Econ 420
Fall, 2022
Li, Hao
UBC

Lecture 7b. SOC: Constrained Optimization

1. Deriving second order sufficient conditions

- Consider $\max_x F(x)$ subject to $G(x) = c$.
  - Graphical method: local separation of contour sets.
  - Algebraic method: no profitable arbitrage.
  - Assume inequality constraint binds – otherwise we have local concavity as second order conditions.

2. A graphical derivation

- Assume $n = 2$ and $m = 1$.

- First order necessary condition for $(\overline{x}_1, \overline{x}_2)$ to be a local maximum: for some λ, and for each $j = 1, 2$,

$$F_j(\overline{x}) = \lambda G_j(\overline{x}).$$

  - Tangency of level curves at $\overline{x}$:

$$\left.\frac{dx_2}{dx_1}\right|_{x=\overline{x}, F(x)=F(\overline{x})} = \left.\frac{dx_2}{dx_1}\right|_{x=\overline{x}, G(x)=G(\overline{x})}.$$

- Second order sufficient condition for $\overline{x}$ to be local maximum.

  – Level curve of F through $\overline{x}$ is "more convex" than level curve of G through $\overline{x}$:

$$\left.\frac{d^2x_2}{dx_1^2}\right|_{x=\overline{x},F(x)=F(\overline{x})} > \left.\frac{d^2x_2}{dx_1^2}\right|_{x=\overline{x},G(x)=G(\overline{x})}.$$

- Graphical condition is local version of separation by maximization.
  - Local separation of upper contour set of $F(x)$ at $\overline{x}$ from lower contour set of $G(x)$ at $\overline{x}$.
  - Illustration: assuming $F_j(\overline{x}) > 0$, $j = 1, 2$.

- Derive second order sufficient condition from graphical condition:

$$\begin{vmatrix} F_{11}(\overline{x}) - \lambda G_{11}(\overline{x}) & F_{12}(\overline{x}) - \lambda G_{12}(\overline{x}) & -G_1(\overline{x}) \\ F_{21}(\overline{x}) - \lambda G_{21}(\overline{x}) & F_{22}(\overline{x}) - \lambda G_{22}(\overline{x}) & -G_2(\overline{x}) \\ -G_1(\overline{x}) & -G_2(\overline{x}) & 0 \end{vmatrix} > 0.$$

  – Determinant of bordered Hessian of Lagrangian is strictly positive.

– Derivation.

3. An algebraic derivation

- $\max_x F(x)$ subject to $G(x) = c$.
  - Allow more than 2 choice variables and more than 1 binding constraint.
  - Apply no profitable arbitrage, as in unconstrained problem.
  - Instead of quadratic forms, we will have constrained quadratic forms.

- Derive constrained quadratic forms.
  - Lagrangian: $L = F(x) + \lambda(c - G(x))$.
  - First order necessary condition: $F_x(\overline{x}) = \sum_{i=1}^{m} \lambda_i G_x^i(\overline{x})$, or in matrix form

    $$F_x(\overline{x}) = \lambda G_x(\overline{x}),$$

    where $F_x(\overline{x})$ is $1 \times n$, λ is $1 \times m$, and $G_x(\overline{x})$ is $m \times n$.

– Taylor expansions:

$$F(\overline{x} + dx) = F(\overline{x}) + F_x(\overline{x})dx + \frac{1}{2}dx^T F_{xx}(\overline{x})dx,$$

and for each $i = 1, \ldots, m$,

$$G^i(\overline{x} + dx) = G^i(\overline{x}) + G^i_x(\overline{x})dx + \frac{1}{2}dx^T G^i_{xx}(\overline{x})dx$$

– Since $G^i(\overline{x} + dx) = G^i(\overline{x}) = c_i$, $i = 1, \ldots, m$,

$$F(\overline{x} + dx) = F(\overline{x}) + \frac{1}{2}dx^T \left(F_{xx}(\overline{x}) - \sum_{i=1}^{m} \lambda_i G^i_{xx}(\overline{x}) \right) dx.$$

- No profitable arbitrage.
  - Second order sufficient condition for $\overline{x}$ to be local maximum: constrained quadratic form $dx^T L_{xx}(\overline{x})dx < 0$ for all dx such that $G^i_x(\overline{x})dx = 0$, $i = 1, \ldots, m$, where

$$L_{xx}(\overline{x}) = F_{xx}(\overline{x}) - \sum_{i=1}^{m} \lambda_i G^i_{xx}(\overline{x})$$

    is the Hessian matrix of the Lagrangian function.

– For $n = 2$, derive second order sufficient condition:

$$\begin{vmatrix} F_{11}(\overline{x}) - \lambda G_{11}(\overline{x}) & F_{12}(\overline{x}) - \lambda G_{12}(\overline{x}) & -G_1(\overline{x}) \\ F_{21}(\overline{x}) - \lambda G_{21}(\overline{x}) & F_{22}(\overline{x}) - \lambda G_{22}(\overline{x}) & -G_2(\overline{x}) \\ -G_1(\overline{x}) & -G_2(\overline{x}) & 0 \end{vmatrix} > 0.$$

- Comparison with unconstrained optimization.

  - Second order sufficient condition: $dx^T F_{xx}(\overline{x}) dx < 0$ for all $dx \neq 0$.

  - For $n = 2$, second order conditions become $F_{11}(\overline{x}) < 0$, and

$$\begin{vmatrix} F_{11}(\overline{x}) & F_{12}(\overline{x}) \\ F_{21}(\overline{x}) & F_{22}(\overline{x}) \end{vmatrix} > 0.$$

  - F may not be locally concave but SOC is still satisfied for constrained optimization.

4. SOC and concave programming

- Concave programming: maximize concave objective $F(x)$, subject to $G(x) \leq c$ where each G^i is convex.
  - First order condition $F_x(\overline{x}) = \lambda G_x(\overline{x})$, together with the complementary slackness conditions for each $G^i(x) \leq c$, is sufficient for $\overline{x}$ to be solution.
  - SOC must be automatically satisfied: $dx^T L_{xx}(\overline{x}) dx < 0$ for all dx such that $G^i_x(\overline{x})dx = 0$ for each binding constraint i.
  - Proof.

5. SOC and quasi-concave programming

- Quasi-concave programming: maximize a strictly quasi-concave objective $F(x)$, subject to linear constraint $G(x) = p \cdot x = c$.
  - First order condition $F_x(\overline{x}) = \lambda p$, together with the binding constraint $p \cdot \overline{x} = c$, is sufficient for $\overline{x}$ to be solution.
  - Second order condition must be automatically satisfied, so we have $dx^T F_{xx}(\overline{x}) dx < 0$ for all dx such that $F_x(\overline{x}) dx = 0$.

- Strict local quasi-concavity.
  - F is strictly quasi-concave at $\overline{x}$, if $F(x) > F(\overline{x})$ implies that $F_x(\overline{x})(x - \overline{x}) > 0$.
  - Illustration of strict local quasi-concavity for $n = 2$.
  - If F is strictly quasi-concave at $\overline{x}$, then $dx^T F_{xx}(\overline{x}) dx < 0$ for all dx such that $F_x(\overline{x}) dx = 0$.
  - Proof.