Econ 421
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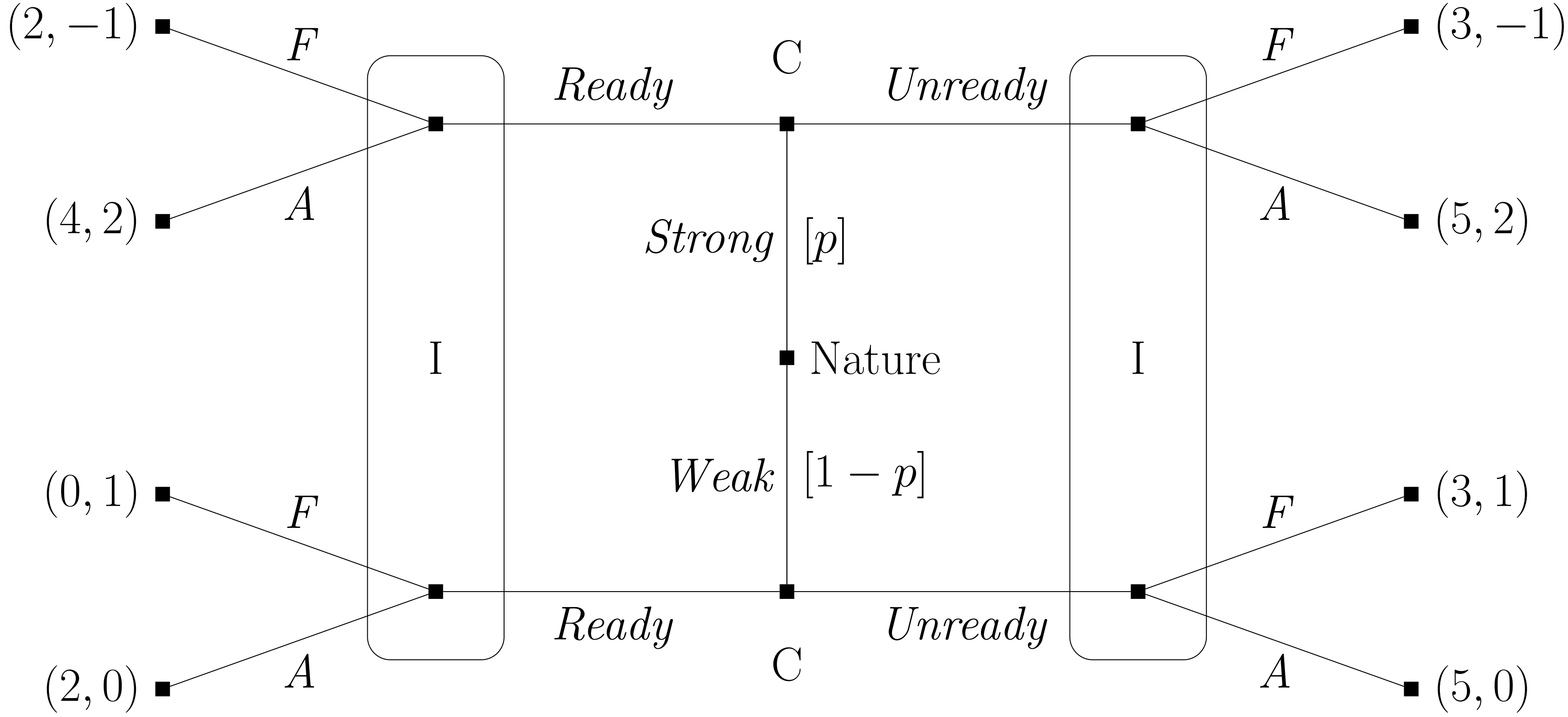
Lecture 9. Perfect Bayesian Equilibrium

1. Dynamic games of incomplete information

- Incomplete information in extensive games with observed actions.
- Observing an opponent's action allows a player to update his belief about the opponent's type.
- PBE refines BNE similarly to how SPE refines NE.

Example (Entry game under incomplete information) A challenger (C) contests an incumbent (I). C is strong with probability p and weak with probability $1 - p$, where $0 < p < 1$; it knows its type, but I does not. C may either ready itself for battle or remain unready. I observes the challenger's readiness, but not its type, and chooses whether to fight (F) or acquiesce (A). An unready C's payoff is 5 if the I acquiesces to its entry. Preparations cost a strong C 1 unit of payoff and a weak C 3 units, and fighting entails a loss of 2 units for each type. I prefers to fight (payoff 1) than acquiesce to (payoff 0) a weak C, and to acquiesce to (payoff 2) than to fight (payoff -1) a strong C.


(2, −1)
F
Ready
C
Unready
F
(3, −1)
(4, 2)
A
A
(5, 2)
Strong
[p]
I
Nature
I
Weak
[1 − p]
(0, 1)
F
F
(3, 1)
Ready
Unready
C
(2, 0)
A
A
(5, 0)

Incumbent observes Challenger's action but not the type.

- Would weak Challenger "bluff" by choosing *Unready*? Would strong Challenger be scared into choosing *Ready*?
- Should Incumbent believe Challenger to be strong if the latter chooses *Unready*? Should Incumbent believe Challenger to be weak if *Ready* is chosen?

- Above game is an example of costly signaling.
  - *Sender* with private information about type moves first and chooses a *message*.
  - *Receiver* observes the message, updates his *belief* about sender's type and chooses an action in response.
  - Signal is costly because it enters directly in sender's payoff.

- Costly signaling is special class of dynamic games with incomplete information.
  - Two players, two stages, and one-sided private information.
  - Sender's strategy specifies his choice of signal for each of his types, and receiver's strategy specifies a response for each signal from the sender.
  - BNE needs to be refined, because it does not check for best responses to signals not sent in equilibrium.
  - Perfect Bayesian equilibrium refines BNE.

2. Signaling games

- Two players: Sender, Receiver.

- Type space of Sender is T, with probability distribution p.

- Message space of Sender is M.

- Action space of Receiver is A.

- vNM payoff $u_S(m, a; t)$ for Sender and $u_R(m, a; t)$ for Receiver.

- Timing: Chance informs Sender of his type $t \in T$; Sender chooses message $m \in M$; Receiver observes m and chooses action $a \in A$.

Strategies in signal games.

- As in Bayesian games, Sender can condition his message on his own type, and so a pure strategy s of Sender is a mapping from T to M.

- As in multi-stage games with observed actions, Receiver conditions his action on the message from Sender (but not on the type of Sender), and so a pure strategy r of Receiver is a mapping from M to A.

Information set.

- Backward induction does not apply here because there is no proper subgame, but concept of information set allows for similar analysis.
  - Information set models information of Receiver, who observes Sender's signal but not Sender's type.
  - Each signal corresponds to an information set, where Receiver cannot distinguish histories (Nature's move) that share the same signal.

Information set in general.

- Information set is a general concept that can be used to model any imperfect information in the game, not just incomplete information generated by Naturer's move.
  - An information set for a given player is a collection of histories assigned to the player by the player function, which have the same action set, and cannot be distinguished by the player.
  - Strategy of the player must be adapted to information sets.
  - Information sets can represent simultaneous moves.

Bayesian Nash equilibrium in pure strategies: (s^*, r^*) such that

- s^* is Sender's optimal strategy against r^*.
  - For any alternative strategy s,

$$\sum_{t \in T} p(t) u_S(s^*(t), r^*(s^*(t)); t) \geq \sum_{t \in T} p(t) u_S(s(t), r^*(s(t)); t).$$

  - Alternatively, for each $t \in T$ and each $m \in M$,

$$u_S(s^*(t), r^*(s^*(t)); t) \geq u_S(m, r^*(m); t).$$

- r^* is Receiver's optimal strategy against s^*.

  – For any strategy r of Receiver,

$$\sum_{t \in T} p(t) u_R(s^*(t), r^*(s^*(t)); t) \geq \sum_{t \in T} p(t) u_R(s^*(t), r(s^*(t)); t).$$

  – Alternatively, for each m such that there exists $t \in T$ with $s^*(t) = m$, and for each $a \in A$,

$$\sum_{t \in T : s^*(t) = m} p(t) u_R(m, r^*(m); t) \geq \sum_{t \in T : s^*(t) = m} p(t) u_R(m, a; t).$$

3. Perfect Bayesian equilibrium

- Bayesian Nash equilibrium has the same problem in a signaling game as Nash equilibrium in an extensive game under complete information.
  - Nash equilibrium does not check any off-the-path part of an equilibrium strategy, and thus may not be subgame perfect.
  - In a BNE (s^*, r^*), Receiver may not play a best response to any m such that there is no $t \in T$ with $s^*(t) = m$.

- Refining Bayesian Nash equilibrium requires explicitly specifying Receiver's beliefs.
  - Beliefs have to be specified in a way consistent with Sender's equilibrium strategy, whenever possible.
  - In the same spirit of subgame perfect equilibrium, Receiver's belief must be specified for each possible message, including those that are not sent by Sender with positive probability (off-the-path).

- A *belief system* of Receiver is a mapping β from each message $m \in M$ to a probability distribution $\beta(\cdot|m)$ over T.

- Receiver's belief system β is *consistent* with Sender's strategy s if β is derived from p and s by Bayes rule whenever possible.

  – For any $m \in M$ such that there is $t' \in T$ with $s(t') = m$ (on-the-path), $\beta(t|m) = p(t)/\sum_{t' \in T: s(t')=m} p(t')$ if $s(t) = m$ and 0 otherwise.

  – For any $m \in M$ such that there is no t with $s(t) = m$ (off-the-path), $\beta(t|m)$ is unrestricted by consistency.

- Given β, Receiver's strategy r is *sequentially rational* if for each $m \in M$, we have $r(m) \in \arg\max_{a \in A} \sum_{t \in T} \beta(t|m) u_R(m, a; t)$.
  - Sequential rationality checks for best response of Receiver's strategy r against Sender's strategy s through beliefs β.
  - For any on-the-path message, Receiver's belief is derived from the prior p according to Bayes' rule, and sequential rationality coincides with optimality of corresponding part of Receiver's strategy against Sender's strategy in BNE.
  - For any off-the-path message, BNE does not check for best response, but β allows for checking sequential rationality.

- **Definition** (PBE in signaling games) A strategy profile (s^*, r^*) together with a belief system β is a PBE if:

  – s^* is optimal for Sender given r^*;

  – r^* is sequentially rational given β;

  – β is consistent with s^*.

- Mixed-strategy PBE, where $s^*(t)$ is a probability distribution over M for some t and/or $r^*(m)$ is a probability distribution over A for some m, can be defined analogously.

- *Separating equilibrium*: perfect Bayesian equilibrium (s^*, r^*, β) such that $t \neq t'$ implies $s^*(t) \neq s^*(t')$.

- Separating equilibrium in the entry signaling game.

  - C's strategy: $s^*(Weak) = Unready$, $s^*(Strong) = Ready$.

  - I's strategy: $r^*(Unready) = F$, $r^*(Ready) = A$.

  - Beliefs: $\beta(Strong|Unready) = 0$, $\beta(Strong|Ready) = 1$.

- Verifying PBE.

- No other separating equilibrium where $s^*(Weak) = Ready$ and $s^*(Strong) = Unready$.

- *Pooling equilibrium*: perfect Bayesian equilibrium (s^*, r^*, β) such that $s^*(t)$ is same for all $t \in T$.

- Pooling equilibrium in the entry signaling game when $p \leq \frac{1}{4}$.

  – $s^*(Weak) = s^*(Strong) = Unready$;

    $r^*(Unready) = r^*(Ready) = F$;

    $\beta(Strong|Unready) = p$, $\beta(Strong|Ready) \leq \frac{1}{4}$.

  – Verifying PBE.

- Pooling equilibria in the entry signaling game when $p \geq \frac{1}{4}$.

  - $s^*(Weak) = s^*(Strong) = Unready$;

    $r^*(Unready) = r^*(Ready) = A$;

    $\beta(Strong|Unready) = p$, $\beta(Strong|Ready) \geq \frac{1}{4}$.

  - $s^*(Weak) = s^*(Strong) = Unready$;

    $r^*(Unready) = A$, $r^*(Ready) = F$;

    $\beta(Strong|Unready) = p$, $\beta(Strong|Ready) \leq \frac{1}{4}$.

- No pooling equilibrium where $s^*(Weak) = s^*(Strong) = Ready$.

- *Semi-separating equilibrium*: perfect Bayesian equilibrium (s^*, r^*, β) such that there is some type $t \in T$ (separating type) that uses some message $m \in M$ exclusively (separating message) and pools on another message $m' \in M$ with some other type $t' \in T$.

- Semi-separating equilibrium in entry signaling game when $p > \frac{1}{4}$.

  - $s^*(Weak) = Unready$; $s^*(Strong) = Ready$ with x; $r^*(Ready) = A$ and $r^*(Unready) = F$ with probability $\frac{1}{2}$; $\beta(Strong|Ready) = 1$, $\beta(Strong|Unready) = \frac{1}{4}$.

  - Finding x and Verifying PBE.

- There is no other semi-separating equilibrium.
  - Only *Strong* type can be separating type.
  - Only *Ready* can be separating message.
  - For *Strong* type to mix between *Ready* and *Unready*, I must randomize between F and A in response to *Unready*.
  - For I to mix between F and A, $\beta(Strong|Unready) = \frac{1}{4}$.
  - There is a unique value of $x \in (0, 1)$ for which Bayes' rule leads to $\beta(Strong|Unready) = \frac{1}{4}$, and only when $p > \frac{1}{4}$.