Econ 421
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Lecture 4. Subgame-perfect Nash Equilibrium

We have modeled one-time independent choices in strategic-form games.

- In many settings, choices occur over time, and players observe past actions chosen by other players before making their own choices.

- Extensive-form games: modeling choices conditional on prior moves.

1. Extensive-form games

Modeling sequential moves: a *history* is an ordered sequence of prior actions $(a^1, \ldots, a^k)$.

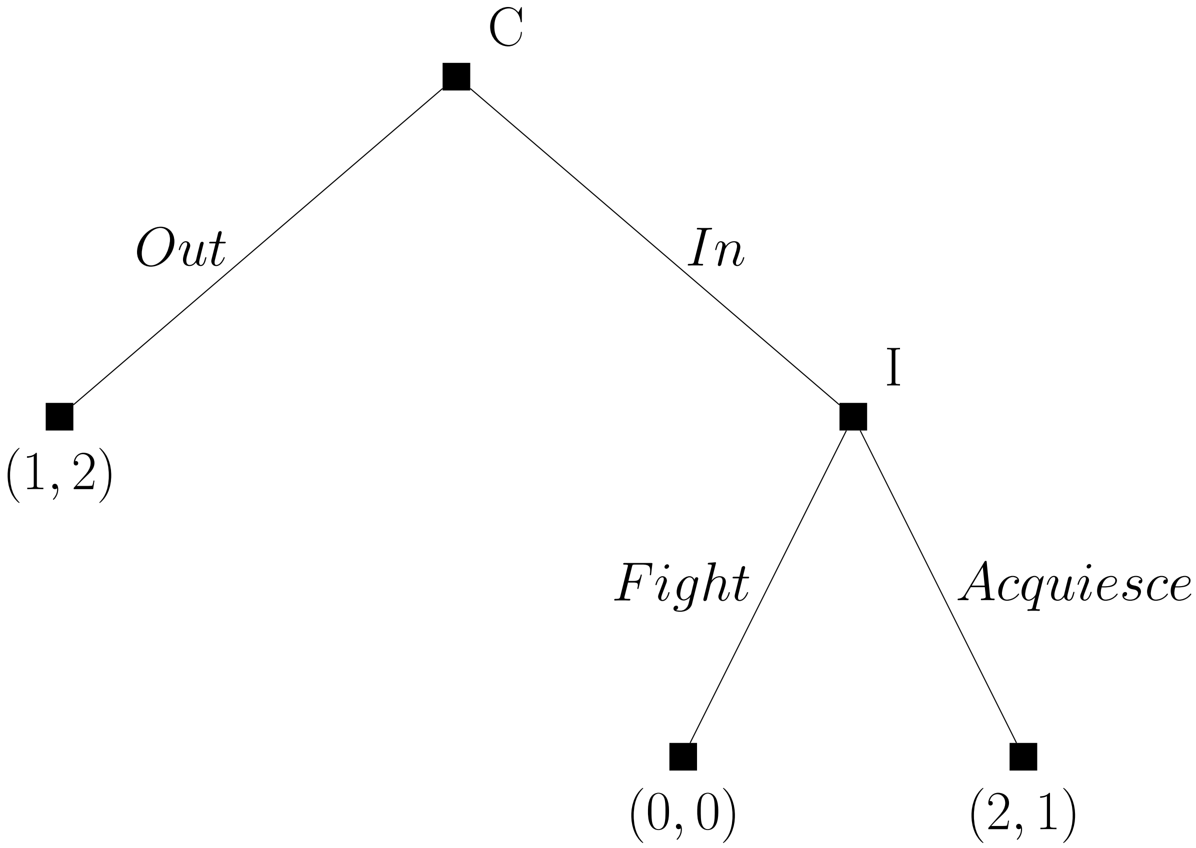
- The outcomes of the game are the *terminal histories* $(a^1, \ldots, a^K)$ or $(a^1, a^2, \ldots)$.

- A *subhistory* of a history $h = (a^1, \ldots, a^k)$ is a history $(a^1, \ldots, a^j)$ consisting of the first j terms of h for $j \leq k$; null history $\emptyset$ if $j = 0$, and *proper subhistory* if $j \neq k$.

An *extensive game* consists of:

- A set of players.
- A set of terminal histories, none of which is a subhistory of any other, with the set of histories consisting of the subhistories of terminal histories.
- A *player function* P that assigns a player $P(h)$ to non-terminal history h, with the actions $A(h)$ available to player $P(h)$ after h given by $\{a \mid (h, a) \text{ is a history}\}$.
- Preferences over terminal histories for each player.

Example (Entry Game). A challenger firm (C) must decide whether or not to enter a monopolistic industry (In or Out). If it enters, the incumbent firm (I) can either fight ($Fight$) or acquiesce ($Acquiesce$). The terminal histories in this game are (In, $Acquiesce$), (In, $Fight$), and Out. The player function P is $P(\emptyset) = \text{C}$ and $P(In) = \text{I}$. We write players' payoffs in the order their first moves.

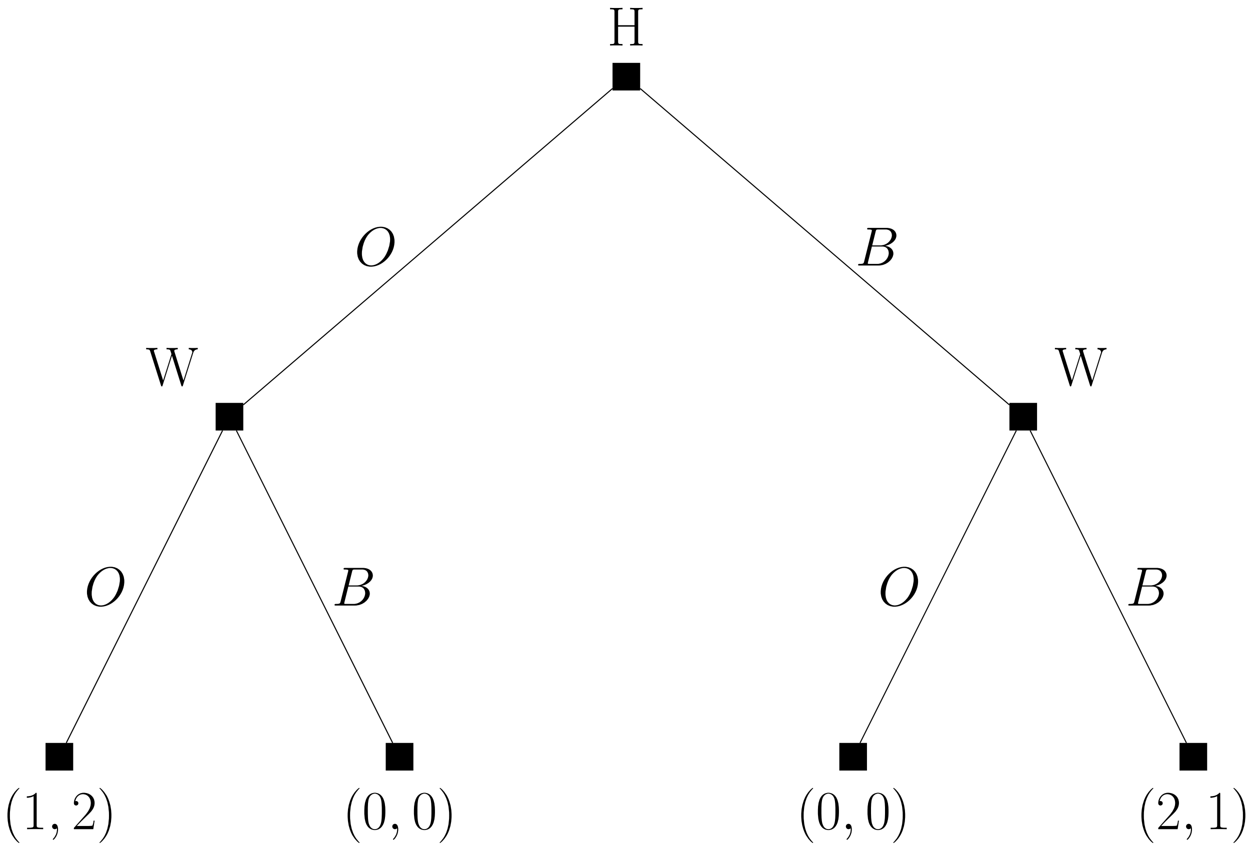


Entry Game.

Example (Sequential Battle of Sexes). Suppose Husband (H) chooses first and Wife (W) observes his choice between O and B and then decides between the two. The terminal histories in this game are (B, B), (B, O), (O, B) and (O, O). The player function P is $P(\emptyset) = \text{H}$ and $P(O) = P(B) = \text{W}$. The payoffs are the same as in the original Battle of sexes game.

Wife

		Opera	*Boxing*
Husband	*Opera*	$1, 2$	$0, 0$
	Boxing	$0, 0$	$2, 1$



Sequential Battle of Sexes.

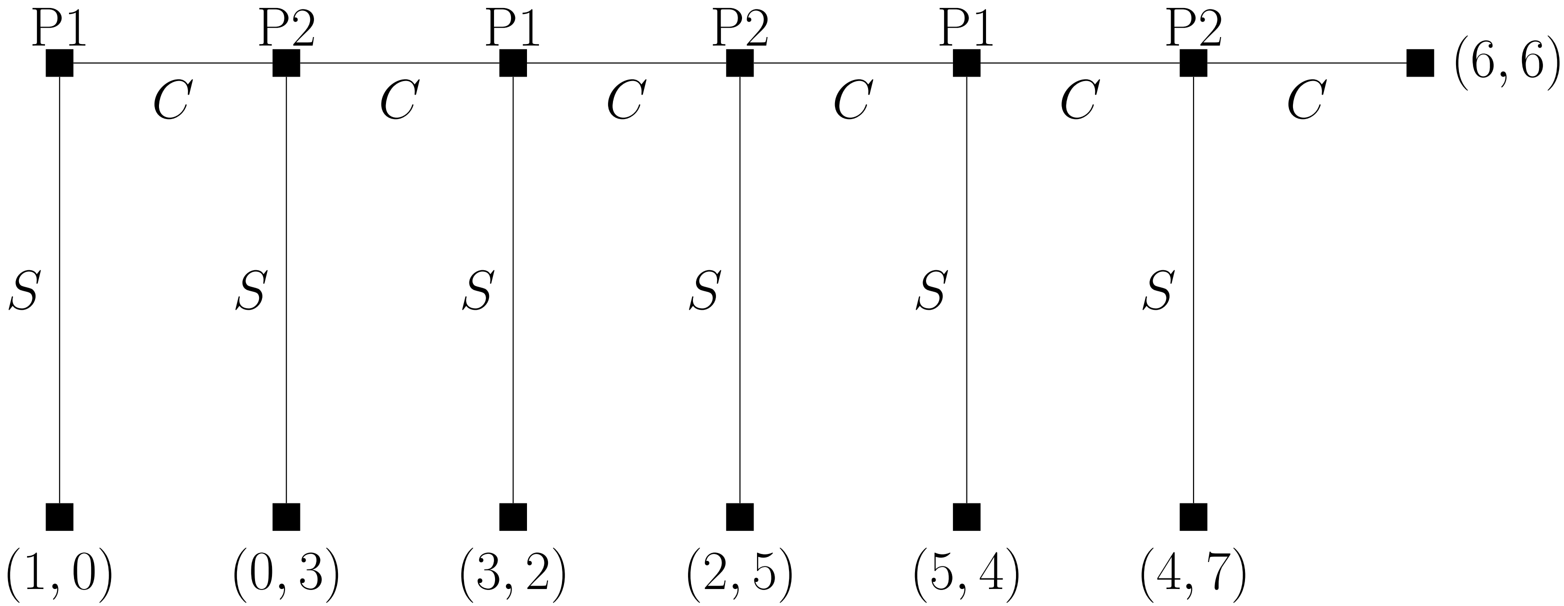
2. Strategies in extensive-form games

A *strategy* for player i is a function $s_i(\cdot)$ that assigns an action $s_i(h)$ in $A(h)$ to each h such that $P(h) = i$.

- A strategy is a complete plan of action that indicates what to do in every conceivable circumstance in which action could be required (even those that are impossible given one's own strategy).

Example (Strategies in Entry Game). *In* and *Out* for C; *Acquiesce* and *Fight* for *I*.

Example (Strategies in Sequential Battle of Sexes). O and B for C, and OO, OB, BO, and BB for W (where first part corresponds to history O, and second part B).



The Centipede Game: player 1 has 8 strategies, 4 of which lead to $(1, 0)$.

3. Nash equilibrium

We can use Nash equilibrium to analyze extensive-form games.

- Once we have all strategies for each player, we can define the strategic-form of the original extensive-form game.

- Nash equilibria can then be defined using the strategic-form.

Example (Nash equilibria in Entry Game). From the strategic form (note that the payoffs are identical for the two profiles (*Out*, *Acquiesce*) and (*Out*, *Fight*)), we can identify two pure-strategy NE: (*In*, *Acquiesce*) and (*Out*, *Fight*).

		I: *Acquiesce*	I: *Fight*
C	*In*	$2, 1$	$0, 0$
C	*Out*	$1, 2$	$1, 2$

Example (Nash equilibria in Sequential Battle of Sexes Game). From the strategic form, we can identify three pure-strategy NE: (O, OO), (B, OB), and (B, BB).

W

H		OO	OB	BO	BB
	O	$1, 2$	$1, 2$	$0, 0$	$0, 0$
	B	$0, 0$	$2, 1$	$0, 0$	$2, 1$

4. Subgame perfect equilibrium

Not all of NE make "reasonable" predictions of the outcome of an extensive-form game.

- What's wrong with the Nash equilibrium $(Out, Fight)$ in Entry Game?

- What's wrong with the Nash equilibrium (O, OO) in Sequential Battle of Sexes?

Need to “refine” NE in extensive-form games.

- Nash equilibrium checks only for best responses *on the equilibrium path*, but does not ask whether parts of strategies *off the path* are credible.

- The right approach in analyzing an extensive-form game has to check best responses after all possible histories.

Extending Nash equilibrium to all possible histories of the game.

- Given a non-terminal history h in an extensive-form game Γ, the *subgame* $\Gamma(h)$ is the game that remains after the history h.
  - $\Gamma(\emptyset)$ is the original game.
- A strategy profile is a *subgame perfect equilibrium* (SPE) if in every subgame strategies are best responses to each other.
  - By definition, A SPE is a Nash equilibrium.
  - A SPE is a strategy profile, not an outcome.

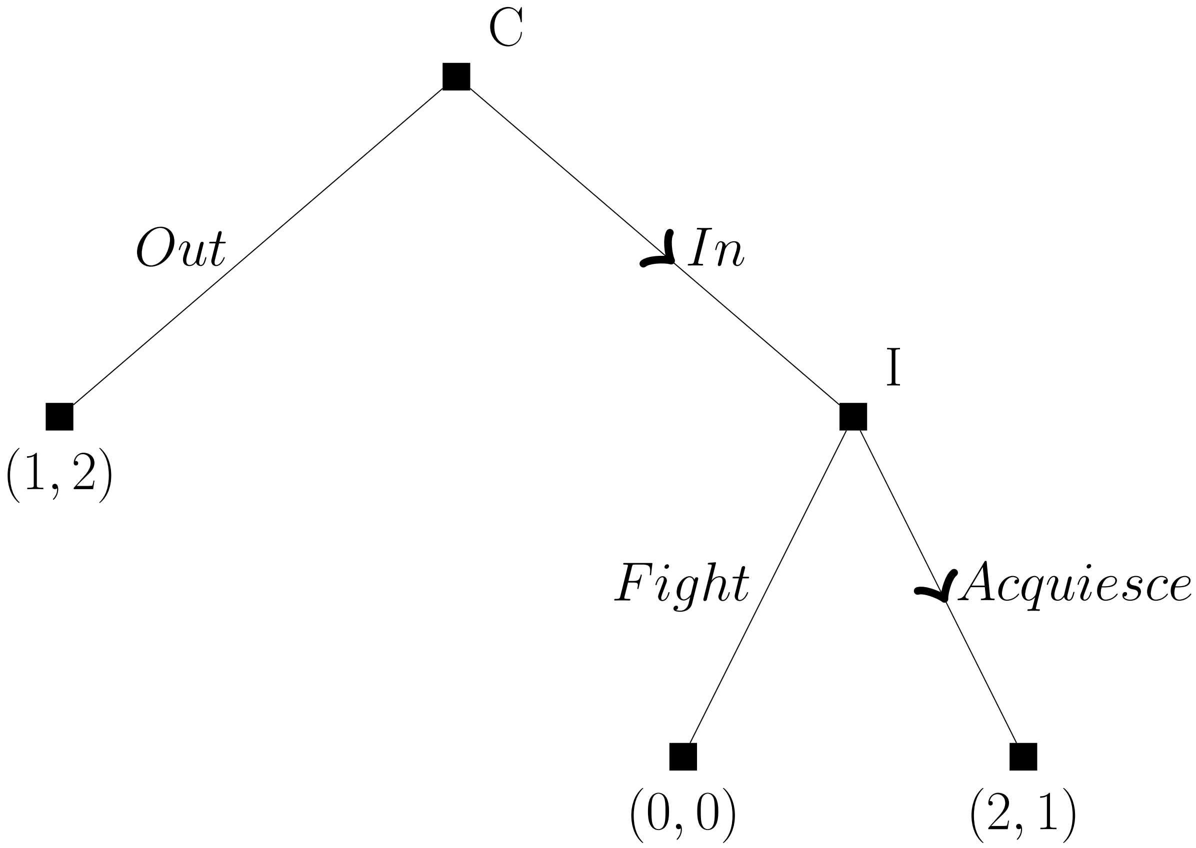
- **Example** (SPE in Entry Game). There are two subgames, namely $\Gamma(\emptyset)$ and $\Gamma(In)$. $(Out, Fight)$ is a NE, but it is not SPE because $Fight$ is not NE in $\Gamma(\text{in})$. $(In, Acquiesce)$ is the unique SPE of the game. The outcome of the game in the SPE is $(In, Acquiesce)$.

- **Example** (SPE in Sequential Battle of Sexes). There are three subgames, $\Gamma(\emptyset)$, $\Gamma(O)$, and $\Gamma(B)$. (O, OO) is NE, but it is not SPE because O is not NE in $\Gamma(B)$. (B, BB) is NE, but it is not SPE because B is not NE in $\Gamma(O)$. (B, OB) is the unique SPE of the game. The outcome of the game in the SPE is (B, B).

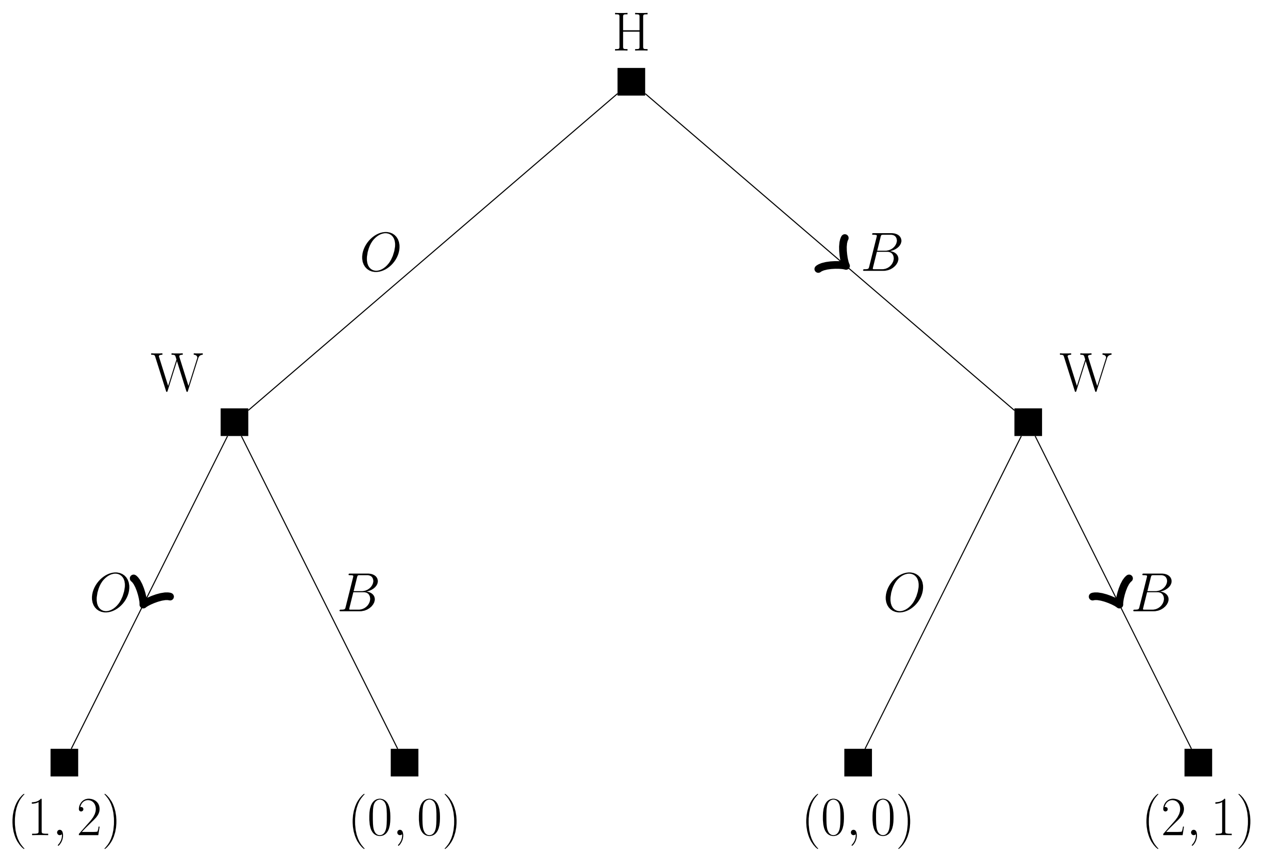
5. Backward induction

SPE can be found by *backward induction.*

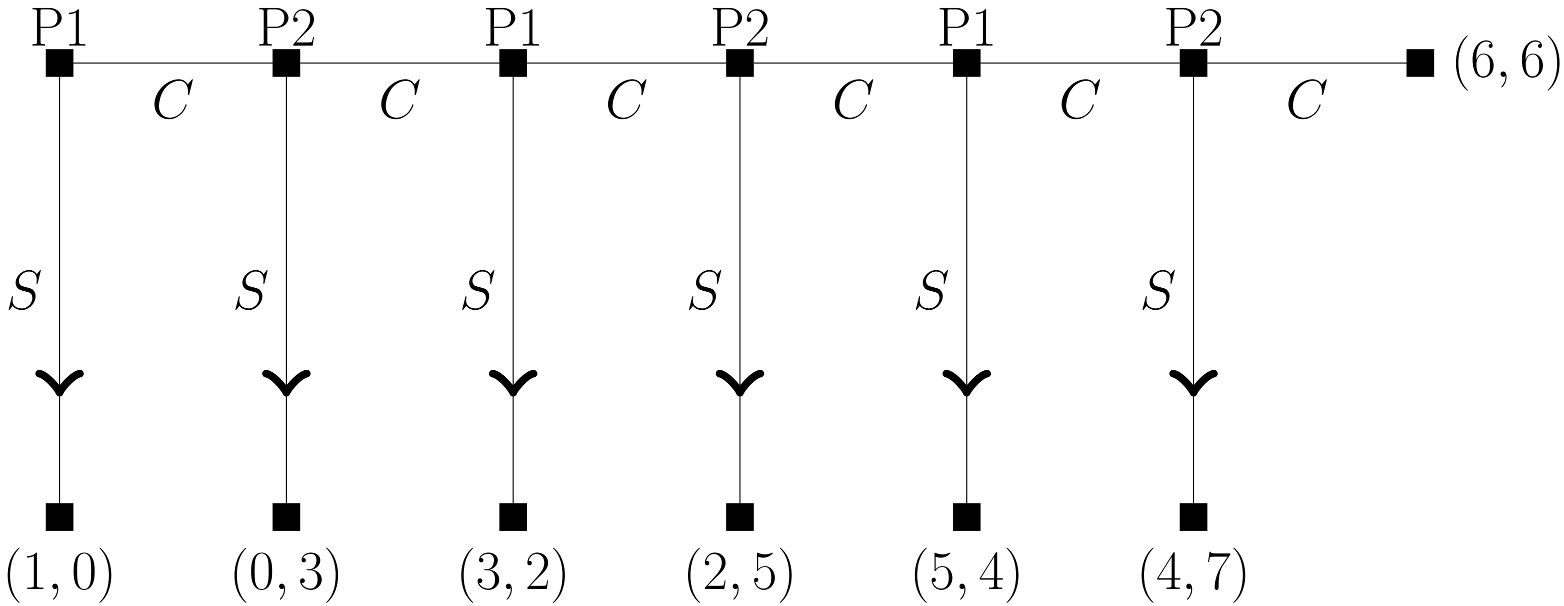
- Each terminal history is a finite sequence of actions.
- Consider the best responses, starting from smallest subgames, that is, the end of the game, and work backwards.
- Backward induction ensures all subgames are considered.
- If best responses are all strict, backward induction leads to a unique SPE, and if there are ties, they are treated separately.



Backward induction in Entry Game.

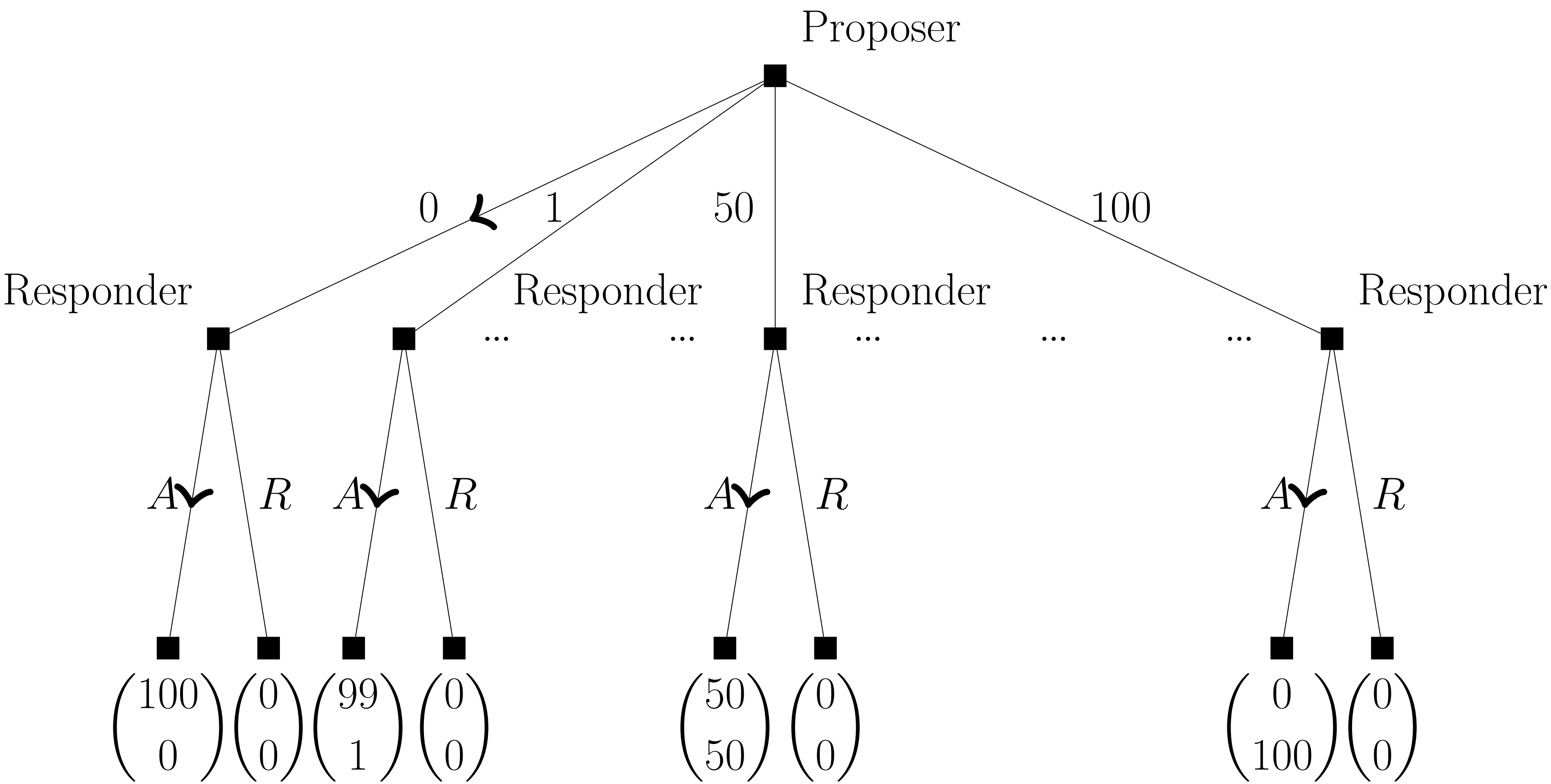


Backward induction in Sequential Battle Of Sexes.

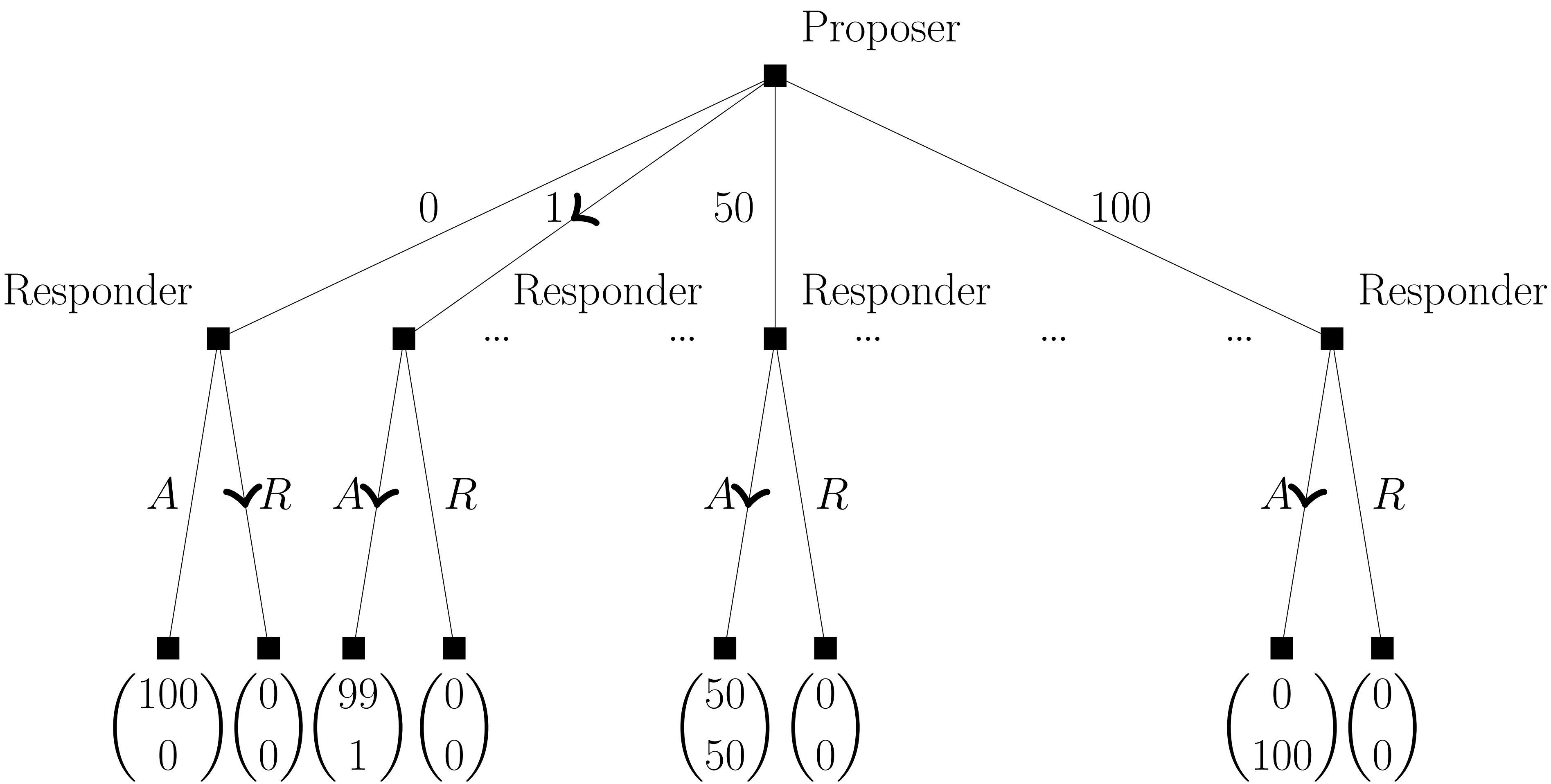


Backward induction in The Centipede Game.

Example (The Discrete Ultimatum Game). Player 1 offers a whole number between 0 and 100 to player 2, who can either accept or reject it (A or R). If accepted, 2 gets x and 1 gets $100 - x$. If rejected, both get 0. Each offer x starts a smallest subgame. Player 2's best response is A if x is least 1, but there is a tie between A and R in best responses for player 2 if $x = 0$.



The Discrete Ultimatum Game: first subgame perfect equilibrium.



The Discrete Ultimatum Game: second subgame perfect equilibrium.

6. Generalizations of extensive form games

We make generalizations in two ways that are important in applications.

- Simultaneous moves.
- Exogenous uncertainty.

The concept of subgame perfect equilibrium applies without change to the two generalizations.

We generalize extensive-form games by allowing simultaneous moves.

- A history h^t at the beginning of period t is a sequence of action *profiles* $(a^1, \ldots, a^{t-1})$, with each a^τ corresponding to period-τ actions simultaneously chosen by more than one player.
- After a history h^t, the player function $P(h^t)$ generally assigns a subset of all N players to make simultaneous moves.
- Let $A_i(h^t)$ be the set of actions for player i in period t after any h^t such that $i \in P(h^t)$.

Example (Battle of Sexes with a Home option) Husband (H) and Wife (W) play Battle of Sexes if H chooses *Out*. If H chooses *Home*, the game ends, with H getting 1.5 and W getting 0.

The terminal histories in this game are $(Out, (B, B))$, $(Out, (B, O))$, $(Out, (O, B))$, $(Out, (O, O))$, and *Home*. The player function P is $P(\emptyset) = \text{H}$ and $P(Out) = \{\text{H}, \text{W}\}$.

Example (Entry Game Meets Cournot Duopoly) A challenger firm (C) must decide whether or not to enter a monopolistic industry (*In* or *Out*). If it does not enter, the incumbent firm (I) remains a monopolist and chooses a quantity. If C enters, C and I play the game of Cournot Duopoly. The inverse demand function is given by $A - Q_C - Q_I$ where Q_C are Q_I are the quantity chosen by C and I respectively, and A is a positive constant. The marginal cost for both C and I is $K < A$, and there is no fixed cost.

The terminal histories in this game are $(In, (Q_C, Q_I))$, and $(Out,\ Q_I)$. The player function P is $P(\emptyset) = \text{C}$ and $P(In) = \{\text{C}, \text{I}\}$.

The concept of subgame perfect equilibrium applies without change.

- A strategy profile is a SPE if in every subgame, strategies are best responses to each other, or equivalently, strategies form a NE.
- Combine backward induction with Nash equilibrium.
- A subgame can be a simultaneous-move game.
- Multiple Nash equilibria in a subgame lead to multiple subgame perfect equilibria.

Example (Battle of Sexes with a Home Option) Two subgame perfect equilibria: $(Home, O)$ for H and O for W, and (Out, B) for H and B for W.

Example (Entry Game Meets Cournot Duopoly) One subgame perfect equilibrium: $(In, (A - K)/3)$ for C, and $(A - K)/3$ for I.

We generalize extensive-form games by allowing exogenous uncertainty.

- At any history h^t such that exogenous uncertainty is resolved, the player function assigns $P(h^t)$ to *Nature*, with $A(h^t)$ equal to the set of possible outcomes from the exogenous uncertainty, and a probability function over the set of possible outcomes.

- Payoffs of players are now required to be von Neumann-Morgenstern.

Example (The Centipede Game with a Trembling Hand) In Centipede Game, each move by a player is subject to small mistakes. This can be modeled by having a Natural's move that with some small probability ϵ switches the player's move and with the remaining probability $1 - \epsilon$ Nature implements the move by the player.

The concept of subgame perfect equilibrium applies without change.

Example (The Centipede Game with a Trembling Hand) For small enough ϵ, it is still a subgame perfect equilibrium for each player to play S whenever it is their turn to move.