Econ 420
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# Problem Set 5

1. (*Properties of Convex Functions*) Let $f$ be a function that maps any point $x$ in $\mathbb{R}^N$ to the real line $\mathbb{R}$.

(a) Show that $f$ is convex if and only if the set $\{(x, y) \in \mathbb{R}^{N+1} | y \geq f(x)\}$ is convex.

(b) Suppose that $f$ is concave from $\mathbb{R}^N$ to $\mathbb{R}$ and $g$ is a concave and non-decreasing function from $\mathbb{R}$ to $\mathbb{R}$. Show that the composite function $g(f(x))$ from $\mathbb{R}^N$ to $\mathbb{R}$ is concave.

(c) Suppose that $N = 1$ and $f$ is twice-differentiable. Show that if $f_{xx}(x) \geq 0$ for all $x$, then $f$ is convex.

2. (*Convexity of a Firm's Profit Function*) A firm chooses vectors $x$ of inputs and $y$ of outputs subject to a production possibility constraint $G(x, y) \leq 0$, to maximize profit $qy - px$, where $q$ is the row vector of output prices and $p$ is the row vector of the input prices. Let $\Pi(q, p)$ be the maximized profit, as a function of $q$ and $p$. Show that $\Pi$ is convex in $(q, p)$.

3. (*Corner Solutions*) Consider an economy with a total labor endowment of $L > 0$ that can used to produce good 1 and good 2. The production possibility constraint is $a_1x_1 + a_2x_2 \leq L$, where each $x_j$, $j = 1, 2$, is the quantity of good $j$ produced, and each $a_j > 0$ is the fixed labor requirement in order to produce one unit of good $j$. The output prices of $p_1 > 0$ and $p_2 > 0$. Consider the problem of maximizing the total value of the

outputs, $p_1x_1 + p_2x_2$, subject to the labor constraint and non-negativity constraints on $x_1$ and $x_2$.

(a) Draw a diagram and solve the maximization problem by separation of two convex sets. (You need to consider two cases, depending on which of $p_1/p_2$ and $a_1/a_2$ is larger.)

(b) Write down the Kuhn-Tucker conditions for the optimization problem. Use the conditions to verify your solution in (a).

(c) Show that the maximal value is given by $\max\{p_1/a_1, p_2/a_2\}L$.