Econ 420
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Problem Set 9

1. (*Maximum Principle and Euler Equation*) Consider the calculus of variations problem of choosing $\{y(t)\}$ to maximize $\int_0^T F(y, \dot{y}, t)dt$ for given $y(0)$ and $y(T)$. Use the Maximum Principle to derive the Euler equation

$$\frac{\partial F(y, \dot{y}, t)}{\partial y} - \frac{d}{dt}\left(\frac{\partial F(y, \dot{y}, t)}{\partial \dot{y}}\right) = 0.$$

[Let the control be $z = \dot{y}$ and the law of motion be $\dot{y} = z$.]

2. (*Life-Cycle Saving*) A worker with a life span T earns wages at a constant rate w, and faces a constant interest rate r with the following equation for the accumulation of his asset (debt) $k(t)$:

$$\dot{k}(t) = w + rk(t) - c(t),$$

where $c(t)$ is his consumption at time t. His objective is to maximize

$$\int_0^T c(t)^{1-\epsilon} e^{-\rho t}/(1-\epsilon)dt,$$

where ϵ is a positive constant and ρ is the positive discount rate.

(a) Use L'Hôpital's Rule to verify that if $\epsilon = 1$ the immediate function, written as $(c(t)^{1-\epsilon} - 1)e^{-\rho t}/(1-\epsilon)$ with a constant added, is given by $\ln(c(t))e^{-\rho t}$.

(b) Use the Hamiltonian to derive a first order ordinary differential equation for $k(t)$.

(c) Solve the equation in (b) using $k(0) = k(T) = 0$.

(d) Solve the equation in (b) using $k(0) = k_0 > 0$ and $k(T) = k_1 > 0$. What is the critical value of k_1 above which there is no feasible solution?

3. (*Optimum Growth*) Consider an economy with a strictly increasing and concave production function $F(k)$ that satisfies $F(0) = 0$ and $F'(0) = \infty$, where k is the capital stock. The rate of depreciation of the capital stock is $\delta > 0$, so the planner of the economy faces the following capital accumulation equation:

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t),$$

where $c(t)$ is the consumption at time t, and the initial capital stock $k(0)$ is fixed. The planner's objective is to maximize

$$\int_0^\infty U(c)e^{-\rho t}dt,$$

where the utility function $U(c)$ is strictly increasing and concave, and ρ is the positive discount rate. Write the Hamiltonian as

$$H(k, c, \pi, t) = U(c)e^{-\rho t} + \pi(F(k) - \delta k - c),$$

and let $\phi = \pi e^{\rho t}$, which is a function of t.

(a) Show that $k(t)$ and $\phi(t)$ satisfy the following pair of first order differential equations

(where t does not appear explicitly)

$$\dot{k} = F(k) - \delta k - G(\phi);$$
$$\dot{\phi} = -\phi(F'(k) - \rho - \delta),$$

where G is the inverse of the function U'.

(b) Use the equations in (a) to draw a phase diagram. Find the stable point.

4. (*Entry-Deterrence*) Consider an industry with a linear demand function $q(t) = a - bp(t)$ at any time t, where $q(t)$ and $p(t)$ are the total quantity supplied and the price at t, and a and b are positive constants. There is a single large firm in the industry. At every time t, the firm sets $p(t)$, which determines the total demand $q(t) = a - bp(t)$, and then a fringe of small firms produce a total quantity of $x(t)$, which determines the sales of the large firm $q(t) - x(t)$. Suppose that the entry and exit of the fringe is governed by the following first order differential equation

$$\dot{x}(t) = k(p(t) - p^*),$$

where p^* is some given positive price and k is a positive constant, and where $x(0)$ is fixed. (The interpretation is that new small firms enter if the large firm sets a price above p^* and existing small firms exit if $p(t)$ is below p^*.) The large firm's objective is to maximize the discounted profits

$$\int_0^\infty (p(t) - c)(a - bp(t) - x(t))e^{-\rho t} dt,$$

where $\rho > 0$ is the interest rate and $c < p^*$ is the constant marginal production cost c (there is no fixed cost).

(a) Use the Hamiltonian to derive a pair of first order ordinary differential equations for x and p that do not involve t explicitly. [The state variable is x and the control variable is p in the optimal control problem.]

(b) Use the equations in (a) to construct the phase diagram.

(c) Discuss how parameters of the problem affect the share of total sales by the fringe at the stable point. Interpret your condition.