Econ 420
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Lecture 3c. Shadow Price of Quota

1. Quota in market economy

- Quota constraints on quantities.
  - Transaction costs can prevent economic agents from quickly adjusting quantities.
  - Relating shadow price of quota to real prices.

2. A producer's problem

- A producer maximizes output $F(z_1, \ldots, z_n)$ by choosing n input quantities.
  - $z_j = x_j + y_j$, $j = 1, \ldots, n$,where x_j and y_j are respectively unregulated and regulated quantities of same input j.
  - Budget constraint: $\sum_{j=1}^{n}(p_j x_j + q_j y_j) \leq B$, where p_j and q_j are respectively price of unregulated input and regulated input j, with $p_j > q_j > 0$.
  - Total quota on regulated inputs: $\sum_{j=1}^{n} y_j \leq K$.

- Lagrangian:

$$L = F(x_1 + y_1, \ldots, x_n + y_n) + \lambda \left(B - \sum_{j=1}^{n} p_j x_j - \sum_{j=1}^{n} q_j y_j \right) + \mu \left(K - \sum_{j=1}^{n} y_j \right).$$

  - λ is multiplier for the budget constraint.
  - μ is multiplier for the quota constraint.

- Kuhn Tucker conditions:

$$\frac{\partial F}{\partial z_j} - \lambda p_j \leq 0,\ x_j \geq 0$$
$$\frac{\partial F}{\partial z_j} - \lambda q_j - \mu \leq 0,\ y_j \geq 0$$
$$\sum_{j=1}^{n} (p_j x_j + q_j y_j) \leq B,\ \lambda \geq 0$$
$$\sum_{j=1}^{n} y_j \leq K,\ \mu \geq 0$$

  all with complementary slackness.

3. Constraint analysis

- Simplification: make sure budget constraint is binding and $\lambda > 0$ at solution.
  - Assume $\partial F/\partial z_j > 0$ for some j and for all z_j.
  - Proof of sufficiency.

- Simplification: make sure non-negativity constraint on each z_j is slack at solution.
  - Assume $\partial F/\partial z_j = \infty$ at $z_j = 0$ for each $j = 1, \ldots, n$.
  - An implication: at solution $x_j > 0$ or $y_j > 0$, or both.
  - Proof of sufficiency.

- Simplification: make sure both unregulated and regulated inputs are used for at most one $j = 1, \ldots, n$.
  - Assume $p_j - q_j \neq p_{\tilde{j}} - q_{\tilde{j}}$ for all $j \neq \tilde{j} = 1, \ldots, n$.
  - Proof of sufficiency.

- Characterization: at solution, if $y_j > 0$ for some $j = 1, \ldots, n-1$, then $x_{\tilde{j}} = 0$ and $y_{\tilde{j}} > 0$ for all $\tilde{j}$ such $p_{\tilde{j}} - q_{\tilde{j}} > p_j - q_j$.

  - Proof.

- Structure of solution.
  - Re-order n inputs in ascending order of $p_j - q_j$.
  - At any solution, there is a critical $\hat{j}$ such that $x_j > 0$ and $y_j = 0$ for all $j < \hat{j}$; $x_{\hat{j}} \geq 0$ and $y_{\hat{j}} > 0$; and $x_j = 0$ and $y_j > 0$ for all $j > \hat{j}$.

4. Shadow prices and real prices

- If quota constraint is slack, then $\mu = 0$ and $\hat{j} = 1$.
  - Proof.

- If $\hat{j} > 1$, then $\mu > 0$, quota constraint is binding, and

$$\lambda(p_{\hat{j}-1} - q_{\hat{j}-1}) \leq \mu \leq \lambda(p_{\hat{j}} - q_{\hat{j}}).$$

  - Proof.
  - Shadow price of quota is equal (or close to) shadow price of budget times price difference of critical input.