Econ 420
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# Lecture 9b. Applications of Maximum Principle

## 1. Applying Maximum Principle

- For each $t$, and for fixed $y(t)$ and $\pi(t)$, solve $\max_{z(t)} H(y(t), z(t), \pi(t), t)$ subject to $G(y(t), z(t), t) \leq 0$.
  - A straightforward static optimization problem.
  - Construct value function $H^*(y(t), \pi(t), t)$.

- A pair of first order differential equations:

$$\dot{y}(t) = H^*_\pi(y(t), \pi(t), t),$$

$$\dot{\pi}(t) = -H^*_y(y(t), \pi(t), t).$$

  - For particular solution, use values of $y(0)$ and $y(T)$.

## 2. Life-cycle saving

- $\max \int_0^T \ln(c(t))e^{-\rho t}dt$, subject to $\dot{k}(t) = w + rk(t) - c(t)$, with $k(0) = k(T) = 0$.
  - $k(t)$ is asset (if positive) or debt (if negative).
  - $c(t)$ is consumption at time $t$.
  - $\rho > 0$ is subjective discount rate.
  - $w > 0$ is constant wage rate.
  - $r$ is constant interest rate.

- Remarks on the model.
  - State variable is $k(t)$.
  - Control is $c(t)$.
  - No feasibility constraint (can borrow and lend any quantity at interest rate $r$).
  - If $c(t) = w$, then $k(t) = k(0)e^{rt}$, and if $c(t) = rk(t)$, then $k(t) = k(0) + wt$.

- $H(k(t), c(t), \pi(t), t) = \ln(c(t))e^{-\rho t} + \pi(t)(w + rk(t) - c(t))$.
  - Solution to $\max_{c(t)} H(k(t), c(t), \pi(t), t)$ is

$$c(t) = \frac{e^{-\rho t}}{\pi(t)}.$$

  - Maximum value function

$$H^*(k(t), \pi(t), t) = -(\rho t + \ln(\pi(t)))e^{-\rho t} + \pi(t)(w + rk(t)) - e^{-\rho(t)}.$$

  - Can also proceed without explicit form of $H^*$.

- Pair of differential equations:

$$\dot{k}(t) = w + rk(t) - e^{-\rho t}/\pi(t),$$

$$\dot{\pi}(t) = -r\pi(t).$$

  – Solution to second equation:

$$\pi(t) = \pi(0)e^{-rt},$$

  with $\pi(0)$ to be determined.

– Substituting and solving first equation,

$$k(t) = \left(k(0) + \int_0^t e^{-rs}\left(w - \frac{e^{(r-\rho)s}}{\pi(0)}\right) ds\right) e^{rt}.$$

– Using $k(0) = k(T) = 0$, we can solve for $\pi(0)$:

$$\pi(0) = \frac{r(1 - e^{-\rho T})}{\rho(1 - e^{-rT})w}.$$

- Economic interpretations of solution.
  - Costate $\pi(t)$ decreases exponentially at the interest rate.
  - Optimal consumption $c(t) = e^{(r-\rho)t}/\pi(0)$.
  - When $r > \rho$, optimal consumption grows overtime, with $c(t) < w$ early in life and $c(t) > w$ later in life.
  - Opposite happens when $r < \rho$.

## 3. Optimal growth

- $\max \int_0^\infty U(c(t))e^{-\rho t}dt$, subject to $\dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$.
  - $c(t)$ is consumption at time $t$; $U(c)$ is utility function, strictly increasing and concave, with $U'(0) = \infty$; and $\rho > 0$ is the subjective discount rate.
  - $k(t)$ is capital stock, with $k(0) > 0$ given; $F(k)$ is production function, strictly increasing and concave, with $F(0) = 0$ and $F'(0) = \infty$; and $\delta > 0$ is constant depreciation rate.

- Remarks on the model.
  - State variable is $k(t)$.
  - Control is $c(t)$.
  - No feasibility constraint.
  - Horizon is infinite.
  - No explicit forms of $U$ and $F$.

- $H(k(t), c(t), \pi(t), t) = U(c(t))e^{-\rho t} + \pi(t)(F(k(t)) - \delta k(t) - c(t))$.
  - First order condition for optimal $c(t)$:

$$U'(c(t))e^{-\rho t} - \pi(t) = 0.$$

  - Optimal $c(t)$ is implicitly determined by $\pi(t)$.

- Pair of differential equations:

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$$

$$\dot{\pi}(t) = -\pi(t)(F'(k(t)) - \delta),$$

where $c(t)$ is implicitly determined by $\pi(t)$ through first order condition.

  - No explicit solutions to differential equations.

  - Need a terminal condition on state.

- Rewriting pair of differential equations in $k(t)$ and $c(t)$:

$$\dot{k} = F(k) - \delta k - c$$

$$\dot{c} = -(F'(k) - (\rho + \delta))U'(c)/U''(c).$$

  - Note $t$ does not appear explicitly.
  - It is also possible to make $t$ disappear by normalizing $\pi(t)$.

- Signs of $\dot{c}$ and $\dot{k}$.
  - $\dot{k} > 0$ if $c < F(k) - \delta k$, and $\dot{k} < 0$ if $c > F(k) - \delta k$.
  - $\dot{c} > 0$ if $k < k^*$, and $\dot{c} < 0$ if $k > k^*$, where $F'(k^*) = \rho + \delta$.

- Phase diagram.
  - Loci of $\dot{k} = 0$ and $\dot{c} = 0$ in $k - c$ diagram.
  - 4 representative paths crossing locus of $\dot{k} = 0$ and locus of $\dot{c} = 0$.

- Stable point: $k^*$ and $c^* = F(k^*) - \delta k^*$.

  – Intersection of loci of $\dot{k} = 0$ and $\dot{c} = 0$.

- Stable paths.
  - For any given $k(0)$, there exists a unique $c(0)$ such that $(k(t), c(t)) \to (k^*, c^*)$.
  - Stable path associated with $k(0)$ is solution.