Econ 421
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Lecture 8. Auctions and Mechanism Design

1. Auctions

- Auction is a commonly used way of allocating indivisible goods among interested buyers.
  - Used cameras, Salvator Mundi, and spectrum auctions.
  - Online platforms (Amazon, eBay) have increased popularity of auctions in the modern digital economy.

Classification of auctions.

- Open outcry versus sealed bid.
  - Best known open outcry: English, Dutch auctions.
- First-price versus second-price.
  - In sealed bid auctions, highest bidder wins but price depends on rule.
- Private value versus common value.
  - Distinction in auction environment rather than rules.

1.1. Private value auctions

- Second price, sealed bid auctions with private values.
  - Each bidder i, $i = 1, \ldots, N$, values an object for sale at v_i; each i knows own valuation v_i, but not any other v_j, $j \neq i$; each i submits a bid b_i independently; bidder i wins the auction if b_i is higher than all other b_j, $j \neq i$, wins with equal probability if b_i is among the highest, and otherwise loses; payoff to each i is $v_i - p$ if i wins, where p is the highest losing bid, and 0 otherwise.

- This is a complex Bayesian game.
  - To set it up, we will need to specify what each bidder i knows about how each v_j, $j \neq i$, is distributed.
  - Type of each bidder i is own valuation v_i.
  - A bidding strategy of each i specifies bid b_i depending on v_i.
- Regardless how we specify the Bayesian game, there is a weakly dominant strategy for each i: bidding $b_i = v_i$ weakly dominates all other bids.

- Bidding one's own valuation is a weakly dominant strategy.
  - Fix any bidder i, and fix any valuation v_i.
  - Denote as b the highest outstanding bid; this is the price i pays if i wins the auction.
  - Bidding $b_i > v_i$ is weakly dominated by $b_i = v_i$: they give the same payoff when $b < v_i$, when $b = v_i$, and when $b > b_i$, but $b_i > v_i$ is strictly worse than $b_i = v_i$ when $v_i < b \leq b_i$.
  - Bidding $b_i < v_i$ is weakly dominated by $b_i = v_i$.

- First price, sealed bid auctions with private values.

  - Each bidder i, $i = 1, \ldots, N$, values an object for sale at v_i; each i knows own valuation v_i, but not any other v_j, $j \neq i$; each i submits a bid b_i independently; bidder i wins the auction if b_i is higher than all other b_j, $j \neq i$, wins with equal probability if b_i is among the highest, and otherwise loses; payoff to each i is $v_i - b_i$ if i wins.

- There is no weakly dominant bidding strategy.
  - Bidding one's own valuation is weakly dominated by bidding below it; so is bidding above it.
- Equilibrium bidding strategy involves shading the bid, i.e., bidding below one's own valuation.
  - To analyze how much one should shade the bid, we need to specify the Bayesian game in greater detail.

- A Bayesian game.
  - Suppose that $N = 2$.
  - Each bidder i, $i = 1, 2$, privately and independently draws valuation v_i from uniform distribution over interval $[0, 1]$.
  - Two properties of uniform distribution: the probability that v_i lies on any subinterval from $[0, 1]$ is given by the length of the interval; and the average is given by the mid point of the subinterval.

- A Bayesian Nash equilibrium: each bidder i uses bidding strategy $b_i = \frac{1}{2}v_i$.
  - Fix any bidder i, and fix any valuation v_i.
  - Any bid b_i wins when $b_i > b_j = \frac{1}{2}v_j$, i.e. when $v_j < 2b_i$, so b_i wins with probability $2b_i$, with expected payoff $2b_i(v_i - b_i)$.
  - The expected payoff is maximized by setting $b_i = \frac{1}{2}v_i$.

1.2. Common values and winner's curse

- Second price, sealed bid auctions with common values.
  - Each bidder i, $i = 1, 2$, receives a private estimate s_i between 0 and 1 of the value of the object for sale; each i observes s_i, and believes that s_j, $j \neq i$, is uniform between 0 and 1; each i's valuation $v_i = s_i + \alpha s_j$, with α a known constant between 0 and 1.
  - Parameter α represents degree of common value, with $\alpha = 1$ being the case of "pure" common value.

- Winner's curse.
    - Suppose each bidder i bids expected valuation given one's own estimate: $b(s_i) = s_i + \alpha\frac{1}{2}$.
    - Fix i and s_i.
    - Probability of winning is s_i.
    - Expected valuation conditional on winning is $s_i + \alpha\frac{1}{2}s_i$.
    - Expected price paid conditional on winning is $\frac{1}{2}s_i + \alpha\frac{1}{2}$.
    - Expected payoff conditional on winning is difference, which is negative for $s_i < \alpha/(1+\alpha)$.

- A Bayesian Nash equilibrium: $b(s_i) = s_i + \alpha s_i$.
  - Fix i and s_i, and consider any b_i.
  - Probability of winning is $b_i/(1+\alpha)$.
  - The expected valuation conditional on winning is given by $s_i + \alpha\frac{1}{2}b_i/(1+\alpha)$.
  - Expected price paid conditional on winning is $\frac{1}{2}b_i$.
  - Expected payoff, which is the probability of winning times the difference of the expected valuation and the expected price conditional on winning, is maximized at $b_i = (1+\alpha)s_i$.

2. Mechanism design

- Consider two-type, independent private value, first price auction.
- Is second price auction better than first price auction for the seller?
- Can any auction do better?
- What is the optimal auction?

2.1. First price auction

- Bayesian game of first price auction:
  - Players: Bidder 1 and Bidder 2.
  - Type space is $T_i = \{v_H, v_L\}$ for $i = 1, 2$, with independence: $p_i(v_H) = p_H$ and $p_i(v_L) = p_L$, for each $i = 1, 2$.
  - Action space $A_i = [0, \infty)$ for each $i = 1, 2$: denote a typical element as b_i.
  - For each $i \neq j = 1, 2$, payoff function $u_i(b_1, b_2; v_i)$ of Bidder i is $v_i - b_i$ if $b_i > b_j$, $\frac{1}{2}(v_i - b_i)$ if $b_i = b_j$, and 0 if $b_i < b_j$.

- Pure strategy Bayesian Nash equilibrium does not exist.
  - Suppose $(b^*(v_H), b^*(v_L))$ is a symmetric BNE.
  - $b^*(v_H)$ solves $\max_{b_1} p_H u_1(b_1, b^*(v_H); v_H) + p_L u_1(b_1, b^*(v_L); v_H)$, and $b^*(v_L)$ solves $\max_{b_1} p_H u_1(b_1, b^*(v_H); v_L) + p_L u_1(b_1, b^*(v_L); v_L)$.
  - $b^*(v_L) \leq v_L$ and $b^*(v_H) \leq v_H$.
  - $b^*(v_H) < v_H$.
  - Profitable unilateral deviation for type v_H to just above $b^*(v_H)$, a contradiction.

- Mixed strategies in Bayesian games are defined as in static games of complete information: m_i is a mixed strategy if $m_i(t_i)$ is a probability distribution over A_i for each $t_i \in T_i$.
  - Mixed strategy BNE $m^* = (m_1^*, \ldots, m_n^*)$ is defined in the same way in static games of complete information.
  - As in static games of complete information, in any mixed strategy Bayesian Nash equilibrium m^*, if type t_i of player i randomizes over A_i, then he must be indifferent among all actions that receive positive probabilities from m_i^* and weakly prefer any such action to all other actions in A_i.

- For present Bayesian game, we will show there is a symmetric mixed strategy Bayesian Nash equilibrium given by:
  - Type v_L bids v_L (pure strategy).
  - Type v_H randomizes according to some continuous function $F(b)$ on $(v_L, \overline{v}]$, where $\overline{v} = p_H v_H + p_L v_L$ and $F(b)$ denotes the probability bid is no greater than some b.

- For type v_H to mix among $(v_L, \overline{v}]$, he must be indifferent among all $b \in (v_L, \overline{v}]$.
  - For any $b \in (v_L, \overline{v}]$, type v_H's expected payoff is

$$p_L(v_H - b) + p_H F(b)(v_H - b).$$

  - Using $b = v_L$ we find $F(b) = [p_L(b - v_L)]/[p_H(v_H - b)]$.
  - Note $F(v_L) = 0$ and $F(\overline{v}) = 1$.
- Verify symmetric Bayesian Nash equilibrium.

- Bidders' equilibrium payoffs.
  - Type v_L's payoff is 0: regardless of getting the object or not.
  - Type v_H's payoff is $p_L(v_H - v_L)$: maximum payoff when meeting type v_L, and zero payoff when meeting type v_H.
- Seller's revenue is $(1 - p_H^2)v_L + p_H^2 v_H$.
  - Total surplus minus 2 times each bidder's expected payoff: $p_L^2 v_L + (1 - p_L^2)v_H - 2p_H p_L(v_H - v_L)$.

2.2. Other auctions

Second price auction: highest bid wins but pays the second highest bid.

- Bidding one's own valuation is weakly dominant regardless of the distribution of opponent's valuation.

- In the two-type example above, seller's expected revenue is same as in first-price auction: $p_L^2 v_L + 2 p_L p_H v_L + p_H^2 v_H = (1 - p_H^2) v_L + p_H^2 v_H$.

- Revenue equivalence: in both first price and second price auctions, types v_L and v_H get same payoffs.

- Consider the following game where bidders announce their type (instead of bidding):
  - If both bidders announce v_L, each pays v_L and gets object with probability $\frac{1}{2}$.
  - If both bidders announce v_H, each pays v_H and gets object with probability $\frac{1}{2}$.
  - If one announces v_H and the other v_L, the former wins and pays $\frac{1}{2}(v_H + v_L)$, and the latter pays nothing.

- Verify truth-telling is a Bayesian Nash equilibrium.
  - Type v_L is strictly better off reporting truthfully, rather than lying.
  - Type v_H is indifferent between telling the truth and lying.
- Seller's revenue is

$$p_L^2 v_L + 2 p_L p_H \frac{1}{2}(v_H + v_L) + p_H^2 v_H = p_H v_H + p_L v_L.$$

  - Greater than in first and second price auctions.

2.3. Optimal auction design problem

An optimization problem.

- Objective is to maximize the seller's revenue.
- Design instrument is *mechanism*, a Bayesian game for bidders.
  - Players, types, feasible allocations, payoff functions, are given.
  - Action space, assignment, payment rules are designed.
- Constraints on optimization problem
  - Bidders participate voluntarily (individual rationality), and play a BNE (incentive compatibility).

2.4. Mechanism design

- Mechanism design problem in general.
  - Players: $i = 1, \ldots, n$.
  - T_i: type space for each i, with $T = T_1 \times \ldots \times T_n$, and probability function p.
  - Y: set of feasible allocations.
  - $u_i(y; t)$: vNM payoff function of player i.

- Mechanism
  - A_i: action space for each player.
  - g: outcome mapping from $A_1 \times \ldots \times A_n$ to Y.
  - Stochastic mechanism: g maps action profiles to distribution over Y.
  - Direct mechanism: $A_i = T_i$ for each i, so each i announces his type.

- Any mechanism, direct or indirect, defines a Bayesian game.
  - Strategy s_i of each player i is a mapping from T_i to to A_i.
  - Outcome $g(s_1(t_1), \ldots, s_n(t_n))$ is a mapping from T to Y.
- **Proposition** (Revelation principle) If s^* is BNE in a mechanism $\langle (A_i), g \rangle$, then there is a truthful BNE in the direct mechanism $\langle (T_i), g(s^*) \rangle$.
  - Suffices to consider direct mechanisms.
  - Suffices to consider truthful BNE in direct mechanisms.

Proof. Since s^* is a Bayesian Nash equilibrium in a mechanism $\langle (A_i), g \rangle$,

$$s_i^*(t_i) \in \arg\max_{a_i} \sum_{t_{-i}} u_i(g(a_i, s_{-i}^*(t_{-i})); t_i, t_{-i}) p(t_{-i}|t_i)$$

for each i and each t_i. Then

$$t_i \in \arg\max_{t_i'} \sum_{t_{-i}} u_i(g(s_i^*(t_i'), s_{-i}^*(t_{-i})); t_i, t_{-i}) p(t_{-i}|t_i).$$

So truth-telling is a BNE in the direct mechanism $\langle (T_i), g(s^*) \rangle$. The outcome of this truth-telling equilibrium is $g(s^*)$, which is the same as the outcome in the BNE s^* in the original mechanism $\langle (A_i), g \rangle$.

3. Optimal auction

- In the two-bidder, two-value auction problem, denote a symmetric direct mechanism as $((x_{HH}, y_{HH}), (x_{HL}, y_{HL}), (x_{LH}, y_{LH}), (x_{LL}, y_{LL}))$.
  - $x_{tt'}$: probability of type t getting the object if announced type profile is $(v_t, v_{t'})$, where $t, t' = H, L$.
  - $y_{tt'}$: payment to the seller by type t if announced type profile is $(v_t, v_{t'})$, where $t, t' = H, L$.
- Objective: maximize $2(p_H(p_H y_{HH} + p_L y_{HL}) + p_L(p_H y_{LH} + p_L y_{LL}))$.

- Constraints:
  - Feasibility: $2x_{HH} \leq 1$, $2x_{LL} \leq 1$, $x_{HL} + x_{LH} \leq 1$.
  - Individual rationality IR_t for each type v_t, where $t = H, L$:

$$p_H(v_t x_{tH} - y_{TH}) + p_L(v_t x_{tL} - y_{tL}) \geq 0.$$

  - Incentive compatibility IC_t for each type v_t, $t' \neq t = H, L$:

$$\begin{aligned} & p_H(v_t x_{tH} - y_{TH}) + p_L(v_t x_{tL} - y_{tL}) \\ \geq & p_H(v_t x_{t'H} - y_{t'H}) + p_L(v_t x_{t'L} - y_{t'L}). \end{aligned}$$

- Rewrite optimal auction problem.
  - Define $X_t = p_H x_{tH} + p_L x_{tL}$, and $Y_t = p_H y_{tH} + p_L y_{tL}$ for each $t = H, L$.
  - Mechanism: $((X_t, Y_t), (X_t, Y_t))$.
  - Objective: $2(p_H Y_H + p_L Y_L)$.
  - Feasibility constraints: $X_H \leq \frac{1}{2}p_H + p_L$; $X_L \leq p_H + \frac{1}{2}p_L$; $p_H X_H + p_L X_L \leq \frac{1}{2}$.
  - IR$_t$: $v_t X_t - Y_t \geq 0$ for each $t = H, L$.
  - IC$_t$: $v_t X_t - Y_t \geq v_t X_{t'} - Y_{t'}$ for each $t \neq t' = H, L$.

- Constraints analysis
  - IR_L and IC_H imply IR_H.
  - IR_L binds at optimum.
  - IC_H binds at optimum.
  - IC_H and IC_L imply $X_H \geq X_L$.
  - $X_H \geq X_L$ and binding IC_H imply IC_L.

- Rewrite optimization problem.
  - IR_L binds: $Y_L = v_L X_L$.
  - IC_H binds: $Y_H = Y_L + v_H(X_H - X_L) = v_L X_L + v_H(X_H - X_L)$.
  - Choice variables are X_H and X_L.
  - Objective is

$$
\begin{aligned}
&2(p_H(v_L X_L + v_H(X_H - X_L)) + p_L v_L X_L) \\
=&2(v_H p_H X_H + (v_L - p_H v_H) X_L).
\end{aligned}
$$

  - Only constraints are feasibility.

Solution: optimal auction.

- Case (i): $v_L > p_H v_H$.
  - Maximizing X_H gives $X_H = \frac{1}{2}p_H + p_L$ and $X_L = \frac{1}{2}p_L$.
  - Revenue under optimal auction: $p_H v_H + p_L v_L$.
- Case (ii): $v_L \leq p_H v_H$.
  - $X_L = 0$ (type v_L is excluded) and $X_H = \frac{1}{2}p_H + p_L$.
  - Revenue under optimal auction: $(1 - p_L^2)v_H$.