Econ 420
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Lecture 6b. Quasi-concave Programming

1. Extension of concave programming

- Consider $\max_x F(x)$ subject to $G(x) \leq c$, where F is quasi-concave and G is vector quasi-convex.
  - Kuhn-Tucker conditions are both necessary and sufficient.
  - Will establish extension of concave programming in special cases only.

2. Special case of quasi-concave programming

- Consider $\max_x F(x)$ subject to $px \leq c$, where F is quasi-concave.
  - A single linear constraint function instead of a vector of quasi-convex constraint functions.
  - Economic application: consumer demand problem.

- Kuhn-Tucker conditions for $\overline{x}$ to be solution: there exists λ such that

  - (i') $F_x(\overline{x}) - \lambda p = 0$;

  - (ii) $\lambda \geq 0$ and $p\overline{x} \leq c$, with complementary slackness.

- Kuhn-Tucker conditions are still necessary.
  - Necessity of (i') and (ii), with no restrictions on F, follows from general argument of no profitable arbitrage, so long as price vector p is not 0.
  - (i) $\overline{x}$ maximizes $F(x) - \lambda px$ without constraint.
  - Necessity of (i) and (ii) generally fails if F is quasi-concave rather than concave.

- Question is whether (i') and (ii) are sufficient.
  - (i) and (ii) are always sufficient, with no restrictions on F.
  - When F is concave, sufficiency of (i') and (ii) follows from sufficiency of (i) and (ii).
  - With F quasi-concave, we show sufficiency of (i') and (ii) directly, because (i) and (ii) are generally not necessary.

- We will establish sufficiency of (i') and (ii) for the case of $\lambda > 0$.
  - (ii) requires $p\overline{x} = c$.
  - Suffices to show that, if for some $\overline{x}$ satisfying $p\overline{x} = c$ there exists $\lambda > 0$ such that $F_x(\overline{x}) = \lambda p$, then $\overline{x}$ solves $\max_x F(x)$ subject to $px \leq c$.

- A characterization of quasi-concave functions.
  - If $F(\cdot)$ is quasi-concave, then for all x^a, x^b, $F(x^b) \geq F(x^a)$ implies $F_x(x^a)(x^b - x^a) \geq 0$.
  - Illustration for $n = 2$.
  - Proof.

- Proof of sufficiency of (i') and (ii).
  - Suppose $F(\widetilde{x}) > F(\overline{x})$.
  - Use characterization to show that $p\widetilde{x} > c$.