Econ 420
Fall, 2022
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Problem Set 4

1. (*The Cobb-Douglas Production Function — The Short Run*) Consider a production function $A\prod_{j=1}^{n} x_j^{\alpha_j}$, where each x_j, $j = 1, \ldots, n$, is an input, and A and α_j are positive constants. Let $w_j > 0$ be the price for input j. Suppose that in the short run there is a minimum input requirement for each input $x_j \geq q_j$ for each $j = 1, \ldots, n$, with q_j a positive constant. Consider the cost minimization problem of $\min \sum_{j=1}^{n} w_j x_j$ subject to meeting a positive target output y, i.e. $A\prod_{j=1}^{n} x_j^{\alpha_j} \geq y$, and subject to the minimum input requirements of $x_j \geq q_j$ for each j.

(a) Write down the Lagrangian, using λ for the multiplier for meeting the target and μ_j for each minimum input requirement. (Make sure that you write it in such a way that all the multipliers are nonnegative.) Provide interpretations of λ and μ_j.

(b) Show that if $A\prod_{j=1}^{n} q_j^{\alpha_j} \leq y$, there is no overshooting the target at the optimum, i.e., the optimal $\overline{x}_j$'s are such that $A\prod_{j=1}^{n} \overline{x}_j^{\alpha_j} = y$. Is the reverse statement true? Prove it or disprove it with a counterexample.

(c) Suppose that $q_{j'} w_{j'}/\alpha_{j'} \geq q_j w_j/\alpha_j$ for two inputs $j' \neq j$. Prove by contradiction that if the minimum input requirement for input j is binding, then so is the input requirement for j'.

2. (*The Cobb-Douglas Production Function— The Long Run*) Consider the same production function $y = A\prod_{j=1}^{n} x_j^{\alpha_j}$ as in the previous question, where y is the output, each x_j,

$j = 1, \ldots, , n$, is an input, and A and α_j are positive constants. Let $w_j > 0$ be the price for input j, let p be the output price, and define $\beta = \sum_{j=1}^{n} \alpha_j$. There are no minimum input requirements in the long run.

(a) Show that the minimum cost function for meeting the target output y is $C(w, y) = \beta(y/A)^{1/\beta} \prod_{j=1}^{n} (w_j/\alpha_j)^{\alpha_j/\beta}$.

(b) Suppose that $\beta < 1$. Use (a) to derive the maximum profit function $\pi(p, w)$.

(c) How does your answer in (b) change if $\beta = 1$, and if $\beta > 1$?

3. (*The CES Expenditure Function*) Suppose that the utility function is $(\alpha x^\rho + \beta y^\rho)^{1/\rho}$, where x and y are quantities of two goods, and $\alpha > 0$, $\beta > 0$, and $\rho < 1$ are given constants.

(a) Derive the the expenditure function $E(p, q, u)$ where p and q are the given prices of the two goods and u is the target utility level.

(b) Find the compensated demand functions $x(p, q, u)$ and $y(p, q, u)$ associated with the expenditure function.

(c) Use your answers in (b) to show that $d \ln(x/y)/d \ln(q/p)$ is a constant. (This expression is called the elasticity of substitution, hence the name CES— constant elasticity of substitution.) Find the condition on ρ that ensures that $d \ln(x/y)/d \ln(q/p)$ is non-negative.