Econ 421
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Lecture 1. Nash Equilibrium

1. About this course

- Self-introduction.
- Student background survey.
- Course outline.

2. About games and game theory

- Game theory is a misnomer; a more descriptively accurate name is "interactive decision theory."
  - A game studied in game theory is a model of interactions among multiple decision-makers (players).
  - Game theory is about making systematic predictions about the outcome in classes of these games.

3. Rationality

- Description of a game should include, for each player:
  - feasible choices;
  - preferences over all possible outcomes (which arise from choices made by all players);
  - information or knowledge.

- For now, a player's preferences must be complete and transitive:
  - for any two outcomes, the player either strictly prefers the first to the second, or strictly prefers the second to the first, or he is indifferent between the two;
  - if he prefers one outcome to a second outcome, and the second to a third outcome, then he prefers the first to the third.

- A player's preferences can be represented by a payoff function that assigns each possible outcome to a number called payoff, such that he strictly prefers one outcome to a second outcome if and only if the payoff from the first outcome is strictly greater than the payoff from the second outcome.

    - Representation is not unique.

    - Preferences are ordinal.

- Game theory assumes that players are rational decision-makers, in that on his own each player would make the choice that leads to his most preferred outcome.
  - When preferences are represented by a payoff function, then a player is assumed to maximize his payoff.
  - Consumers in consumer theory are rational.

4. Game theory vs economic theory

- General equilibrium in intermediate micro also deals with how multiple decision-makers interact, what does game theory add?
  - Edgeworth box as an exchange economy with two consumers.
  - Competitive equilibrium is prediction of market interactions, based on price-taking and market clearing.
  - Game theory provides different predictions, which are close only when there are many consumers.

5. Non-cooperative vs cooperative game theory

- In this course we study non-cooperative game theory.
  - Game must specify details of interactions: who makes what move when.
  - Analysis is based on individual player's perspective.
  - Bargaining games in Edgeworth box.

- Cooperative game theory abstracts from such details, and makes prediction based on desired properties of how outcome changes with primitives of interactions.
  - Nash bargaining solution in Edgeworth Box.

6. Games and solution concepts

- There are four classes of games, depending on two details – whether all players move once and simultaneously or not, and whether all players have all information about the game:
  - Static games of complete information;
  - Dynamic games of complete information;
  - Static games of incomplete information;
  - Dynamic games of incomplete information.

- Main analytical tool, known as solution concept, for each class of games is
  - Nash equilibrium;
  - Subgame perfect Nash equilibrium;
  - Bayesian Nash equilibrium;
  - Perfect Bayesian Nash equilibrium.

7. Strategic-form games

- Model of static strategic analysis under complete information, in words.
  - All players simultaneously choose strategies and the game ends.
  - Players have preferences over the strategies chosen by all of the players.
  - All available strategies, preferences are common knowledge to all players.

- Until games with incomplete information, we will not be explicit about what we mean by *common knowledge*.
  - Each players knows available strategies and preferences of all players.
  - Each player knows all players know available strategies and preferences of all players.
  - Each player knows all players know all players know available strategies and preferences of all players.
  - ...

- Formally, a *strategic-form game* consists of
  - A set of players $\{1, \ldots, N\}$.
  - A set S_i of strategies for each player i.
  - Payoffs $u_i(s)$ for each player i, and for each *strategy profile* $s = (s_1, \ldots, s_N)$.
- The chosen strategy profile is the *outcome* of the game.
  - Game theory: predict outcome of the game.

- Matrix form to graphically represent strategic-form games with two players and a small number of strategies.
  - Players: row player, column player.
  - Strategies: rows for row player, columns for column player.
  - Strategy profile: box in the matrix.
  - Payoffs in each box: row player's payoff first, column player's payoff second.

- **Example** (Prisoner's Dilemma).

Prisoner B

		Confess	*Not confess*
Prisoner A	*Confess*	$1, 1$	$3, 0$
	Not confess	$0, 3$	$2, 2$

- **Example** (Battle of Sexes).

		Wife	
		Opera	*Boxing*
Husband	*Opera*	$1, 2$	$0, 0$
	Boxing	$0, 0$	$2, 1$

- **Example** (Chicken, aka Hawk and Dove).

Buzz

		Straight	*Swerve*
Jim	*Straight*	$-1, -1$	$2, 0$
	Swerve	$0, 2$	$1, 1$

- **Example** (Stag Hunt).

  There are $N \geq 2$ hunters ambushing a stag. Each hunter must independently decide whether to stay in his position or hunt for a rabbit instead. They can successfully catch a stag only if all hunters stay in their positions. An equal share of the stag is worth $x > 1$ times a rabbit. For $n = 2$:

Hunter 2

Hunter 1		*Stag*	*Rabbit*
	Stag	x, x	$0, 1$
	Rabbit	$1, 0$	$1, 1$

- **Example** (Matching Pennies).

		Child 2	
		Heads	*Tails*
Child 1	*Heads*	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

8. Dominated strategies

- In some games, some players can eliminate some strategies from consideration without having to predict other players' choices.

- We say that s_i *strictly dominates* s_i' for player i if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for all } s_{-i},$$

  where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$ is profile of strategies of all players except i.

- A strategy is *strictly dominant* if it strictly dominates all others.

Rational players will never choose strictly dominated strategies.

- In the Prisoner's Dilemma, *Confess* is strictly dominant for each player.
- In any game with a strictly dominant strategy for each player, rationality requires each player to choose the strictly dominant strategy, without any need to predict the choices of other players.

Most games are not so simple, but common knowledge of rationality allows iterated elimination of strictly dominated strategies.

- **Example** (Airline Pricing) Air Canada and West Jet choose among a high price (H), medium price (M) and low price (L) on some route.

$$
\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{\text{West Jet}} \\
 & & L & M & H \\
\cline{3-5}
 & H & 0,5 & 0,6 & 4,4 \\
\cline{3-5}
\text{Air Canada} & M & 1,4 & 3,3 & 6,0 \\
\cline{3-5}
 & L & 2,2 & 4,1 & 5,0 \\
\cline{3-5}
\end{array}
$$

(L, L) is prediction based on rationality and common knowledge of rationality.

However, iterated elimination of strictly dominated strategies may not provide a unique prediction.

- Only the Prisoner's Dilemma has a strictly dominated strategy to start with.

- Will later introduce main solution concept in static games with complete information, Nash equilibrium, which is more powerful but requires more than common knowledge of rationality.

Strategy s_i *weakly dominates* s_i' if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \text{for all } s_{-i}, \text{ with strict inequality for some } s_{-i}.$$

Example (Restaurant Voting) Three friends go out for dinner. One already chose Italian. The other two, who both prefer Chinese instead, have to vote, with the final choice decided by the majority among the three.

Friend 3

		Chinese	*Italian*
Friend 2	*Chinese*	$1, 1$	$0, 0$
	Italian	$0, 0$	$0, 0$

We can use weak dominance in a similar way as strict dominance.

- A strategy is *weakly dominant* if it strictly dominates all others.
- Iterated elimination of weakly dominated strategies.
- Unlike the strict version, rationality does not rule out a weakly dominated strategies.

9. Nash equilibrium

- A strategy profile $s^* = (s_1^*, \ldots, s_N^*)$ is a *Nash equilibrium* if, for each i,

$$s_i^* \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}^*).$$

- Nash equilibrium s^* is *strict* if each s_i^* is the unique maximizer.

- Useful ways of describing Nash equilibrium.

  – No profitable unilateral deviations.

  – Predictions that are validated by rationality.

- Examples
  - Prisoner's Dilemma: $(Confess, Confess)$ is the unique NE.
  - Battle of Sexes: $(Opera, Opera)$ and $(Boxing, Boxing)$ are both NE.
  - Chicken: $(Straight, Swerve)$ and $(Swerve, Straight)$ are both NE.
  - Stag Hunt: all N hunters choosing $Stag$ and all choosing $Rabbit$ are both NE.
  - Matching Pennies: no NE.

- For player i, denote *best responses* to s_{-i} as

$$B_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}.$$

  $B_i(\cdot)$ is called *best response function.* (We use lower case b when there is a unique best response.)

- We can rewrite the definition of Nash equilibrium as:

$$s^* \text{ is a NE if } s_i^* \in B_i(s_{-i}^*) \text{ for all } i.$$

  so Nash equilibrium is an intersection of best response functions.

In Airline Pricing, we can find the unique Nash equilibrium (L, L) by identifying best response functions of West Jet and Air Canada.

		West Jet		
		L	M	H
	H	$0, 5$	$0, 6^*$	$4, 4$
Air Canada	M	$1, 4^*$	$3, 3$	$^*6, 0$
	L	$^*2, 2^*$	$^*4, 1$	$5, 0$

Example (A Tournament) Two employees each choose effort to win a share of bonus $R > 0$. If each employee i chooses a_i, $i = 1, 2$, then i wins a share equal to $a_i/(a_1 + a_2)$, at a personal monetized cost of a_i, and so i's payoff is $Ra_i/(a_1 + a_2) - a_i$.

- The best response of 1 to a_2 solves $\max_{a_1} Ra_1/(a_1 + a_2) - a_1$.
- This gives best response function $b_1(a_2) = \max\{\sqrt{Ra_2} - a_2, 0\}$ for all $a_2 > 0$, while $b_1(0)$ does not exist.
- Nash equilibrium is intersection of $b_1(a_2)$ and $b_2(a_1)$.
- There is a unique NE: $(R/4, R/4)$.

10. Nash equilibrium and dominance

Nash equilibrium is more powerful than iterated elimination of strictly dominated strategies as a solution concept.

- Additional predictive power of NE comes from assuming players have "correct" guesses about how others play.
- Even though NE relaxes common knowledge of rationality, the assumption of having correct guesses ensures it never contradicts iterated elimination of strictly dominated strategies.

Proposition (Nash equilibrium and iterated elimination of strictly dominated strategies).

(i) If $s^* = (s_1^*, \ldots, s_N^*)$ is a Nash equilibrium, then each strategy s_i^* survives iterated elimination of strictly dominated strategies.

(ii) If s^* is the only profile that survives iterated elimination of strictly dominated strategies, then s^* is a Nash equilibrium, and there are no other Nash equilibria.

Proof of (i).

- Suppose that s_i^* is among the first Nash equilibrium strategies eliminated.

- Then there is a strategy for player i, say s_i', that strictly dominates s_i^*, that is, $u_i(s_i', s_{-i}) > u_i(s_i^*, s_{-i})$ for all profiles s_{-i} that have not yet been eliminated.

- Since s_i^* is among the first, the above implies $u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$, contradicting the definition of Nash equilibrium.

Proof of (ii).

- Suppose that s^* is not a Nash equilibrium.

- There exist i and s_i' such that $u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$.

- Since s^* is the only profile that survives iterated elimination of strictly dominated strategies, s_i' is eliminated at some iteration, by say s_i'', and so $u_i(s_i'', s_{-i}) > u_i(s_i', s_{-i})$ for all profiles s_{-i} not yet been eliminated.

- Since s^* is never eliminated, $u_i(s_i'', s_{-i}^*) > u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$.

- If $s_i'' = s_i^*$, we have a contradiction; if not, repeat with s_i''.

Nash equilibrium can involve weakly dominated strategies, i.e., it may not survive iterated elimination of weakly dominated strategies.

- In Restaurant Voting, (*Itailian*, *Italian*) is a Nash equilibrium, although *Italian* is weakly dominated by *Chinese* for both players.

Friend 3

		Chinese	*Italian*
Friend 2	*Chinese*	$1, 1$	$0, 0$
	Italian	$0, 0$	$0, 0$

In economic applications, we select Nash equilibria that survive iterated eliminated of weakly dominated strategies.

- Playing a weakly dominated strategy never strictly benefits the player, and can be "risky."

- If s^* is the only profile that survives iterated elimination of weakly dominated strategies, then s^* is a Nash equilibrium.