Econ 420
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Lecture 8c. First-order Linear Differential Equations

1. Difference equations vs differential equations

- Consider law of motion in discrete-time model.
  - $y_{t+1} - y_t = Q(y_t, z_t, t)$, where y_t is state and z_t is control at date t.
  - Given any $\{z_t\}_{t=0}^{T}$, we can solve difference equation for $\{y_{t+1}\}_{t=0}^{T}$ starting from any initial y_0 by repeated substitutions.

- Consider law of motion in continuous-time model.
  - $\dot{y}(t) = Q(y(t), z(t), t)$, where $y(t)$ is state, $z(t)$ is control at date t, and $\dot{y}(t)$ is time derivative of $y(t)$.
  - Given any control $z(t)$ for $t = 0$ to $t = T$, and initial state $y(0)$, we have a first-order differential equation in $y(t)$.
  - Repeated substitutions do not work.
  - Even with explicit functional form for Q, it is often impossible to solve for $y(t)$ explicitly.

2. First-order linear differential equations

- Suppose Q is linear in y.
  - $Q(y(t), z(t), t) = q_1(z(t), t)y(t) + q_0(z(t), t)$, where $q_1(z(t), t)$ is the linear coefficient and $q_0(z(t), t)$ is the constant.
  - Law of motion becomes first order linear differential equation in $y(t)$

$$\dot{y}(t) = Q_1(t)y(t) + Q_0(t),$$

where $Q_1(t) = q_1(z(t), t)$ and $Q_0(t) = q_0(z(t), t)$.

- Special case I: suppose $Q_1(t) = 0$ for all t.
  - Solution can be obtained by integrating both sides of

$$dy(t) = Q_0(t)dt.$$

  - $y(t) = y(0) + \int_0^t Q_0(s)ds.$

- Special case II: suppose $Q_0(t) = 0$ for all t.
  - Solution can be obtained by integrating both sides of

$$\frac{dy(t)}{y(t)} = Q_1(t)dt.$$

  - $y(t) = y(0) \exp\left(\int_0^t Q_1(s)ds\right)$.

- In general, conjecture $y(t) = C(t)\exp\left(\int_0^t Q_1(s)ds\right)$ for some function $C(t)$.
  - Use differential equation to verify conjecture and show

$$\dot{C}(t) = Q_0(t)\exp\left(-\int_0^t Q_1(s)ds\right).$$

  - Show $C(0) = y(0)$, and so

$$C(t) = y(0) + \int_0^t Q_0(s)\exp\left(-\int_0^s Q_1(r)dr\right)ds.$$

3. General versus particular solutions

- What does it mean to solve a differential equation?
  - Consider $\dot{y}(t) = Q(y(t), t)$.
  - General solutions: any $y(t)$ that satisfies differential equation.
  - Particular solution, or solution: general solution $y(t)$ pinned down by a single point $(y(\tau), \tau)$ for any $\tau \in [0, T]$.

- General solutions in first order linear differential equations.
  - Special case I: $y(t) = x + \int_0^t Q_0(s)ds$ for any x.
  - Special case II: $y(t) = x \exp\left(\int_0^t Q_1(s)ds\right)$ for any x.
  - General case:

$$y(t) = \left(x + \int_0^t Q_0(s) \exp\left(-\int_0^s Q_1(r)dr\right) ds\right) \exp\left(\int_0^t Q_1(s)ds\right)$$

for any x.