Econ 420
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Problem Set 8

1. (*Second Time Derivatives of the State Variable*) Consider a calculus of variations problem in which the immediate function F depends also on the second time derivative of the state variable y. The objective function is $\int_{t=0}^{T} F(y, \dot{y}, \ddot{y})dt$, with $y(0)$, $\dot{y}(0)$, $y(T)$ and $\dot{y}(T)$ all fixed. Show that the Euler equation is given by

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{y}}\right) = 0.$$

2. (*Linear Immediate Functions*) The immediate function F in a calculus of variations problem is linear in the first time derivative of the state variable y if there exist functions $A(y, t)$ and $B(y, t)$ such that

$$F(y, \dot{y}, t) = A(y, t) + B(y, t)\dot{y},$$

where

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial t}.$$

Show that the Euler equation is automatically satisfied (and hence provides no way of solving the problem) if the immediate function is linear in $\dot{y}$.

3. (*Resource Extration*) Consider a small resource-based economy that chooses the extraction rate to maximize the profit over a fixed short period of time, from $t = 0$ to $t = T$.

Let $y(t)$ denote the economy's remaining amount of resource at time t, with $y(0) = y_0 > 0$ given. Denote $\dot{y}(t) < 0$ as the rate of extraction. The unit price p that the economy faces is constant in this time period. The revenue flow is then $-p\dot{y}(t)dt$ in the (infinitesimal) time interval $(t, t+dt)$. The flow of cost of extraction is given by $K(\dot{y}(t))dt$, with $K(0) = 0$ (the cost is zero if $\dot{y}(t) = 0$), $K' < 0$ (the cost is greater if $\dot{y}(t)$ is more negative), and $K'' > 0$ (the cost increases faster if $\dot{y}(t)$ is more negative). The objective function is therefore

$$\int_0^T \left(-p\dot{y}(t) - K(\dot{y}(t)\right) dt,$$

where there is no discounting because the time period is short. Use the Euler equation to find the solution to the above dynamic maximization problem when $y(T) = y_T < y_0$ is fixed.