Econ 420
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Lecture 1b. Lagrange's Method: General Case

1. General statement of problem

- We have derived first order necessary condition for the special case of 2 choice variables and 1 equality constraint.
- More choice variables, and more equality constraints.

- Maximization subject to equality constraints.
  - Objective function is $F(x)$, where x_j, $j = 1, \ldots, n$, are choice variables :

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  - Constraints: $G^i(x) = c_i$, $i = 1, \ldots, m$.
  - Assume $m < n$.

2. Lagrange's method

- Define Lagrangian function

$$L(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m) = F(x_1,\ldots,x_n)+\sum_{i=1}^{m}\lambda_i(c_i-G^i(x_1,\ldots,x_n)),$$

where each λ_i, $i=1,\ldots,m$, is multiplier of i-th constraint.

- First order necessary conditions for $\overline{x}$ to be solution are: there exist $\lambda_1, \ldots, \lambda_m$ such that: for each $j = 1, \ldots, n$,

$$\frac{\partial L}{\partial x_j} = F_j(\overline{x}) - \sum_{i=1}^{m} \lambda_i G_j^i(\overline{x}) = 0,$$

and for each $i = 1, \ldots, m$,

$$\frac{\partial L}{\partial \lambda_i} = c_i - G^i(\overline{x}) = 0.$$

- Remarks.
  - Special case of $n = 2$ and $m = 1$.
  - Lagrange's method converts a constrained problem into an unconstrained problem.
  - Sign of multipliers.
  - $m + n$ equations, $m + n$ unknowns.

3. First order conditions in vector-matrix form

- Notation.
  - Let λ be the row vector of multipliers $\lambda = [\lambda_1 \ \ldots \ \lambda_m]$.
  - Write m constraints as $G(x) = c$, where $G(x)$ is column vector of functions and c is column vector of parameters:

$$G(x) = \begin{bmatrix} G^1(x) \\ \vdots \\ G^m(x) \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = c.$$

– For each $i = 1, \ldots, m$, let $G_x^i(\overline{x})$ be the gradient row vector of G^i at $\overline{x}$, given by

$$G_x^i(\overline{x}) = [G_1^i(\overline{x}) \ \ldots \ G_n^i(\overline{x})].$$

– Let $G_x(\overline{x})$ be $m \times n$ matrix of partial derivatives of vector G of functions with respect to x, evaluated at $\overline{x}$, given by

$$G_x(\overline{x}) = \begin{bmatrix} G_1^1(\overline{x}) \ldots & G_n^1(\overline{x}) \\ \vdots & \vdots \\ G_1^m(\overline{x}) \ldots & G_n^m(\overline{x}) \end{bmatrix}$$

- An illustration of gradient vector of $G(x_1, x_2)$ at $(\overline{x}_1, \overline{x}_2)$.

  – Tangent vector $[dx_1\ dx_2]$ of level curve of $G(x)$ at $\overline{x}$ satisfies $G_1(\overline{x})dx_1 + G_2(\overline{x})dx_2 = 0$.

  – Gradient vector $[G_1(\overline{x})\ \ G_2(\overline{x})]$ of G at $\overline{x}$ is orthogonal to tangent vector.

- First order conditions in vector-matrix form.

  – Lagrangian function:

$$L(x, \lambda) = F(x) + \lambda(c - G(x)).$$

  – First order conditions:

$$L_x(\overline{x}, \lambda) = F_x(\overline{x}) - \lambda G_x(\overline{x}) = 0;$$

$$L_\lambda(\overline{x}, \lambda) = c - G(\overline{x}) = 0.$$

- Lagrange's method in vector-matrix form.

  – First order necessary condition for $\overline{x}$ to solve maximization subject to equality constraints is: gradient vector of objective function at $\overline{x}$ is a linear combination of gradient vectors of the constraint functions at $\overline{x}$.

  – $F_x(\overline{x}) - \lambda G_x(\overline{x}) = 0$.

4. Constraint qualification

- In special case of $n = 2$ and $m = 1$, we require

$$G_x(\overline{x}) = [G_1(\overline{x}) \;\; G_2(\overline{x})] \neq [0 \;\; 0].$$

- In general, we require $G_x(\overline{x})$ has maximum rank.
  - Since $m < n$, maximum rank is m.
  - Rows of $G_x(\overline{x})$ are linearly independent.
  - Gradient vectors of the m constraint functions are linearly independent.