 < a - c$, and 0 otherwise.

(c) Argue that the strategy profile given in (b) is not subgame perfect.

(d) Find the subgame perfect equilibrium of the game. What is the price in the subgame perfect equilibrium?

6. (*Inattention in duopoly*) Two firms produce homogenous goods in the same market. Firm 1 chooses quantity first. With probability $\alpha \in (0, 1)$, Firm 2 makes its own quantity choice without observing Firm 1's choice; with the complementary probability $1 - \alpha$, Firm 2 makes its quantity choice after observing Firm 1's choice. If Firm 1 produces q_1 and Firm 2 produces q_2 , then the price p is $A - (q_1 + q_2)$ when $q_1 + q_2 \leq A$ and 0 otherwise, where $A > 0$ is a positive constant. With the price thus determined, Firm 1's payoff is pq_1 and Firm 2's payoff is pq_2 (there is no production cost).

(a) Find the unique subgame perfect equilibrium of the game when α is strictly between 0 and 1. By considering how it changes with α , relate this equilibrium to the Nash equilibrium of the corresponding Cournot duopoly game and to the subgame perfect equilibrium of the corresponding Stackelberg duopoly game.

(b) Now suppose that Firm 2 does not observe Firm 1's quantity (that is, $\alpha = 0$), but it can incur a small cost $k > 0$ to observe Firm 1's quantity before making its own choice. (The analysis should not depend on the precise magnitude of k so long as it is strictly positive and sufficiently small.) If Firm 2 chooses not to observe Firm 1's quantity, its payoff is $\max\{A - (q_1 + q_2), 0\}q_2$ from choosing q_2 when Firm 1 has chosen q_1 ; if Firm 2 observes that Firm 1 has chosen q_1 , its payoff is $\max\{A - (q_1 + q_2), 0\}q_2 - k$

from choosing q_2 . Find all pure-strategy subgame perfect equilibria of this game. Compare your answers to this question and question (a) above, and give an intuitive explanation of the comparison.

7. (*Tax avoidance and tax evasion*) A taxpayer (P) chooses between *Cheat* and *Don't*, and an auditor (A) chooses between *Audit* and *Don't*. Choosing *Don't* for P means reporting his entire income for tax at a fixed rate of τ (a parameter strictly between 0 and 1), with a payoff of $1 - \tau$ regardless of A chooses *Audit* or *Don't*. Choosing *Cheat* means P reporting zero income, with a payoff of 1 if A chooses *Don't*, and a payoff of 0 if A chooses *Audit* (the entire income is confiscated as penalty for cheating). The cost of choosing *Audit* is κ for A, also a parameter strictly between 0 and 1. For questions (a) and (b), suppose that P and A play a simultaneous-move game, with the following payoff table:

		A	
		<i>Audit</i>	<i>Don't</i>
P	<i>Cheat</i>	0, $1 - \kappa$	1, 0
	<i>Don't</i>	$1 - \tau$, $\tau - \kappa$	$1 - \tau$, τ

- (a) Does there exist a pure-strategy Nash equilibrium? Explain your answer. [Hint: Use the assumptions of $0 < \tau < 1$ and $0 < \kappa < 1$.]
- (b) Find the mixed-strategy Nash equilibrium. Does the equilibrium payoff of P depend on κ ? Explain your answer.

For questions (c)-(e), suppose that P can engage in costly tax avoidance by making it more difficult for A to audit his income. Specifically, consider a two-stage game where in the first stage P chooses the amount of cost c he incurs in order to add c to A's cost of choosing *Audit*. For each c he chooses, in the second stage he plays a simultaneous-move game with A, given by

		A	
		<i>Audit</i>	<i>Don't</i>
P	<i>Cheat</i>	$-c, 1 - \kappa - c$	$1 - c, 0$
	<i>Don't</i>	$1 - \tau - c, \tau - \kappa - c$	$1 - \tau - c, \tau$

We use backward induction to find all subgame perfect equilibria in the two-stage game. There are two kinds of subgames.

- (c) For all subgames where $c < 1 - \kappa$, show that there is a unique Nash equilibrium and it is in mixed strategies. How does the equilibrium payoff of P depend on c ? [Hint: You may want to use your answers to (a) and (b).]
- (d) For all subgames where $c \geq 1 - \kappa$, use strict and weak dominance to show that there is a unique Nash equilibrium and it is in pure strategies. How does the equilibrium payoff of P depend on c ?
- (e) [20 points] Find the unique subgame perfect equilibrium outcome when $\kappa > 1 - \tau$, and when $\kappa < 1 - \tau$.

For the remaining questions, we return to the setup of simultaneous moves by P and A, and suppose that P can choose any fraction $r \in [0, 1]$ of the income to report for taxation. We thus have a simultaneous-move game in which P has a continuum of strategies and A still has the same two strategies, *Audit* and *Don't*. When A chooses *Audit*, the probability of P is caught and loses all income is $1 - r$, and with the remainder probability P gets away from paying only tax on the reported income, and so P's payoff is $r(1 - r\tau)$, and A's payoff is $1 - r + r^2\tau - \kappa$. When A chooses *Don't*, P's payoff is $1 - r\tau$ and A's payoff is $r\tau$. Note that $r = 0$ corresponds to *Cheat* in questions (a) and (b), and $r = 1$ corresponds to *Don't*.

- (e) Show that when $\tau > 1/2$ and $\kappa < (2\tau - 1)/(4\tau)$, it is a pure-strategy Nash equilibrium for P to choose $r = 1/(2\tau)$ and A to choose *Audit*. [Hint: What is P's best response to *Audit* by A? You are free to use the fact that $r(1 - r\tau)$ is maximized at $r = 1/(2\tau)$ when $\tau > 1/2$.]
- (f) Show that when $\tau < 1/2$, or when $\tau > 1/2$ but $\kappa > (2\tau - 1)/(4\tau)$, there is a Nash equilibrium where A randomizes between *Audit* and *Don't*, and P plays a pure strategy. [Hint: A mixes in such a way that, when P plays the best response to it A is indifferent between *Audit* and *Don't*].

8. (*Rational escalation*) Two countries, C and P, vie for an uninhabited island in the middle of the ocean between them. The island is worth $V > 0$ to both countries. For questions (a)-(f) below, in any time period $t = 1, \dots, T$, with $T \geq 2$, the two countries simultaneously choose to escalate, or quit. Escalating at any t requires the country to commit to 1 additional unit of resources without actually consuming it; quitting means the loss of the total amount of resources previously committed, which is equal to $t - 1$. As a result, the game ends if both countries quit at t , with each getting a payoff of $-(t - 1)$, or if only one country quits, which gives that country a payoff of $-(t - 1)$ and the other country V . When both escalate, the game continues to the next period $t + 1$ if $t \leq T - 1$, and if $t = T$, the game ends with a catastrophe, with a payoff of $-L < -(T - 1)$ to each country — L is the loss in a catastrophe, and is assumed to be independent of T .

We will find the subgame perfect equilibria of this game by backward induction. The simultaneous-move game in period T is the smallest subgame, with a game table:

		P	
		<i>Quit</i>	<i>Escalate</i>
C	<i>Quit</i>	$-(T-1), -(T-1)$	$-(T-1), V$
	<i>Escalate</i>	$V, -(T-1)$	$-L, -L$

- (a) Find the two pure-strategy Nash equilibria in the subgame in period T . What is the probability of catastrophe in each equilibrium?
- (b) Find the symmetric mixed-strategy Nash equilibrium in the subgame at T . Show that the equilibrium payoff is $-(T-1)$. What is the equilibrium probability of catastrophe?

Now let's rollback to period $T-1$. Consider the subgame where both countries have already committed to $T-2$ units of their resources. By assumption, the payoff to a country that chooses *Quit* is $-(T-2)$ regardless of what the other country chooses. If both countries choose *Escalate*, the game continues to the period- T subgame that we have already analyzed. Since there are three Nash equilibria the period- T subgame, we do backward induction separately.

- (c) Find all subgame perfect equilibria in the subgame starting in period $T-1$, using each of the pure-strategy equilibria you have found in question (a). Be sure to state each subgame perfect equilibrium as a profile of complete plans.
- (d) Find a symmetric subgame perfect equilibrium in the subgame starting in period $T-1$, where both countries mix between *Escalate* and *Quit* in period $T-1$, using the mixed strategy equilibrium you have found in question (b). Show that the equilibrium payoff is $-(T-2)$. What is the equilibrium probability of catastrophe?

If $T = 2$, we have already finished analyzing the game. Now, suppose that $T \geq 3$. [Hint: You may want to skip (e) and (f), and answer the remaining questions before coming back when you have time at the end.]

- (e) Find two pure-strategy subgame perfect equilibria. Does the outcome in each equilibrium depend on the potential length of game T ?
- (f) Find a symmetric subgame perfect equilibrium in which both countries mix between *Escalate* and *Quit* in each period $t = 1, \dots, T$. Does the equilibrium payoff to each country depend on the potential length of game T ? Does the equilibrium probability of catastrophe depend on T ?

For the remaining questions, we consider a simultaneous-move version of the escalation game, where the two countries, C and P, choose the amount of resources they commit to, at the same time and once for all. Let t_C and t_P be the choice of C and P respectively, which can take any integer value from $0, 1, \dots, T$. We continue to assume that $T \geq 2$, so there are at least 3 strategies for each country in the game. If $t_C > t_P$, then C's payoff is V and P's payoff is $-t_P$; that is, as in the escalation game we have analyzed above, country C claims the island without actually consuming the resource t_C it has committed to, while P loses all its committed resources. Conversely, if $t_C < t_P$, then C's payoff is $-t_C$ and P's payoff is V . If $t_C = t_P = t \leq T - 1$, neither country claims the island and each gets a payoff of $-t$; this corresponds to the outcome of both countries quitting at the same time in the escalation game. Finally, if $t_C = t_P = T$, catastrophe strikes and the payoff to each country is $-L < -(T - 1)$.

- (g) Is any strategy strictly or weakly dominated for each country?
- (h) Show there is no pure-strategy equilibrium (t_C, t_P) such that $1 \leq t_C \leq T - 1$ or $1 \leq t_P \leq T - 1$.

- (i) Find two pure-strategy Nash equilibria (t_C, t_P) where t_C and t_P are either 0 or T .
How does the outcome in each of the two equilibria compare with your answer in question (c) (assuming that $T = 2$) or (e) if you completed that question?
- (j) Prove by contradiction that in any symmetric mixed-strategy Nash equilibrium of the game, 0 is chosen with a positive probability, and the equilibrium payoff is 0. Then show that T is also chosen with a positive probability.
- (k) Is there a symmetric mixed-strategy Nash equilibrium in which both countries mix only between 0 and T ?
- (l) Use (j) to find the probability of each country choosing T in a symmetric mixed-strategy Nash equilibrium. How does the equilibrium probability of catastrophe differ from your answer in (c) assuming that $T = 2$ or (f) if you completed that question?