Econ 420
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Lecture 8b. Calculus of Variations

1. Statement of problem

- Unconstrained dynamic optimization.
  - Any time derivative of state is feasible control, and so law of motion is entirely subject to control.
  - The only dynamic link is today's control changes tomorrow's payoff.

- Statement of terminal-time problem.
  - State variable $y(t)$, with initial state $y(0) = y_0$.
  - Control variable $z(t) = \dot{y}(t)$.
  - Terminal time T and terminal state $y(T) = y_T$ are fixed.
  - Objective is to maximize $\int_0^T F(y(t), \dot{y}(t), t)dt$.

- Economic example: optimal resource extraction.
  - State is remaining resource.
  - Control is extraction rate.
  - Objective is to maximize present value of sales revenue of extracted resource minus cost of extraction.

2. Euler equation

- Necessary condition for time path of state $y(t)$ from $t = 0$ to T to be solution is there is no profitable arbitrage.
  - Consider $\tilde{y}(t) = y(t) + \epsilon\eta(t)$, where $\eta(t)$ is any continuous function of time with $\eta(0) = \eta(T) = 0$, and ϵ is a number.
  - $\tilde{y}(t)$ is arbitrage for any $\eta(t)$: feasible for all ϵ and $\tilde{y}(t) = y(t)$ when $\epsilon = 0$.

- Illustration of arbitrage.

- No profitable arbitrage.
  - Define

$$J(\epsilon) = \int_0^T F(y(t) + \epsilon\eta(t), \dot{y}(t) + \epsilon\dot{\eta}(t), t)dt.$$

  - No profitable arbitrage: $J'(0) = 0$ for all $\eta(t)$ such that $\eta(0) = \eta(T) = 0$.

- Euler equation:

$$F_y - \frac{d}{dt} F_{\dot{y}} = 0.$$

  - Derivation.

- Special case I.
  - $F_{\dot{y}} = 0$.
  - Euler equation implies $F_y = 0$.
  - No dynamic link at all.
  - Instant-by-instant static optimization.

- Special case II.
  - $F_y = 0$.
  - Euler equation implies $F_{\dot{y}}$ is constant in t.
  - If further $F_t = 0$, then $\dot{y}$ is constant in t.

- Special case III.
  - $F_t = 0$.
  - Euler equation implies $F - \dot{y}F_{\dot{y}}$ is constant in t, which is a first-order differential equation instead of second-order.
  - Proof: Euler equation can be rewritten as

$$\frac{d}{dt}(F - \dot{y}F_{\dot{y}}) - F_t = 0.$$

- Special case IV.
  - Immediate function $F(y, \dot{y})e^{-\rho t}$.
  - Euler equation is an autonomous second-order differential equation:

$$F_y + F_{\dot{y}}\rho - F_{y\dot{y}}\dot{y} - F_{\dot{y}\dot{y}}\ddot{y} = 0.$$

3. Second order necessary condition

- For path $y(t)$ from $t = 0$ to T to be optimal, $F_{\dot{y}\dot{y}} \leq 0$, for all $t \in [0, T]$.
  - Proof.

- Second order sufficient condition for local optimality of path $y(t)$, $t = 0$ to T, is $F_{\dot{y}\dot{y}} < 0$, for all $t \in [0, T]$.
  - Counterparts of second order conditions in unconstrained static optimization problems.

4. Transversality condition

- Consider terminal surface problem.
  - Objective is to maximize $\int_0^\tau F(y, \dot{y}, t)dt$.
  - Choice variables are $\dot{y}(t)$, τ and $y(\tau)$, with $y(0)$ fixed and $\Omega(y(\tau), \tau) = 0$.

- Arbitrage for $y(t)$, terminal time T and state $y(T)$.
  - $\tilde{y}(t) = y(t) + \epsilon\eta(t)$, where $\eta(0) = 0$, $\tau(0) = T$, and for all ϵ close enough to zero, $\Omega(\tilde{y}(\tau(\epsilon)), \tau(\epsilon)) = 0$.
  - Illustration.

- No profitable arbitrage.
  - Define $J(\epsilon) = \int_0^{\tau(\epsilon)} F(y + \epsilon\eta, \dot{y} + \epsilon\dot{\eta}, t)dt$.
  - $J'(0) = 0$.

- Euler equation holds as before.
  - Follows from $J'(0) = 0$ when $\tau(\epsilon) = T$ for all ϵ.

- New condition:

$$\tau'(0)F(y(T),\dot{y}(T),T) + \eta(T)F_{\dot{y}}(y(T),\dot{y}(T),T) = 0.$$

  - Compute $\tau'(0)$ from $\Omega(y(\tau(\epsilon)) + \epsilon\eta(\tau(\epsilon)),\tau(\epsilon)) = 0$.
  - Use slope of terminal surface.
  - Condition becomes:

$$\left.\frac{dy}{d\tau}\right|_{\Omega(y,\tau)=0,\tau=T} = \dot{y}(T) - \frac{F(y(T),\dot{y}(T),T)}{F_{\dot{y}}(y(T),\dot{y}(T),T)}$$