Econ 421
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Lecture 3. Mixed-strategy Nash Equilibrium

1. Randomization in games

Recall Matching Pennies has no NE.

Child 2

Child 1		*Heads*	*Tails*
	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

- In order not to lose this game, players must be unpredictable, by randomizing over actions.
  - Randomization is intentional.
  - Randomization makes outcome uncertain.
- Allowing for randomization requires us to specify preferences over uncertain outcomes.

2. Expected payoff

Choices by players when facing uncertainty.

- We model uncertain outcomes as lotteries with known odds over certain outcomes.

- Our objective is to specify preferences over lotteries, so as to define rational choice under uncertainty.

Fix a finite number k of possible outcomes, denoted by $o_1, \ldots, o_k$.

- A *lottery* is a vector $L = (p_1, \ldots, p_k)$ such that $p_j \geq 0$ for each j and $\sum_{j=1}^{k} p_j = 1$, with each p_j representing the probability that the outcome o_j occurs in the lottery L.

- Each certain outcome is a degenerate lottery: o_1 is $(1, 0, \ldots, 0)$, ..., and o_k is $(0, \ldots, 0, 1)$.

Example: in Matching Pennies, there are 4 certain outcomes, and a lottery specifies 4 non-negative numbers that sum up to 1.

A representation U of a player's preferences over all lotteries is called an *expected-payoff representation* if there is a payoff function u from $\{o_1, \ldots, o_k\}$ to $\mathbb{R}$ such that for each lottery $L = (p_1, \ldots, p_k)$,

$$U(L) = \sum_{j=1}^{k} p_j u(o_j).$$

- Representation means that the player prefers L_1 to L_2 if and only if $U(L_1) > U(L_2)$.
- Expected-payoff representation extends the payoff function u over certain outcomes to payoff function U over uncertain outcomes.

Under expected payoff representation, rational choice under uncertainty is maximizing expected payoff.

- u is called a von Neumann-Morgenstern (vNM) payoff function.
- $\sum_{j=1}^{k} p_j u(o_j)$ is the expected value of the function u under the lottery $L = (p_1, \ldots, p_k)$.

Example: in Matching Pennies game, using payoff function u is given in the matrix, we can compute and evaluate any random outcome.

Expected payoff representation combines linearity and multiplicative separability.

- Not unique: if $u(\cdot)$ is the vNM payoff function representing some preferences, then $v(\cdot) = au(\cdot) + b$ for any $a > 0$ represents the same preferences.

- More than ordinal: if $u(\cdot)$ is the vNM payoff function representing some preferences, then $v(\cdot) = f(u(\cdot))$ for an increasing function f does not generally represent the same preferences.

3. Mixed strategies

- A *mixed strategy* for i is a probability distribution over S_i.
  - If S_i is finite, then a mixed strategy m_i assigns a non-negative probability $m_i(s_i)$ to each strategy s_i in S_i, such that

$$\sum_{s_i \in S_i} m_i(s_i) = 1.$$

  - A *pure strategy* $s_i \in S_i$ of player i is just m_i such that $m_i(s_i) = 1$.
  - Implementation of mixed strategies.

- Write $U_i(m)$ for player i's expected payoff when the strategy profile is $m = (m_1, \ldots, m_N)$.
  - Randomizations by players are assumed to be independent of each other.
  - Under independence the probability of outcome $s = (s_1, \ldots, s_N)$ is $m_1(s_1) \times \ldots \times m_N(s_N)$, or simply $\prod_{j=1}^{N} m_j(s_j)$.
  - Expected payoff $U_i(m) = \sum_s \prod_{j=1}^{N} m_j(s_j) u_i(s)$.
- Example: expected payoffs in Matching Pennies.

- A useful way of rewriting:

$$U_i(m) = \sum_{s_i \in S_i} m_i(s_i) U_i(s_i, m_{-i}),$$

where

$$U_i(s_i, m_{-i}) = \sum_{s_{-i}} \prod_{j \neq i} m_j(s_j) u_i(s_i, s_{-i})$$

is i's expected payoff when i uses pure strategy s_i and others mix according to profile m_{-i}.

- Example: Matching Pennies.

4. Mixed-strategy Nash equilibrium

Nash equilibrium in mixed strategies.

- Nash equilibrium is defined in the same way as before except with expected payoffs instead of just payoffs: m^* is a Nash equilibrium if $U(m_i^*, m_{-i}^*) \geq U(m_i, m_{-i}^*)$ for all i and all mixed strategy of i.

- The above definition covers Nash equilibria in pure strategies.

Proposition (Mixed-strategy Nash equilibrium). If m^* is a NE and $m_i^*(s_i) > 0$ then $U_i(s_i, m_{-i}^*) \geq U_i(s_i', m_{-i}^*)$ for all $s_i' \in S_i$.

- *Proof.* Follows from linearity of U_i in $m_i^*(s_i)$ and independence of m_i^* and m_{-i}^*.

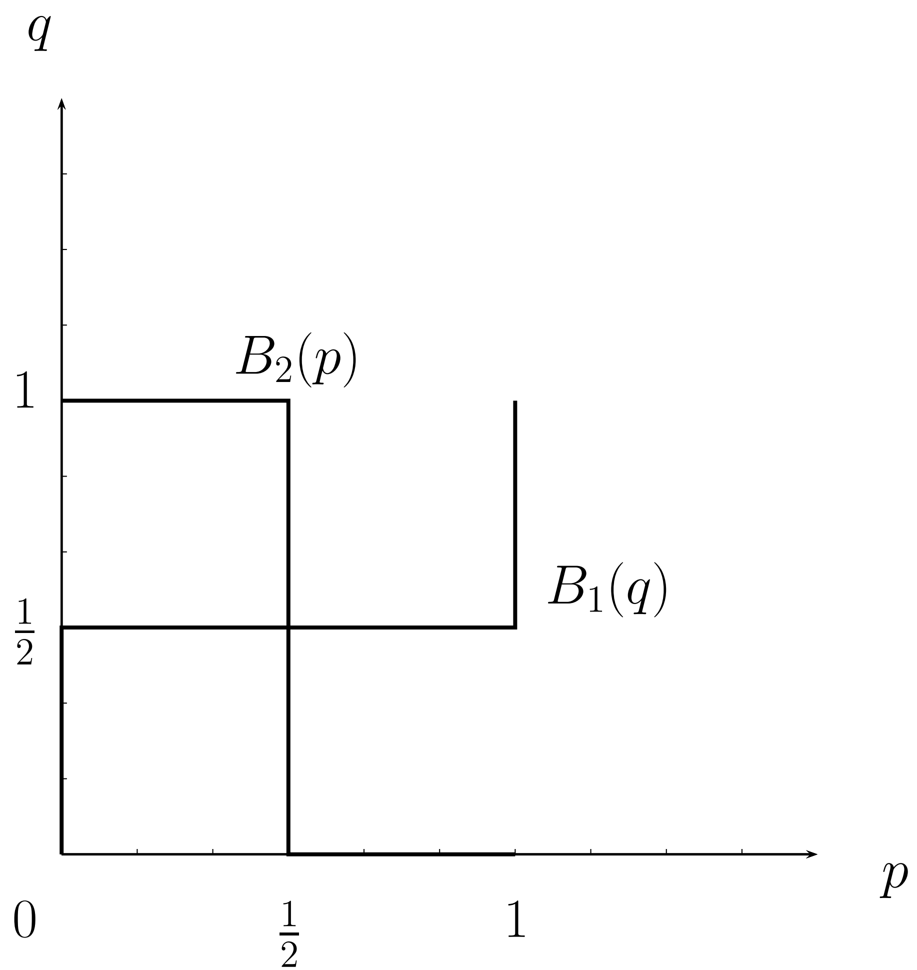
- Implication: If m^* is a NE, then $m_i^*(s_i) > 0$ and $m_i^*(s_i') > 0$ imply $U_i(s_i, m_{-i}^*) = U_i(s_i', m_{-i}^*)$.
  - Player i must be indifferent between s_i and s_i' in order to mix between them in a Nash equilibrium.
  - A mixed-strategy Nash equilibrium is not strict.
  - In applications, above implication imposes a restriction on m_{-i}^* in order for i to mix between s_i and s_i', and can be used to find mixed-strategy Nash equilibrium.

Example (Matching Pennies). We can find NE by intersection of best response functions.

- The best response of 1, in terms of the probability p of playing H, to the probability q that 2 chooses H, is $p = 0$ if $q < \frac{1}{2}$; $p = 1$ if $q > \frac{1}{2}$; and $p \in [0, 1]$ if $q = \frac{1}{2}$.

- The best response of 2, in terms of the probability q of playing H, to the probability p that 1 chooses H, is $q = 1$ if $p < \frac{1}{2}$; $q = 0$ if $p > \frac{1}{2}$; and $q \in [0, 1]$ if $p = \frac{1}{2}$.

- There is a unique intersection at $p = q = \frac{1}{2}$.



Mixed-strategy Nash equilibrium in Matching Pennies.

We can also find the unique NE by indifference.

- For 1 to mix between H and T, player 2 must choose q to make 1 indifferent between H and T, which gives $q = \frac{1}{2}$.

- Symmetrically, for 2 to mixed between H and T, player 1 must choose p to make 2 indifferent between H and T, which gives $p = \frac{1}{2}$.

Example (Battle of Sexes).

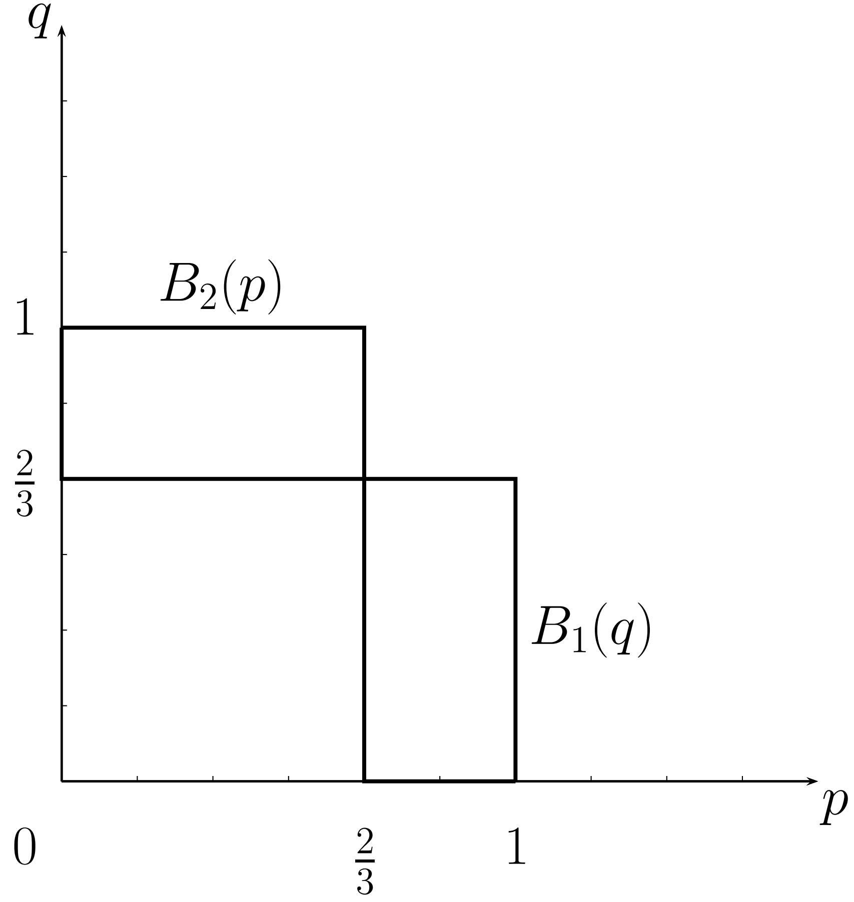
		Wife	
		Opera	*Boxing*
Husband	*Opera*	$1, 2$	$0, 0$
	Boxing	$0, 0$	$2, 1$

Find all NE by intersection of best response function.

- The best response of Husband, in terms of the probability p of playing B, to the probability q that Wife chooses O, is $p = 1$ if $q < \frac{2}{3}$; $p = 0$ if $q > \frac{2}{3}$; and $p \in [0, 1]$ if $q = \frac{2}{3}$.

- The best response of Wife, in terms of the probability q of playing O, to the probability p that 1 chooses B, is $q = 1$ if $p < \frac{2}{3}$; $q = 0$ if $p > \frac{2}{3}$; and $q \in [0, 1]$ if $p = \frac{2}{3}$.

- There are three intersections, $(1, 0)$, $(0, 1)$ and $\left(\frac{2}{3}, \frac{2}{3}\right)$.



Mixed-strategy Nash equilibria in Battle of Sexes.

We can also find the mixed-strategy NE by indifference.

- For Husband to mix between B and O, Wife must randomize to make Husband indifferent, which gives $q = \frac{2}{3}$.
- For Wife to mix between B and O, Husband must randomize to make Wife indifferent, which gives $p = \frac{2}{3}$.

Example (Reporting a Crime). Consider n citizens, all witnesses to a crime, who independently choose whether or not to call the police. Each gets payoff of 0 if nobody calls, $v > 0$ if some other citizen or citizens call, and $v - c > 0$ if he calls (regardless of whether others also call).

- The pure strategy Nash equilibria are the profiles in which there is exactly one caller.
  - These equilibria are reasonable predictions only if players know which one to play.

- There is a symmetric NE in mixed strategies.
  - By the indifference condition, the equilibrium probability p that each citizen calls satisfies $v - c = v\left(1 - (1-p)^{n-1}\right)$, which gives $p = 1 - (c/v)^{1/(n-1)}$.
  - As n grows, p falls, and in fact the probability no one calls rises as n increases.
  - This is another example of under-provision of public good.

5. Comparative statics

In a mixed-strategy Nash equilibrium, there is no strict incentive for any player to use a particular mix.

- The equilibrium mix of a player is determined to make opponents indifferent so as to be willing to mix.
- Comparative statics of mixed-strategy of Nash equilibrium can be counter-intuitive for this reason.

Example (Penalty Kicks). A penalty kicker and a goalkeeper play the following zero-sum game: a smaller α means an improvement in Keeper's skill in reducing Kicker's advantage.

Keeper

		Left	*Right*
Kicker	*Left*	1, 0	$\alpha, 1-\alpha$
	Right	0, 1	1, 0

Mixed-strategy Nash equilibrium depends on α, and can be found by using indifference.

- To make Kicker indifferent, Keeper chooses *Left* with q such that $q \cdot 1 + (1-q) \cdot \alpha = q \cdot 0 + (1-q) \cdot 1$, which gives $q = (1-\alpha)/(2-\alpha)$.

- To make Keeper indifferent, Kicker chooses *Left* with p such that $p \cdot 0 + (1-p) \cdot 1 = p \cdot (1-\alpha) + (1-p) \cdot 0$, which gives $p = 1/(2-\alpha)$.

- When α decreases, Keeper gets better with *Right* but achieves a higher payoff by using *Right* less often.