Econ 420
Fall, 2022
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Lecture 3a. Shadow Prices

1. Comparative statics

- Critical to economic analysis: understanding how solution changes with parameters in constrained optimization.
  - Comparative? Statics?
  - Marginal parameter changes.
  - Start with how value of problem changes with parameters.

2. Equality constraints

- Special case of equality constraints: $n = 2$ and $m = 1$.
  - Consider $\max_{x_1,x_2} F(x_1, x_2)$ subject to $G(x_1, x_2) = c$.
  - Let $\overline{x}$ be the solution, and $v = F(\overline{x})$ value of problem.
  - Let dc be infinitesimal change in parameter c.
  - First order comparative statics: $dv/dc =?$

- $dv/dc = \lambda$, where λ is value of multiplier for $G(x_1, x_2) = c$.
  - Proof: let $\overline{x} + d\overline{x}$ be new solution.

- Shadow price interpretation of λ.
  - λ is the rate of change in value of problem with respect to parameter c.
  - λ measures how much dc is worth in terms of the value of problem.

- Deriving $dv/dc = \lambda$ through Lagrangian.
  - $L = F(x) + \lambda(c - G(x))$.
  - The Lagrangian method transforms $\max_x F(x)$ subject to $G(x) = c$ to $\max_x L$.
  - $dv = dL = \lambda dc$.
  - Does solution $\overline{x}$ change? If so, how does it change?

3. Inequality constraints

- General case of n choice variables and m inequality constraints.
  - $\max_x F(x)$ subject to $G(x) \leq c$.
  - Let $\overline{x}$ be the solution, and $v = F(\overline{x})$ value of problem.
  - Let dc be infinitesimal change in $m \times 1$ column vector c.
  - First order comparative statics: relate dv to dc.

- $dv = \lambda dc$, where λ is $1 \times m$ row vector of multipliers at $\overline{x}$.
  - Proof: by Kuhn Tucker theorem, $F_x(\overline{x}) = \lambda G_x(\overline{x})$.

- For each $i = 1, \ldots, m$, from $dv = \sum_{i=1}^{m} \lambda_i dc_i$ and complementary slackness, there are two cases of comparative statics of value of problem with respect to c_i.
  - $\lambda_i > 0$, and so $G^i(\overline{x}) = c_i$.
  - $G^i(\overline{x}) < c_i$, and so $\lambda_i = 0$.

- Sign of multipliers.
  - Each $\lambda_i \geq 0$: increasing c_i by dc_i relaxes inequality constraint $G^i(x) \leq c_i$ and can not reduce value v of problem.
  - $dv > 0$ if $\lambda_i > 0$, and $dv = 0$ if $\lambda_i = 0$.

4. Constrained minimization

- Consider $\min_x F(x)$ subject to $G(x) \geq c$.
  - $\max_x -F(x)$ subject to $G(x) \geq c$.
  - Lagrangian $-F(x) + \lambda(G(x) - c)$.
  - First order conditions: $-F_x(\overline{x}) + \lambda G_x(\overline{x}) = 0$.
  - Comparative statics: $-dv = -\lambda dc \implies dv = \lambda dc$.

- Interpretation of λ_i.
  - $\lambda_i \geq 0$.
  - λ_i is the rate of change in value of problem with respect to parameter c_i.
  - Increasing c_i by dc_i tightens $G^i(x) \geq c_i$ and can not reduce value of minimization problem.
  - $dv > 0$ if $\lambda_i > 0$, and $dv = 0$ if $\lambda_i = 0$.

- Direct treatment of constrained minimization problems.
  - Lagrangian $F(x) + \lambda(c - G(x))$.
  - Comparative statics: $dv = \lambda dc$.