Econ 420
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Lecture 6a. Concave Programming

1. Special class of constrained optimization

- Consider $\max_x F(x)$ subject to $G^i(x) \leq c_i$, $i = 1, \ldots, m$.
  - F is concave.
  - Each G^i is convex.

- Kuhn-Tucker conditions are both necessary and sufficient.
  - Necessity follows from separating hyperplane theorem, using concavity of F and convexity of G.
  - Sufficiency follows from more general result about maximizing the Lagrangian, and then uses concavity of F and convexity of G.

2. Necessary conditions for concave programming

- Economic context of $\max_x F(x)$ subject to $G(x) \leq c$.
  - F is revenue function.
  - x is n-dimensional output vector.
  - G is m-dimensional vector of input requirements.
  - c is vector of input supplies.

- Properties of maximum value function $V(c)$.
  - $V(c)$ is non-decreasing.
  - $V(c)$ is concave if F is concave and G is convex.

- Convex sets.

  - $A \equiv \{(c, v) \in \mathbb{R}^{m+1} | v \leq V(c)\}$, all combinations of input supplies and achievable revenue: Proof of convexity of A.

  - $B \equiv \{(c, v) \in \mathbb{R}^{m+1} | c \leq c^*, v \geq v^*\}$ for some (c^*, v^*) such that $v^* = V(c^*)$, set of (weakly) unachievable combinations at (c^*, v^*).

- Separation of A and B.
  - A and B have no common interior points.
  - Illustration.

- Separating hyperplane.
  - There exist $\lambda \in \mathbb{R}^m$ and $\iota \in \mathbb{R}$, $(-\lambda, \iota) \neq 0$, such that

$$\iota v - \lambda c \begin{cases} \leq \iota v^* - \lambda c^* & \text{if } (c, v) \in A \\ \geq \iota v^* - \lambda c^* & \text{if } (c, v) \in B \end{cases}$$

  - $\iota \geq 0$.
  - $\lambda_i \geq 0$ for each $i = 1, \ldots, m$.

- Slater's condition: $\exists x^0$ such that $G(x^0) \ll c^*$.
  - Constraint qualification to ensure $\iota > 0$.
  - An illustration of a separating hyperplane independent of v^*.
  - Economic interpretation of Slater's condition.
  - Proof of $\iota > 0$ under Slater's condition.

- If $\overline{x}$ maximizes $F(x)$ subject to $G(x) \leq c$, where F is concave, G is vector-convex, and if there exists x^0 such that $G(x^0) \ll c$, then a row vector λ can be found such that:
  - (i) $\overline{x}$ maximizes $F(x) - \lambda G(x)$ without constraint;
  - (ii) $\lambda \geq 0$ and $G(\overline{x}) \leq c$, with complementary slackness.

- Saddle point interpretation of conditions (i) and (ii) for solution pair $(\overline{x}, \overline{\lambda})$.

  – Define Lagrangian $L(x, \lambda) = F(x) + \lambda(c - G(x))$.

  – Condition (i): $\overline{x}$ maximizes $L(x, \overline{\lambda})$ among all x.

  – Condition (ii): $\overline{\lambda}$ minimizes $L(\overline{x}, \lambda)$ among all $\lambda \geq 0$.

- Proof.

  – Let V be the maximum value function.

  – There exists an m-dimensional row vector $\lambda \geq 0$ such that $\tilde{v} - \lambda \tilde{c} \leq V(c) - \lambda c$ for all $(\tilde{c}, \tilde{v}) \in \mathbb{R}^{m+1}$ satisfying $\tilde{v} \leq V(\tilde{c})$.

  – Apply separation inequality to $(G(\overline{x}), F(\overline{x}))$ to get (ii), and then to $(G(x), F(x))$ to get (i).

- Kuhn-Tucker conditions are necessary for concave programming.
  - (i') $F_x(\overline{x}) - \lambda G_x(\overline{x}) = 0$.
  - Proof: (i') is necessary for (i).
  - This is an alternative proof in the special case of concave programming.

- Kuhn-Tucker conditions (i') and (ii) are always necessary, but (i) and (ii) are generally not necessary outside concave programming.
  - Example: $\max_x e^x$ subject to $x \leq 1$.

2. Sufficient conditions for concave programming

- $\overline{x}$ maximizes $F(x)$ subject to $G(x) \leq c$ if there exists m-dimensional row vector λ such that
  - (i) $\overline{x}$ maximizes $F(x) - \lambda G(x)$ without constraint;
  - (ii) $\lambda \geq 0$ and $G(\overline{x}) \leq c$, with complementary slackness.
- Proof.

- Kuhn-Tucker conditions are sufficient for concave programming.
  - (i') $F_x(\overline{x}) - \lambda G_x(\overline{x}) = 0$.
  - (i') is sufficient for (i) if F is concave and G is vector convex.
  - Proof: $H(x) \equiv F(x) - \lambda G(x)$ is concave, and satisfies

$$H(x^b) - H(x^a) \leq H_x(x^a)(x^b - x^a)$$

for all x^a, x^b.

- Kuhn-Tucker conditions (i') and (ii) are always necessary, but are generally insufficient outside concave programming.
  - Reason: (i) always implies (i'), but (i') generally does not imply (i).