Econ 420
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Lecture 7c. SOC: Comparative Statics

1. Comparative statics in the special case of $m = 1$

- Consider $\max_x F(x, \theta)$ subject to $G(x, \theta) \leq 0$.
  - θ is a scalar parameter that can generally appear in objective and/or constraint.
  - Inequality constraint binds.
  - FOC and SOC are satisfied.

- Lagrangian $L = F(x, \theta) - \lambda G(x, \theta)$.

  - First order conditions for $\overline{x}$ to be local maximum:

$$F_{x_j}(\overline{x}, \theta) - \lambda G_{x_j}(\overline{x}, \theta) = 0, \ \ j = 1, \ldots, n$$

$$G(\overline{x}, \theta) = 0.$$

- Totally differentiating FOC's, treating $\overline{x}_j$, $j = 1, \ldots, n$, and λ as functions of θ.

$$
\begin{bmatrix}
F_{x_1x_1}(\overline{x}) - \lambda G_{x_1x_1}(\overline{x}) & F_{x_1x_2}(\overline{x}) - \lambda G_{x_1x_2}(\overline{x}) & -G_{x_1}(\overline{x}) \\
F_{x_2x_1}(\overline{x}) - \lambda G_{x_2x_1}(\overline{x}) & F_{x_2x_2}(\overline{x}) - \lambda G_{x_2x_2}(\overline{x}) & -G_{x_2}(\overline{x}) \\
-G_{x_1}(\overline{x}) & -G_{x_2}(\overline{x}) & 0
\end{bmatrix}
\begin{bmatrix}
d\overline{x}_1 \\
d\overline{x}_2 \\
d\lambda
\end{bmatrix}
$$

$$
= -\begin{bmatrix}
F_{x_1\theta}(\overline{x}, \theta) - \lambda G_{x_1\theta}(\overline{x}, \theta) \\
F_{x_2\theta}(\overline{x}, \theta) - \lambda G_{x_2\theta}(\overline{x}, \theta) \\
-G_{\theta}(\overline{x}, \theta)
\end{bmatrix} d\theta
$$

- By second order condition, bordered Hessian matrix of Lagrangian is invertible.
  - Since $y^T L_{xx}(\overline{x}) y < 0$ for all $y \neq 0$ such that $G_x(\overline{x}) y = 0$,

$$\begin{bmatrix} L_{xx}(\overline{x}) & -G_x(\overline{x})^T \\ -G_x(\overline{x}) & 0 \end{bmatrix}$$

    is invertible.
  - Proof.

- Inverting bordered Hessian matrix,

$$\begin{bmatrix} d\overline{x}_1 \\ d\overline{x}_2 \\ d\lambda \end{bmatrix} = - \begin{bmatrix} F_{x_1x_1}(\overline{x}) - \lambda G_{x_1x_1}(\overline{x}) & F_{x_1x_2}(\overline{x}) - \lambda G_{x_1x_2}(\overline{x}) & -G_{x_1}(\overline{x}) \\ F_{x_2x_1}(\overline{x}) - \lambda G_{x_2x_1}(\overline{x}) & F_{x_2x_2}(\overline{x}) - \lambda G_{x_2x_2}(\overline{x}) & -G_{x_2}(\overline{x}) \\ -G_{x_1}(\overline{x}) & -G_{x_2}(\overline{x}) & 0 \end{bmatrix}^{-1} \begin{bmatrix} F_{x_1\theta}(\overline{x},\theta) - \lambda G_{x_1\theta}(\overline{x},\theta) \\ F_{x_2\theta}(\overline{x},\theta) - \lambda G_{x_2\theta}(\overline{x},\theta) \\ -G_{\theta}(\overline{x},\theta) \end{bmatrix} d\theta$$

2. Comparative statics in consumer theory

- Substitution effect in Hicksian demand function.
  - Expenditure $E(p, u) = \min_x\{px|U(x) \geq u\}$ is a concave function of p.
  - By envelope theorem, Hicksian demand $C(p, u) = E_p(p, u)$.
  - $\partial C_j/\partial p_j = E_{jj} \leq 0$ for all j: own substitution effect is non-positive.
- Alternative proof: revealed preference.

- Substitution effect in Slutsky compensated price change.
  - Consider consumer problem $\max_x U(x)$ subject to $px \leq I$.
  - Suppose $\overline{x}$ is a local maximum, satisfying $U_x(\overline{x}) = \lambda p$ and $p\overline{x} = I$ for some multiplier $\lambda > 0$.
  - Consider dp and dI such that $dI = dp\overline{x}$, and thus $pd\overline{x} = 0$.
  - Comparative statics:

$$\begin{bmatrix} U_{xx}(\overline{x}) & -p^T \\ -p & 0 \end{bmatrix} \begin{bmatrix} d\overline{x} \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp^T \\ 0 \end{bmatrix}.$$

– Own substitution effect is negative:

$$
\begin{aligned}
d\overline{x}^T dp^T =& \frac{1}{\lambda}\begin{bmatrix} d\overline{x}^T & d\lambda \end{bmatrix} \begin{bmatrix} \lambda dp^T \\ 0 \end{bmatrix} \\
=& \frac{1}{\lambda}\begin{bmatrix} d\overline{x}^T & d\lambda \end{bmatrix} \begin{bmatrix} U_{xx}(\overline{x}) & -p^T \\ -p & 0 \end{bmatrix} \begin{bmatrix} d\overline{x} \\ d\lambda \end{bmatrix} \\
=& d\overline{x}^T U_{xx}(\overline{x}) d\overline{x} \\
<& 0.
\end{aligned}
$$

- Alternative proof: revealed preference.

3. Comparative statics in producer theory

- Long run versus short run.
  - Long run value $V(\theta) = \max_{y,z}\{F(y, z, \theta)|G(y, z) \leq c\}$.
  - Short run value $v(z, \theta) = \max_{y}\{F(y, z, \theta)|G(y, z) \leq c\}$.
  - Denote $Z(\theta)$ as long run solution.
  - Known envelope properties: $v(Z(\theta'), \theta) \leq V(\theta)$ for all θ, θ', with equality if $\theta = \theta'$, and $v_\theta(Z(\theta), \theta) = V_\theta(\theta)$ for all θ.

- New envelope property: $v_{\theta\theta}(Z(\theta), \theta) \leq V_{\theta\theta}(\theta)$ for all θ.
- Short run value function is less convex than long run value function.
- Illustration.
- Proof.

- Derived demand in constant returns to scale industry.
  - Consider minimizing cost subject to achieving output target.
  - Minimum cost $C(w, y) = \min_x\{wx|f(x) \geq y\}$ is concave function of input prices w, and is $yc(w)$ for some $c(w)$ when production function f has constant returns to scale.
  - By envelope theorem, solution is $x = C_w(w, y) = yc_w(w)$.
  - In an industry with downward sloping aggregate demand $D(p)$, competitive equilibrium price $p = c(w)$.

– Derived demand for inputs

$$x = D(c(w))c_w(w).$$

– Own price effect is negative for two reasons:

$$\frac{\partial x_j}{\partial w_j} = Dc_{jj} + D'c_j^2 < 0.$$

- Substitution effect in optimal input demand.
  - Consider firm choosing input vector x to maximize profit $R(f(x)) - wx$, where f and R are production and revenue functions.
  - Comparative statics of local maximum $\overline{x}$ with respect to w:

$$d\overline{x} = -F_{xx}(\overline{x})^{-1}F_{xw}(\overline{x}, w)dw^T = F_{xx}(\overline{x})^{-1}dw^T.$$

- By second order condition, $F_{xx}(\overline{x})$ is negative definite.
- $F_{xx}(\overline{x})^{-1}$ exists and is also negative definite.
- Own substitution effect is negative:

$$dw d\overline{x} = dw F_{xx}(\overline{x})^{-1} dw^T < 0,$$

  for all $dw \neq 0$.