Econ 420
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Lecture 5a. Separating Convex Sets

1. Maximization by separation

- A sufficient condition for optimum.
  - Introduce a separating hyperplane theorem.
  - Apply the theorem to constrained optimization.

- Illustration of maximization by separation ($n = 2$, $m = 1$).
  - Consider $\max_x F(x)$ subject to $G(x) \leq c$.
  - $\overline{x}$ is solution if there is a line through $\overline{x}$ separating convex sets $\{x : F(x) \geq F(\overline{x})\}$ and $\{x : G(x) \leq c\}$.

2. Convex contour sets

- Convex set.
  - A set S is convex if $x^a, x^b \in S$ implies $\theta x^a + (1-\theta)x^b \in S$ for all $\theta \in [0, 1]$.
  - Illustration of convex combination.

- Quasi-convex functions.
  - G is quasi-convex if lower contour sets $\{x : G(x) \leq c\}$ are convex.
  - Equivalently, G is quasi-convex if for all x^a, x^b and $\theta \in [0, 1]$, $G(\theta x^a + (1-\theta)x^b) \leq \max\{G(x^a), G(x^b)\}$.
  - Proof of equivalence.

- Economics of quasi-convexity of G: diminishing marginal rate of transformation.
- Illustration.

- Quasi-concave functions.
  - F is quasi-concave if upper contour sets $\{x : F(x) \geq v\}$ are convex.
  - Equivalently, F is quasi-concave if for all x^a, x^b and $\theta \in [0, 1]$, $F(\theta x^a + (1 - \theta)x^b) \geq \min\{F(x^a), F(x^b)\}$.
  - Proof of equivalence.

- Economics of quasi-concavity of F: diminishing marginal rate of substitution.
- Illustration.

- Convex and concave functions.
  - G is convex if $G(\theta x^a + (1-\theta)x^b) \leq \theta G(x^a) + (1-\theta)G(x^b)$

    for all x^a, x^b and $\theta \in [0,1]$.
  - Illustration.
  - Economics of convexity.

- Convexity implies quasi-convexity but reverse is not true.
- Quasi-convexity is invariant to monotone transformation, but convexity is not.

– F is concave if $F(\theta x^a + (1-\theta)x^b) \geq \theta F(x^a) + (1-\theta)F(x^b)$ for all x^a, x^b and $\theta \in [0,1]$.

– Illustration.

– Economics of concavity.

- Concavity implies quasi-concavity but reverse is not true.
- Quasi-concavity is invariant to monotone transformation, but concavity is not.

3. A separating hyperplane theorem

- If sets A and B are both convex, at least one of them has a non-empty interior, and they have no interior points in common.
  - Then there exist a non-zero vector p and a number b, such that $px \leq b$ for all $x \in A$ and $px \geq b$ for all $x \in B$.

- Interior versus boundary points.
  - $x^a \in S$ is interior if there exists $r > 0$ such that all points within r radius around x^a are in S.
  - Example: any x^a such that $G(x^a) < c$ is interior point of $\{x : G(x) \leq c\}$.

- $x^a \in S$ is a boundary point if there for any $r > 0$ there are points within r radius around x^a are not in S.
- Example: any x^a such that $F(x^a) = v$ is a boundary point of $\{x : F(x) \geq v\}$.

- Illustration of separating hyperplane theorem.

- Separation is impossible if A and B have no interior points.
  - Illustration.

- Separation is impossible if one of A and B is not convex.
  - Illustration.

- Separation is impossible if A and B have common interior points.
  - Illustration.