Econ 420
Fall, 2022
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Lecture 2b. Kuhn Tucker Theorem in Practice

1. Using Kuhn Tucker theorem

- Use first order necessary conditions to find solutions to constrained optimization problems.
  - Assume a solution exists.
  - Solving first order necessary conditions.

- First order necessary conditions for $\overline{x}$ to solve $\max_x F(x)$ subject to $G(x) \leq c$ and $x \geq 0$:
  - Define Lagrangian $L(x, \lambda) = F(x) + \lambda(c - G(x))$, where λ is m-dimensional row vector.
  - There exists λ such that

$$L_x(\overline{x}, \lambda) \leq 0, \; \overline{x} \geq 0,$$

$$L_\lambda(\overline{x}, \lambda) \geq 0, \; \lambda \geq 0,$$

    both with complementary slackness.

- Solving inequalities with complementary slackness.
  - $m + n$ pairs of weak inequalities.
  - By complementary slackness, we can select 1 equation from each pair to get $m + n$ equations for $m + n$ unknowns.
  - 2^{m+n} different systems of equations.
  - Need constraint analysis to reduce number of systems to solve.

2. Quasi-linear preferences

- Consumer problem: $\max_{x,y} \alpha \ln(x) + y$, subject to $px + qy \leq I$ and $x, y \geq 0$, where $\alpha, p, q, I > 0$.
  - $\alpha > 0$ represents relative importance of good X.
  - x and y are quantities of two goods, say X and Y.
  - p and q are prices of X and Y respectively.
  - I is income.

- Illustrating quasi-linearity: marginal rate of substitution between X and Y is independent of y.

- Define Lagrangian

$$L(x, y, \lambda) = \alpha \ln(x) + y + \lambda(I - px - qy).$$

  - First order necessary conditions are

$$\frac{\alpha}{x} - \lambda p \leq 0,\ \ x \geq 0$$

$$1 - \lambda q \leq 0,\ \ y \geq 0,$$

$$I - px - qy \geq 0, \lambda \geq 0,$$

    all with complementary slackness.

- Constraint analysis.

  - 3 unknowns, x, y and λ.

  - 2^3 ways of selecting 3 equations.

  - Use economic intuition to guide us to eliminate 6 out of 8 cases.

– Budget constraint binds at solution: this eliminates 4 cases.

– First order condition with respect to x holds as equality: this eliminates further 2 cases.

- First remaining case: $y = 0$.
  - Solve for unknowns: $x = I/p$ by budget constraint, and $\lambda = \alpha/I$ by first order condition with respect to x.
  - Check dropped first order condition with respect to y: need $I \leq \alpha q$ for above solution to be valid.

- Second remaining case: $1 - \lambda q = 0$.
  - Solve for unknowns: with $\lambda = 1/q$, $x = \alpha/p$ by first order condition with respect to x, and $y = I/q - \alpha$ from budget constraint.
  - Check dropped non-negativity constraint on y: need $I \geq \alpha q$ for above solution to be valid.

Solution depends on parameters.

- When $I \leq \alpha q$, solution is $x = I/p$ and $y = 0$.
- When $I \geq \alpha q$, solution is $x = \alpha q/p$ and $y = I/q - \alpha$.

- Knife edge case: $I = \alpha q$.
  - Both first order condition with respect to y and non-negativity constraint $y \geq 0$ hold as equalities.
  - Exceptional case not ruled by complementary slackness.

- Economic interpretation of solution.
  - Good X is necessity, and good Y is luxury.
  - Optimal quantity x is always positive, while optimal quantity y is 0 if I is too low, α is too high, or q is too high.

3. Technological unemployment

- Planner's production problem: $\max_{x,y} \alpha \ln(x) + \beta \ln(y)$ subject to $2x + y \leq 300$, $x + 2y \leq 450$, and $x, y \geq 0$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.
  - 300: total available units of labor.
  - 450: total available units of land.
  - x: units of wheat, labor intensive.
  - y: units of beef, land intensive.
  - α: relative weight on wheat consumption.

- Illustration of production possibility set.
  - Full labor and land employment at intersection of $x = 50$ and $y = 200$ only.
  - Otherwise unemployment for at least one resource.

- Define Lagrangian

$$L(x, y, \lambda, \mu) = \alpha \ln(x) + \beta \ln(y) + \lambda(300 - 2x - y) + \mu(450 - x - 2y).$$

  - First order necessary conditions are, all with complementary slackness,

$$\alpha/x - 2\lambda - \mu \leq 0,\ \ x \geq 0,$$

$$\beta/y - \lambda - 2\mu \leq 0,\ \ y \geq 0,$$

$$300 - 2x - y \geq 0, \lambda \geq 0,$$

$$450 - 2 - 2y \geq 0, \mu \geq 0,$$

- Constraint analysis.
  - First order condition with respect to x holds as equality.
  - First order condition with respect to y holds as equality.
  - Case of $\lambda = \mu = 0$ can be ruled out.
  - 3 remaining cases.

- First remaining case: $\lambda = 0$ and $450 - x - 2y = 0$.

  - Solve 3 equations for 3 unknowns: $x = 450\alpha$, $y = 225\beta$, and $\mu = 1/450$.

  - Check dropped first order condition $300 - 2x - y \geq 0$: need $\beta \geq 8/9$ for solution to be valid.

- Second remaining case: $300 - 2x - y = 0$ and $\mu = 0$.
  - Solve 3 equations for 3 unknowns: $x = 150\alpha$, $y = 300\beta$, and $\lambda = 1/300$.
  - Check dropped first order condition $450 - x - 2y \geq 0$: need $\beta \leq 2/3$ for solution to be valid.

- Third remaining case: $300 - 2x - y = 0$ and $450 - x - 2y = 0$.
  - Now solve 4 equations for 4 unknowns: $x = 50$, $y = 200$, $\lambda = 1/125 - 9\beta/200$, $\mu = \beta/100 - 1/150$.
  - Check dropped first order conditions $\lambda \geq 0$ and $\mu \geq 0$: need $2/3 \leq \beta \leq 8/9$ for solution to be valid.

Solution depends on parameters.

- When $\beta > 8/9$, unemployment for labor and full employment for land.
- When $\beta < 2/3$, unemployment for land and full employment for labor.
- When $2/3 \leq \beta \leq 8/9$, full employment for both land and labor.

- Illustration of the 3 cases of solution.

- Economics behind the 3 cases.