Econ 420
Fall, 2022
Li, Hao
UBC

Problem Set 10

1. (*Investment with Adjustment Cost*) A firm's revenue is given by a quadratic function $ak_t - \frac{1}{2}bk_t^2$ in each time period $t = 0, 1, \ldots$, where k_t the capital stock in period t, and a and b are positive constants, with the initial stock k_0 given. Suppose that it is costly for the firm to adjust its capital stock: if the firm chooses k_{t+1} in period t for the capital stock in the next period given that it is k_t now, it must pay a cost of $\frac{1}{2}c(k_{t+1} - k_t)^2$ in period t, where c is a positive constant. The firm's dynamic optimization problem is to choose an infinite sequence of (non-negative) capital stocks $\{k_{t+1}\}_{t=0}^{\infty}$ to maximize the present value of the profit, where the interest rate r is constant.

(a) Write down the Bellman equation for the value function v.

(b) Show that the function $v(k) = \alpha k - \frac{1}{2}\beta k^2 + \gamma$ satisfies the Bellman equation, where

$$\alpha = a + \frac{\delta \alpha c}{\delta \beta + c};$$
$$\beta = b + \frac{\delta \beta c}{\delta \beta + c};$$
$$\gamma = \delta \gamma + \frac{1}{2} \frac{(\delta \alpha)^2}{\delta \beta + c}$$

where $\delta = 1/(1 + r)$.

(c) Show that the three equations in (b) uniquely determine the values of α, β and γ, with $\beta \in (b, b + c)$. [Start with the second equation in (b).]

(d) Construct three infinite sequences of coefficients, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ for a sequence of value functions $\{v_n\}$ by solving the maximization problem in the Bellman equation repeatedly, starting with some arbitrary $v_0 = \alpha_0 k - \frac{1}{2}\beta_0 k^2 + \gamma_0$. Use the convergence of the three sequences to provide another proof of the results in (b) and (c). [To show that the sequence $\{\beta_n\}$ converges, you may need to use the fact that a sequence converges if it is increasing and bounded from above, or decreasing and bounded from below.]

2. (*A Cake-eating Problem*) In each time period $t = 0, 1, \ldots$, the owner of a cake chooses how much of the remaining cake to consume. The cake depreciates at the rate of δ per time period; that is, if the size of the cake at the beginning of some time period t is s_t and c_t is consumed that period, then the size of the cake at the beginning of period $t+1$ is $(1-\delta)(s_t - c_t)$. The initial size of the cake $s_0 > 0$ is fixed. The objective is to maximize the discounted sum of the utilities from cake consumption, $\sum_{t=0}^{\infty} \beta^t \ln(c_t)$, where $\beta \in (0, 1)$ is the discount factor.

(a) Write down the Bellman equation for this dynamic optimization problem.

(b) Use the conjecture that a constant share of the remaining cake is consumed in each period to find the solution to the Bellman equation.

(c) How does the optimal share of cake consumption depend on the depreciation rate? Interpret your result.

3. (*Linear Utility in Optimum Growth*) Consider the infinite horizon problem of maximizing the discounted sum of consumption over time, $\sum_{t=0}^{\infty} \beta^t c_t$, where $\beta \in (0, 1)$ is the discount factor. Each period consumption c_t is non-negative and bounded from above by the output in period t, which is given by the production function $f(k_t)$, where k_t is the capital stock at

period t, with a given positive initial capital stock k_0. Suppose that f is strictly increasing and concave, with $f(0) = 0$, $f'(0) > \beta^{-1}$ and $f'(\infty) = 0$. Define k^* to be the unique maximizer of $\beta f(k) - k$.

(a) Show that there is a unique value $\underline{k} \in (0, k^*)$ such that $f(k) \geq k^*$ if and only if $k \geq \underline{k}$.

(b) Suppose that $k_0 \geq \underline{k}$. Write down the Bellman equation for the dynamic optimization problem where feasible capital stocks are restricted to above $\underline{k}$.

(c) Show that the following value function satisfies the Bellman equation:

$$v(k) = f(k) - k^* + \frac{\beta}{1-\beta}(f(k^*) - k^*)$$

for all $k \geq \underline{k}$. Give an interpretation of this solution. [What is the corresponding optimal policy? Compare the present problem with the the case of log utility.]

4. (*A Tree-cutting Problem*) Consider a tree whose growth is described by the function h. That is, if k_t is the size of the tree in any period t, then $k_{t+1} = h(k_t)$. The initial size is given by $k_0 > 0$. We assume that h is strictly increasing and concave, with $h(0) = 0$, $h'(0) > \beta^{-1}$ and $h'(\infty) = 0$, where $\beta \in (0, 1)$ is the discount factor. The price of wood is the same in every period. The problem is simply to choose the time to cut the tree and sell the wood to maximize the present value of the revenue.

(a) Argue that the value function V satisfies the Bellman equation $V(k) = \max\{k, \beta V(h(k))\}$.

(b) Show that there is a unique $k^* > 0$ such that $\beta h(k^*) = k^*$, and $\beta h(k) > (<)k$ when $k < (>)k^*$.

(c) Use "guess-and-verify" to solve the Bellman equation in (a). [Hint: $V(k) = k$ for $k \geq k^*$—for such tree it is optimal to cut it today; $V(k) = \beta h(k)$ for $k \in [k_1^*, k^*)$ where

$k_1^* = h^{-1}(k^*)$—it is optimal to cut it tomorrow; $V(k) = \beta^2 h(h(k))$ for $k \in [k_2^*, k_1^*)$ where $k_2^* = h^{-1}(k_1^*)$—it is optimal to cut it day after tomorrow; and so on...]