Econ 420
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Lecture 2a. Kuhn Tucker Theorem

1. Non-negativity constraints

- Inequality constraints and Kuhn-Tucker theorem via non-negativity constraints.
  - Modify the arbitrage argument.
  - Special role of non-negativity constraints in economics.

- Statement of problem
  - x: n dimensional choice variable.
  - F: scalar-valued objective function of x.
  - G: m-dimensional function of x.
  - c: m-dimensional parameter.
  - $G(x) = c$: m equality constraints.
  - $x \geq 0$: component-wise non-negativity constraints.

- Want to find necessary first order condition for $\overline{x}$ to be solution.
  - If $\overline{x}_j > 0$ for all $j = 1, \ldots, n$, same first order necessary conditions hold.
  - If $\overline{x}_j = 0$ for some j, need to derive new first order necessary conditions by modifying the arbitrage argument.

- Special case of $n = 2$ and $m = 1$ again.
  - Suppose $\overline{x}_1 = 0$ and $\overline{x}_2 > 0$, with $G_1(\overline{x}), G_2(\overline{x}) \neq 0$.
  - Modify the arbitrage argument to show first order necessary condition is

$$F_1(\overline{x}) - \frac{F_2(\overline{x})}{G_2(\overline{x})} G_1(\overline{x}) \leq 0.$$

– Derivation of first order necessary condition.

- Rewrite with multiplier for equality constraint

$$\frac{\partial L}{\partial x_1} = F_1(\overline{x}) - \lambda G_1(\overline{x}) \leq 0,$$
$$\frac{\partial L}{\partial x_2} = F_2(\overline{x}) - \lambda G_2(\overline{x}) = 0.$$

- First order necessary condition for $\overline{x}_1 > 0$ and $\overline{x}_2 = 0$ to be solution is symmetric:

$$\frac{\partial L}{\partial x_1} = F_1(\overline{x}) - \lambda G_1(\overline{x}) = 0,$$
$$\frac{\partial L}{\partial x_2} = F_2(\overline{x}) - \lambda G_2(\overline{x}) \leq 0.$$

- The two cases can be combined: for each $j = 1, 2$,

$$\frac{\partial L}{\partial x_j} = F_j(\overline{x}) - \lambda G_j(\overline{x}) \leq 0, \ \overline{x}_j \geq 0,$$

with at least one of the two inequalities holding as an equality.

  – Complementary slackness.

- With n non-negative choice variables and m equality constraints, first order necessary conditions for $\overline{x}$ to be solution are: for each $j = 1, \ldots, n$,

$$\frac{\partial L}{\partial x_j} = F_j(\overline{x}) - \sum_{i=1}^{m} \lambda_i G^i_j(\overline{x}) \leq 0,\ \overline{x}_j \geq 0,$$

  with complementary slackness, and for each $i = 1, \ldots, m$,

$$\frac{\partial L}{\partial \lambda_i} = c_i - G^i(\overline{x}) = 0.$$

  - Will prove this as a special case of Kuhn-Tucker theorem.

- Understanding complementary slackness.

  – Equivalent statement:

$$L_j(\overline{x}) \leq 0,\ \overline{x}_j \geq 0,\ L_j(\overline{x})\overline{x}_j = 0.$$

  – Ruled out: $L_j(\overline{x}) < 0$ and $\overline{x}_j > 0$.

  – What's slackness? What's complementary?

2. Inequality constraints

- Convert an inequality constraint into an equality constraint with a new choice variable constrained to be non-negative.
  - Derive first order necessary conditions for the new problem.
  - Convert back to original problem.

- Original problem: $\max F(x)$ subject to $G^1(x) \leq c_1$ and $G^i(x) = c_i$ for $i = 2, \ldots, m$.
  - Define $x_{n+1} = c_1 - G^1(x)$.
  - New problem: $\max F(x)$ subject to $G^1(x) + x_{n+1} = c_1$, $G^i(x) = c_i$ for $i = 2, \ldots, m$ and $x_{n+1} \geq 0$.

- Lagrangian $\hat{L}(x_1, \ldots, x_n, x_{n+1}, \lambda_1, \ldots, \lambda_m)$ for the new problem:

$$F(x_1, \ldots, x_n) + \lambda_1(c_1 - G^1(x_1, \ldots, x_n) - x_{n+1}) + \sum_{i=2}^{m} \lambda_i(c_i - G^i(x_1, \ldots, x_n)).$$

  – First order necessary conditions for $\overline{x}$, x_{n+1} to be solution:

$$\frac{\partial \hat{L}}{\partial x_j} = 0, \; j = 1, \ldots, n,$$
$$\frac{\partial \hat{L}}{\partial \lambda_i} = 0, \; i = 2, \ldots, m,$$
$$\frac{\partial \hat{L}}{\partial \lambda_1} = c_1 - G^1(\overline{x}) - x_{n+1} = 0,$$
$$\frac{\partial \hat{L}}{\partial x_{n+1}} = -\lambda_1 \leq 0, \; x_{n+1} \geq 0, \; -\lambda_1 x_{n+1} = 0.$$

- Define Lagrangian for the original problem as

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = F(x_1, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i (c_i - G^i(x_1, \ldots, x_n)).$$

  - First order necessary conditions for $\overline{x}$ to be solution:

$$\frac{\partial L}{\partial x_j} = 0, \; j = 1, \ldots, n,$$
$$\frac{\partial L}{\partial \lambda_i} = 0, \; i = 2, \ldots, m,$$
$$\frac{\partial L}{\partial \lambda_1} = c_1 - G^1(\overline{x}) \geq 0, \; \lambda_1 \geq 0$$

    with complementary slackness.

- Complementary slackness for the inequality constraint.
  - If inequality constraint is slack, then multiplier is 0.
  - If multiplier is strictly positive, then inequality constraint is binding.

- Multiplier for inequality constraint is non-negative.
  - For equality constraints, multipliers have no sign restriction.
  - Why sign restriction for multiplier associated with inequality constraint?

3. Kuhn Tucker theorem

- Statement of problem
  - x: n dimensional choice variable.
  - F: scalar-valued objective function of x.
  - G: m-dimensional function of x.
  - c: m-dimensional parameter.
  - $G(x) \leq c$: m inequality constraints.
  - $x \geq 0$: non-negative constraint.

- First order necessary conditions for $\overline{x}$ to be solution:
  - Define Lagrangian $L(x, \lambda) = F(x) + \lambda(c - G(x))$, where λ is m-dimensional row vector.
  - There exists λ such that

$$L_x(\overline{x}, \lambda) \leq 0, \ \overline{x} \geq 0,$$

$$L_\lambda(\overline{x}, \lambda) \geq 0, \ \lambda \geq 0,$$

both with complementary slackness.