Econ 420
Fall, 2022
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Lecture 2c. A Proof of Kuhn Tucker Theorem

1. Kuhn Tucker theorem reformulated

- Treat any non-negativity constraint as inequality constraint.
  - $x_j \geq 0$ becomes $G^i(x) \leq 0$ with $G^i(x) = -x_j$.
- For any fixed $\overline{x}$, re-label all m inequality constraints.
  - $G^i(x) \leq c_i$, $i = 1, \ldots, k$, binds at $\overline{x}$.

- Kuhn Tucker theorem restated: Suppose that $\overline{x}$ solves $\max_x F(x)$ subject to $G^i(x) \leq c_i$, $i = 1, \ldots, m$, and that gradient vectors $G^i_x(\overline{x})$ of binding constraints $i = 1, \ldots, k$ are linearly independent. Then there exist $\lambda_i \geq 0$, $i = 1, \ldots, k$, such that

$$F_x(\overline{x}) = \sum_{i=1}^{k} \lambda_i G^i_x(\overline{x}).$$

  - Choose $\lambda_i = 0$ for $i = k+1, \ldots, m$.
  - Complementary slackness.

2. Constraint qualification

- Require $k \times n$ matrix $G_x(\overline{x})$ to have rank k.
  - Otherwise, first order Taylor approximations of the binding constraints do not represent constraints on arbitrage.
  - Arbitrage dx may satisfy $G_x^i(\overline{x}) \cdot dx \leq 0$ for all $i = 1, \ldots, k$ without satisfying $G^i(\overline{x} + dx) \leq c_i$ for all $i = 1, \ldots, k$.

- An illustration of failure of constraint qualification.

  - $G^1(x) = x_1^3 - x_2 \leq 0$, $G^2(x) = x_1^3 + x_2 \leq 0$, with $\overline{x} = 0$.

3. Illustration of the proof

- Want to prove there exist $\lambda_i \geq 0$, $i = 1, \ldots, k$, such that

$$F_x(\overline{x}) = \sum_{i=1}^{k} \lambda_i G_x^i(\overline{x}).$$

  - Gradient vector $F_x(\overline{x})$ is in the "cone" formed by k gradient vectors $G_x^i(\overline{x})$ of binding constraint functions.

- An illustration with $n = 2$ and $k = 2$.

  – If $F_x(\overline{x})$ is not in the cone formed by $G_x^i(\overline{x})$, $i = 1, \ldots, k$, then there is a profitable arbitrage.

4. A proof

- Let K be the set of all vectors in $\mathbb{R}^n$ that are non-negative linear combinations of $G_x^i(\overline{x})$, $i = 1, \ldots, k$.
  - K is a cone: if $x \in K$ then $tx \in K$ for all $t \geq 0$.
  - K is convex: if $x_1, x_2 \in K$ then $tx_1 + (1-t)x_2 \in K$ for all $t \in [0, 1]$.
  - K is closed: if $x \notin K$ then there exists $\epsilon > 0$ such that $\{y \in \mathbb{R}^n : |x - y| < \epsilon\} \cap K = \emptyset$.

- Suppose that $F_x(\overline{x}) \notin K$.
  - Since K is closed, there is $y \in K$ that minimizes $|F_x(\overline{x}) - z|$ among all $z \in K$.
  - $F_x(\overline{x}) - y \neq 0$.

- Claim: $(F_x(\overline{x}) - y) \cdot y = 0$ since K is a cone.

  - Proof and illustration.

- Claim: $(F_x(\overline{x}) - y) \cdot z \leq 0$ for all $z \in K$ since K is convex.
  - Proof and illustration.

- Consider arbitrage dx proportional to $F_x(\overline{x}) - y$.
  - Feasible by the second claim.
  - Profitable by the first claim.