Econ 420
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LECTURE 9A. OPTIMAL CONTROL: MAXIMUM PRINCIPLE

1. Discrete-time model

- $\max_{\{y_t\}_{t=1}^T, \{z_t\}_{t=0}^T} \sum_{t=0}^T F(y_t, z_t, t)$, subject to:
  - Law of motion $y_{t+1} - y_t = Q(y_t, z_t, t)$, for all $t = 0, 1, \ldots, T$, with y_0 and y_{T+1} given.
  - Feasibility $G(y_t, z_t, t) \leq 0$, for all $t = 0, 1, \ldots, T$.

- Will use discrete-time model to derive Hamiltonian function and Maximum Principle.
  - Approximations of continuous-time counterparts.
  - Applications will all use continuous-time.

2. Hamiltonian function

- Define Lagrangian

$$L = \sum_{t=0}^{T} \Big(F(y_t, z_t, t) + \pi_{t+1} \left(y_t + Q(y_t, z_t, t) - y_{t+1} \right) - \lambda_t G(y_t, z_t, t) \Big).$$

  - π_{t+1} is multiplier for law of motion at t, or shadow value of state y_t.
  - λ_t is multiplier for feasibility constraint at t, or shadow value of relaxing the constraint.

- First order conditions.
  - With respect to z_t, $t = 0, 1, \ldots, T$:

$$F_z(y_t, z_t, t) + \pi_{t+1} Q_z(y_t, z_t, t) - \lambda_t G_z(y_t, z_t, t) = 0.$$

  - With respect to y_t, $t = 1, 2, \ldots, T$:

$$\pi_{t+1} - \pi_t = -\Big(F_y(y_t, z_t, t) + \pi_{t+1} Q_y(y_t, z_t, t) - \lambda_t G_y(y_t, z_t, t)\Big),$$

where π_t arises from dynamic link.

- Define Hamiltonian function

$$H(y, z, \pi, t) = F(y, z, t) + \pi Q(y, z, t).$$

  - For fixed $t = 1, \ldots, T$, y_t and π_{t+1}, first order condition with respect to z_t of maximizing $H(y_t, z_t, \pi_{t+1}, t)$ subject to $G(y_t, z_t, t) \leq 0$ is identical to the original.
  - Proof.

- Interpretation of Hamiltonian.
  - Optimal z_t does not maximize immediate function $F(y_t, z_t, t)$ due to dynamic link.
  - Hamiltonian is the correct objective for optimal control z_t, by incorporating effect on future state y_{t+1} through multiplier π_{t+1}.

3. Maximum Principle

- Consider $\max_{z_t} H(y_t, z_t, \pi_{t+1}, t)$ subject to feasibility constraint $G(y_t, z_t, t) \leq 0$.
  - Let $H^*(y_t, \pi_{t+1}, t)$ be maximum value function.
  - By envelope theorem,

$$H_y^*(y_t, \pi_{t+1}, t) = F_y(y_t, z_t, t) + \pi_{t+1} Q_y(y_t, z_t, t) - \lambda_t G_y(y_t, z_t, t),$$

$$H_\pi^*(y_t, \pi_{t+1}, t) = Q(y_t, z_t, t),$$

where z_t is solution, λ_t is multiplier for feasibility constraint.

- Maximum Principle: if $\{y_t\}_{t=1}^T$ and $\{z_t\}_{t=0}^T$ are solution, then there exist $\{\pi_{t+1}\}_{t=0}^T$ such that
  - z_t maximizes $H(y_t, z_t, \pi_{t+1}, t)$ subject to $G(y_t, z_t, t) \leq 0$ for each $t = 1, \ldots, T$, with maximum value function $H^*(y_t, \pi_{t+1}, t)$.
  - $\{y_t\}_{t=1}^T$ and $\{\pi_{t+1}\}_{t=0}^T$ satisfy

$$y_{t+1} - y_t = H^*_\pi(y_t, \pi_{t+1}, t)$$

$$\pi_{t+1} - \pi_t = -H^*_y(y_t, \pi_{t+1}, t).$$

  - Proof.

- Remarks on Maximum Principle.
  - A restatement of first order necessary condition that is both insightful and useful.
  - Practical use: maximizing the Hamiltonian function subject to feasibility is a straightforward static problem; and two first order difference equations with two boundary conditions can be informative even if not explicitly solvable (phase diagram in continuous time).

- Transversality condition.

  - Inequality constraint $y_{T+1} \geq \hat{y}$ instead of equality $y_{T+1} = \hat{y}$.

  - No change to first order conditions, or Maximum Principle.

  - Except law of motion at T, given by $y_T + Q(y_T, z_T, T) \geq \hat{y}$, and $\pi_{T+1} \geq 0$, with complementary slackness.

4. Continuous-time model

- Same Hamiltonian:

$$H(y, z, \pi, t) = F(y, z, t) + \pi Q(y, z, t).$$

  - $\pi(t)$ is multiplier for law of motion $\dot{y}(t) = Q(y(t), z(t), t)$ at time t.
  - $\pi(t)$ is often referred to as co-state.

- Maximum Principle: if $y(t)$ and $z(t)$, $t = 0$ to T, are solution, then there exist $\pi(t)$, $t = 0$ to T, such that
  - $z(t)$ maximizes Hamiltonian $H(y(t), z(t), \pi(t), t)$ subject to $G(y(t), z(t), t) \leq 0$ for all t, with maximum value function $H^*(y(t), \pi(t), t)$.
  - $y(t)$ and $\pi(t)$, $t = 0$ to T, satisfy

$$\dot{y}(t) = H^*_\pi(y(t), \pi(t), t)$$

$$\dot{\pi}(t) = -H^*_y(y(t), \pi(t), t).$$