Econ 420
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Lecture 7a. SOC: Unconstrained Optimization

1. Local versus global optimum

- Global maximum in concave programming.
  - Necessary and sufficient conditions in concave programming are for global maximum.
  - But concave programming is restrictive.

- Local optimum instead of global maximum is often sufficient for comparative statics.
  - Comparative statics is local.
  - For comparative statics, we just need problem to be locally concave at the solution.
  - Second order conditions for optimum.
  - Comparative statics of solution, not just value function.

2. Unconstrained problems: Scalar choice variable

- Consider $\max_x F(x)$.
  - x is a scalar choice variable.
  - Suppose $\overline{x}$ is a local maximum.
  - $x - \overline{x}$ is arbitrage.
  - Use no profitable arbitrage to derive second-order necessary and sufficient conditions for $\overline{x}$ to be a local maximum.

- Taylor expansion around $\overline{x}$.

$$F(x) = F(\overline{x}) + F'(\overline{x})(x - \overline{x}) + \frac{1}{2}F''(\overline{x})(x - \overline{x})^2.$$

  - First-order necessary condition: $F'(\overline{x}) = 0$.
  - Second-order necessary condition: $F''(\overline{x}) \leq 0$.
  - Second-order sufficient condition: $F''(\overline{x}) < 0$.

- F is called locally (strictly) concave at $\overline{x}$ if $F''(\overline{x}) \leq (<) 0$.
  - Differentiation: global concavity implies local concavity at any $\overline{x}$.
  - Integration: local concavity at all $\overline{x}$ implies global concavity.
  - F can be locally concave at some $\overline{x}$ without being globally concave.

- Regular local maximum.
  - When $F'(\overline{x}) = 0$ and $F''(\overline{x}) < 0$.
  - Irregular local maximum?

- Comparative statics: $\max_x F(x, \theta)$, where θ is scalar parameter.
  - Envelope theorem gives comparative statics of value function:

$$V'(\theta) = F_\theta(X(\theta), \theta).$$

  - Comparative statics of solution function $X(\theta)$:

$$X'(\theta) = -\frac{F_{x\theta}(X(\theta), \theta)}{F_{xx}(X(\theta), \theta)},$$

    so $X'(\theta)$ has same sign as $F_{x\theta}(X(\theta), \theta)$ at regular maximum.

- Comparative statics: an example.

  - Consider maximizing revenue $F(x, \theta)$, where x is quantity produced and θ is a demand parameter, and $F_\theta(x, \theta) > 0$ so that a greater θ increases demand and raises revenue function.

  - An increase in θ necessarily increases the maximum revenue.

  - It does not however necessarily increase optimal quantity.

3. Unconstrained problems: Vector choice variables

- Consider $\max_x F(x)$, where x is n-dimensional choice variable.
  - Suppose x is local maximum.
  - Arbitrage $x - \overline{x}$.
  - Taylor expansion around $\overline{x}$:

$$F(x) = F(\overline{x}) + F_x(\overline{x})(x - \overline{x}) + \frac{1}{2}(x - \overline{x})^T F_{xx}(\overline{x})(x - \overline{x}).$$

- $F_x(\overline{x})$ is gradient row vector of F at $\overline{x}$.

- $(x - \overline{x})^T$ is n-dimensional row vector, transpose of $(x - \overline{x})$.

- $F_{xx}(\overline{x})$ is $n \times n$ symmetric matrix of second order derivatives of F at $\overline{x}$ (Hessian matrix), with j-th row and k-column

$$F_{jk}(\overline{x}) = \frac{\partial^2 F(x)}{\partial x_j \partial x_k}\Big|_{x=\overline{x}}.$$

- Quadratic form in $(x_j - \overline{x}_j)$, $j = 1, \ldots, n$:

$$(x - \overline{x})^T F_{xx}(\overline{x})(x - \overline{x}) = \sum_{j=1}^{n} \sum_{k=1}^{n} (x_j - \overline{x}_j) F_{jk}(\overline{x})(x_k - \overline{x}_k).$$

- No profitable arbitrage.
  - First order necessary condition: $F_x(\overline{x}) = 0$.
  - Second order necessary condition: $(x-\overline{x})^T F_{xx}(\overline{x})(x-\overline{x}) \leq 0$ for all x.
  - Second order sufficient condition: $(x-\overline{x})^T F_{xx}(\overline{x})(x-\overline{x}) < 0$ for all $x \neq \overline{x}$.

- Local (strict) concavity.
  - F is locally concave at $\overline{x}$ if $(x - \overline{x})^T F_{xx}(\overline{x})(x - \overline{x}) \leq 0$ for all x, and is locally strictly concave at $\overline{x}$ if the inequality is strict for all $x \neq \overline{x}$.
  - Global concavity implies local concavity at any $\overline{x}$ (convert to scalar case and differentiate), and conversely local concavity at all $\overline{x}$ implies global concavity (convert to scalar case and integrate).
  - F can be locally concave at some $\overline{x}$ without being globally concave.

- Negative (semi) definiteness.
  - Sign of quadratic form $(x-\overline{x})^T F_{xx}(\overline{x})(x-\overline{x})$ is independent of magnitude of arbitrage $(x-\overline{x})$, and is a property of Hessian matrix $F_{xx}(\overline{x})$.
  - Symmetric $n \times n$ matrix M is negative semidefinite if quadratic form $y^T M y \leq 0$ for all $y \in \mathbb{R}^n$, and is negative definite if $y^T M y < 0$ for all $y \neq 0$.

- Some properties of negative definite Hessian matrix $F_{xx}(\overline{x})$.
  - $F_{jj}(\overline{x}) < 0$ for each i ($F''(\overline{x}) < 0$ in scalar case).
  - $(x - \overline{x})^T(-F_{xx}(\overline{x}))(x - \overline{x}) > 0$ for all $x \neq \overline{x}$ ($-F_{xx}(\overline{x})$ is called positive definite, F is locally convex at $\overline{x}$, and second order sufficient condition for local minimum $\overline{x}$ is satisfied).
  - $F_{xx}^{-1}(\overline{x})$ exists (needed for comparative statics).

- Comparative statics: consider $\max_x F(x, \theta)$, where θ is s-dimensional column vector of parameters.
  - First order necessary condition for local maximum $\overline{x}$:

$$F_x(\overline{x}, \theta) = 0.$$

  - Differentiating and using second order sufficient condition,

$$d\overline{x} = -F_{xx}^{-1}(\overline{x}, \theta) F_{x\theta}(\overline{x}, \theta) d\theta,$$

    where $F_{x\theta}(\overline{x}, \theta)$ is $n \times s$ matrix of partial derivatives of F.