Econ 420
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Lecture 4a. Maximum Value Function

1. Comparative statics everywhere

- Extending comparative statics analysis to parameters in objective function.
  - Application to producer theory (short run vs long run costs).
  - Application to consumer theory (duality).

2. Parameters in objective function

- Consider $\max_x F(x, \theta)$ subject to $G(x) \leq c$.
  - θ is a column vector of k parameters.
  - Comparative statics: how does value of problem change with marginal changes in θ?
  - As θ changes to $\theta + d\theta$, solution $\overline{x}$ changes to $\overline{x} + d\overline{x}$, how is the change in value $dv = F(\overline{x} + d\overline{x}, \theta + d\theta) - F(\overline{x}, \theta)$ related to $d\theta$ and $d\overline{x}$?

- First order conditions.
  - Lagrangian $L = F(x, \theta) + \lambda(c - G(x))$.
  - First order conditions:

$$F_x(\overline{x}, \theta) - \lambda G_x(\overline{x}) = 0,$$

$$G(\overline{x}) \leq c, \ \ \lambda \geq 0,$$

with complementary slackness.

- Use first order conditions to show $dv = F_\theta(\overline{x}, \theta)d\theta$.
  - F_θ is row vector of partial derivatives of F with respect to θ.
  - Proof.

- Understanding $dv = F_\theta(\overline{x}, \theta)d\theta$.
  - Changes in v are due to changes in θ alone.
  - Solution $\overline{x}$ does change, but $F_x(\overline{x})d\overline{x} = 0$.

- Understanding $dv = F_\theta(\overline{x}, \theta)d\theta$.
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3. An example: Cost minimization

- Consider $\min_x \theta x$ subject to $F(x) \geq Q$.
  - x is a column vector of n inputs.
  - θ is a row vector of input prices.
  - $F(x)$ is production function.
  - Q is output target.

- Comparative statics: $dv = d\theta\overline{x}$.
  - Interpretation: a marginal increase in input price θ_j increases minimum cost of producing Q by $\overline{x}_j$, as if solution $\overline{x}_j$ does not change when θ_j increases.

4. Envelope theorem

- Maximum value function.
  - Define

$$V(\theta) = \max_x \{F(x, \theta) | G(x) \leq c\}$$

as the maximized value of the objective function as a function of parameters θ.

  - Define solution function

$$X(\theta) = \arg\max_x \{F(x, \theta) | G(x) \leq c\}.$$

- Suppose θ is a scalar.
  - $V(\theta_1) = F(X(\theta_1), \theta_1)$ for all θ_1.
  - For any fixed θ_1, we have $V(\theta) \geq F(X(\theta_1), \theta)$ for all θ.
  - Thus $V'(\theta_1) = F_\theta(X(\theta_1), \theta_1)$.
  - This is precisely $dv = F_\theta(\overline{x}, \theta)d\theta$.

- Envelope theorem.
  - Graphical illustration of $dv = F_\theta(\overline{x}, \theta)d\theta$.
  - $V(\theta)$ is upper envelope of family of functions $\{F(X(\theta'), \theta)\}_{\theta'}$.

- Cost minimization problem again.
  - Minimum cost (as a function of an input price) is a lower envelope of cost lines for fixed input mix.

5. Generalized envelope theorem

- We now allow parameters to appear in objective function as well as constraints.
  - Consider $\max_x F(x, \theta)$ subject to $G(x, \theta) \leq c$.
  - Want to know how value of problem change with marginal changes in θ.

- Generalized envelope theorem: $dv = L_\theta d\theta$.
  - L_θ is partial derivative of L with respect to θ.
  - Proof.