Econ 420
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Lecture 1c. A General Proof of Lagrange's Method

1. Lagrange's first order necessary condition

- Without direct reference to Lagrangian function or multipliers.
  - If $\overline{x}$ maximizes $F(x)$ subject to $G^i(x) = c_i$, $i = 1, \ldots, m$, and if the m gradient vectors $G^i_x(\overline{x})$ are linearly independent, then gradient vector $F_x(\overline{x})$ is a linear combination of $G^i_x(\overline{x})$, $i = 1, \ldots, m$.

- Proof method

  - A contradiction argument based on arbitrage.

  - Want to show if $F_x(\overline{x})$ is not a linear combination of $G^i_x(\overline{x})$, $i = 1, \ldots, m$, then there is profitable arbitrage.

- An illustration of the proof for special case of $n = 2$ and $m = 1$.

- An illustration of the proof for $n = 3$ and $m = 2$.

- Notation

  - Let S be the space of all vectors in $\mathbb{R}^n$ spanned by $G_x^i(\overline{x})$, $i = 1, \ldots, m$:

$$S = \left\{ y \in \mathbb{R}^n : y = \sum_{i=1}^{m} \lambda_i G_x^i(\overline{x}) \text{ for some } \lambda_1, \ldots, \lambda_m \right\}.$$

  - For any $y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, define the dot product $y \cdot z = \sum_{j=1}^{n} y_j z_j$.

- Construct arbitrage.
  - Decompose $F_x(\overline{x}) = y + z$, where $y \in S$ and $z \cdot w = 0$ for all $w \in S$.
  - Since $F_x(\overline{x}) \notin S$, we have $z \neq 0$.
  - Let dx be infinitesimal and proportional to z.

- Profitable arbitrage.
  - dx is feasible: $z \cdot w = 0$ for all $w \in S$ and $G_x^i(\overline{x}) \in S$ imply that for each $i = 1, \ldots, m$,

$$dx \cdot G_x^i(\overline{x}) = 0.$$

  - dx is profitable: $z \cdot w = 0$ for all $w \in S$ and $y \in S$ imply that

$$dF = F_x(\overline{x}) \cdot dx = (y + z) \cdot dx = z \cdot dx > 0.$$