Econ 420
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Lecture 1a. Lagrange's Method: Special Case

1. Statement of problem

- Verbally, optimization subject to equality constraints.
  - Maximization vs. minimization.
  - Equality constraints vs inequality constraints.
- Economics: consumer choice theory.

- Notation for a special case.

  - Two choice variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

  - Objective function: $F(x)$.

  - One constraint: $G(x) = c$.

- Formal statement of problem: find value of x that maximizes $F(x)$ subject to $G(x) = c$.

2. The arbitrage argument

- Necessary condition for $\overline{x}$ to be solution to optimization problem: no profitable arbitrage.
  - What is arbitrage?
  - Why arbitrage?
  - What is profitable arbitrage?

- Formal statement of no profitable arbitrage.

  – Notation: infinitesimal change to x is

$$dx = \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

  value of x after change is $\overline{x} + dx$.

  – Necessary for $\overline{x}$ to be solution: there does not exist dx such that $G(\overline{x} + dx) = c$ and $F(\overline{x} + dx) > F(\overline{x})$.

3. First order necessary condition

- By Taylor expansion,

$$G(\overline{x} + dx) - G(\overline{x}) = G_1(\overline{x})dx_1 + G_2(\overline{x})dx_2,$$

where for each $j = 1, 2$,

$$G_j(\overline{x}) = \frac{\partial G(x)}{\partial x_j}\bigg|_{x=\overline{x}}.$$

  - Any dx such that $G_1(\overline{x})dx_1 + G_2(\overline{x})dx_2 = 0$ satisfies the constraint.

- By Taylor expansion,

$$F(\overline{x} + dx) - F(\overline{x}) = F_1(\overline{x})dx_1 + F_2(\overline{x})dx_2,$$

where for each $j = 1, 2$,

$$F_j(\overline{x}) = \frac{\partial F(x)}{\partial x_j}\bigg|_{x=\overline{x}}.$$

- Assume for now $G_j(\overline{x}) \neq 0$, $j = 1, 2$.

- A necessary condition for $\overline{x}$ to be solution is

$$\frac{F_1(\overline{x})}{G_1(\overline{x})} = \frac{F_2(\overline{x})}{G_2(\overline{x})}.$$

  - Proof by no profitable arbitrage: if the above condition fails, there exists dx satisfying $G_1(\overline{x})dx_1 + G_2(\overline{x})dx_2 = 0$ such that $F(\overline{x} + dx) - F(\overline{x}) > 0$.

- Proof.

- Rewrite first order necessary condition as

$$F_j(\overline{x}) = \lambda G_j(\overline{x})$$

for $j = 1, 2$.

  - λ is called multiplier.
  - We will derive first order necessary conditions for more than 2 choice variables and more than 1 equality constraint.

4. Constraint qualification

- If $G_1(\overline{x}) = 0$ but $G_2(\overline{x}) \neq 0$, then $F_1(\overline{x}) = 0$ by the same arbitrage argument, and we can still write

$$\frac{F_1(\overline{x})}{G_1(\overline{x})} = \frac{F_2(\overline{x})}{G_2(\overline{x})}.$$

  - Economics.
  - Proof.

- If $G_1(\overline{x}) = G_2(\overline{x}) = 0$, then our approach is invalid.
  - First order Taylor expansion of constraint does not represent the constraint.
  - Example: $G(x) = (p_1x_1 + p_2x_2 - c)^3 = 0$.
  - Economics.

5. Tangency argument

- Illustration of first-order condition: if $\overline{x}$ maximizes $F(x)$ subject to $G(x) = c$, then $\overline{x}$ is tangency point of level curves $F(x) = F(\overline{x})$ and $G(x) = c$.

6. Necessary but not sufficient

- First order condition can be satisfied at a minimum of the objective function rather than a maximum.

- First order condition can be satisfied at a local maximum rather than a global maximum.

- First order condition can be satisfied at a point that is neither maximum nor minimum.