Econ 420
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Lecture 8a. Introduction to Dynamic Optimization

1. Dynamic optimization

- Economic examples.
  - Economic growth: repeated choices between consumption and investment.
  - Search: choice between stopping with current best alternative and continuing to search for an even better one.

- Dynamic link.
  - What makes dynamic optimization different from a sequence of static optimization problems is that dynamic optimization requires forward looking.
  - Reason is dynamic link: choice made today affects tomorrow's choice problem, in terms of feasible choice and payoff.
  - Dynamic link is modeled by a state variable.
- Alternative view.
  - Dynamic optimization as one-time choice of entire plan.

- Conceptually different approaches.
  - Open loop: solution is a “plan” of controls that depends on calendar time only, and is simple to implement but not so reliable.
  - Close loop: solution is a “policy” that gives current control as function of current state, and is more complex but more reliable.

- Solution methods.
  - Calculus of variation: unconstrained dynamic optimization, and Euler equation.
  - Optimal control: co-state as Lagrange multiplier of state, and Hamiltonian function.
  - Dynamic programming: backward induction, and Bellman equation.

2. Discrete-time model

- Decision time is discrete, $t = 0, 1, \ldots, T$.
  - T can be infinity, and can be a choice variable.
  - State variable y_t at beginning of each t, with y_0 given.
  - Control variable z_t in each t.
  - Feasibility constraint $G(y_t, z_t, t) \leq 0$ for each t.

- Law of motion $y_{t+1} - y_t = Q(y_t, z_t, t)$ for each t.
  - Captures dynamic link.
  - y_t is a stock variable, and z_t is a flow variable.
  - Given z_t, $t = 0, \ldots, T$, law of motion is difference equation in y_t, and can be solved by repeated substitutions, starting from y_0.

- Terminal constraints.
  - Terminal time problems where T is fixed, with either equality or inequality constraint on terminal state y_{T+1}.
  - Terminal surface problems where both y_{T+1} and T are choice variables, satisfying $\Omega(y_{\tau+1}, \tau) = 0$.

- Objective function $\sum_{t=0}^{T} F(y_t, z_t, t)$.
  - $F(y_t, z_t, t)$: immediate function in period t depends only on state and control in period t.
  - Objective is additively separable in time.
  - Special immediate function $F(y_t, z_t)(1+r)^{-t}$ or $F(y_t, z_t)\beta^t$: maximize present value with interest rate r or discounted sum of payoffs with subjective discount factor β.

3. Continuous-time model

- Decision time is continuous variable, running from $t = 0$ to $t = T$.
  - State variable at t is now denoted as $y(t)$.
  - Control variable at t is now $z(t)$.
  - Law of motion is $\dot{y}(t) = Q(y(t), z(t), t)$, where $\dot{y}(t) = dy(t)/dt$.
  - Objective function is now $\int_0^T F(y(t), z(t), t)dt$, with special immediate function $F(y(t), z(t))e^{-\rho t}$ where $\rho > 0$ is the continuous-time discount rate.

- Continuous-time versus discrete-time.
  - Continuous-time allows differentiation with respect to time, and will be used in calculus of variations and optimal control.
  - Discrete-time allows incorporating more details of dynamic optimization, and will be used in dynamic programming.