Econ 420
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# Lecture 4c. Duality in Consumer Theory

## 1. Indirect utility function

- Consider consumer problem: $\max_x U(x)$ subject to $px \leq I$.
  - $U$ is utility function over quantities $x$ of $n$ goods.
  - $p$ is vector of prices.
  - $I$ is budget.

- First order necessary conditions.

  - Lagrangian

$$L = U(x) + \lambda(I - px).$$

  - Assume non-negativity constraints on $x$ are slack and $\lambda > 0$.

  - First order conditions:

$$U_x(\overline{x}) = \lambda p,$$

$$I = p\overline{x}.$$

- Indirect utility function $V(p, I)$: the maximum value function of consumer problem.
  - "Indirect" as opposed to "direct" utility function $U$.
  - $V_I(p, I) = \lambda$: marginal utility of income.
  - $V_p(p, I) = -\lambda \overline{x}$: marginal impact on the budget times marginal utility of income.

- Marshallian (uncompensated) demand function $D(p, I)$: solution function of consumer problem.
  - Deriving $D(p, I)$ from $V(p, I)$:

$$D(p, I) = -\frac{V_p(p, I)}{V_I(p, I)}.$$

## 2. Expenditure function

- Consider "dual problem" $\min_x px$ subject to $U(x) \geq u$.
  - Same choice variables as in "primal" problem.
  - Same price vector $p$.
  - Same utility function $U$, but appearing in constraint.
  - $u$ is utility target.

- First order necessary conditions.

  - Lagrangian

$$L = px + \mu(u - U(x))$$

  - Assume non-negativity constraints on $x$ are slack and $\mu > 0$.

  - First order conditions:

$$p - \mu U_x(\overline{x}),$$

$$U(\overline{x}) = u.$$

- Expenditure function $E(p, u)$: the minimum value function of the dual problem.
  - $E_u(p, u) = \mu$: inverse of marginal utility of income.
  - $E_p(p, u) = \overline{x}$: Impact of price increase on expenditure.

- Hicksian (compensated) demand function $C(p, u)$: the solution function of the dual problem.
  - "Compensated"?
  - Deriving $C(p, u)$ from $E_p(p, u)$:

$$C(p, u) = E_p(p, u).$$

## 3. Duality

- Illustration of duality.

- $E(p, V(p, I)) = I$.
  - Identity in parameters $p$ and $I$.
  - Implies $\lambda\mu = 1$.
  - Proof: $E(p, V(p, I)) \leq I$, and $E(p, V(p, I)) \geq I$.

- $V(p, E(p, u)) = u$.
  - Identity in parameters $p$ and $u$.
  - Implies $\lambda\mu = 1$.
  - Proof: $V(p, E(p, u)) \geq u$, and $V(p, E(p, u)) \leq u$.

- Link between uncompensated and compensated demand functions:

$$D(p, I) = C(p, V(p, I)),$$

$$D(p, E(p, u)) = C(p, u).$$

– Implies Slutsky equation:

$$D_k^j(p, I) = C_k^j(p, V(p, I)) - D_I^j(p, I) D^k(p, I).$$

– Illustration.

– Proof.