Econ 420
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Lecture 10c. Solving Bellman Equations

1. Bellman equation as a fixed-point problem

- Define value function of dynamic programming

$$V(y_0) \equiv \max_{\{y_t\}_{t=1}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t F(y_t, y_{t+1}) \middle| y_{t+1} \in \Gamma(y_t), t = 0, 1, \ldots, \infty \right\}.$$

- $V(\cdot)$ necessarily satisfies Bellman equation

$$V(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V(y).$$

- If $V_*(\cdot)$ satisfies Bellman equation, and if for all feasible $\{y_t\}_{t=1}^{\infty}$,

$$\lim_{T \to \infty} \beta^T V_*(y_T) = 0,$$

then $V_* = V$.

- Bellman equation is equation in value function of a static problem.
  - For any V, the right-hand side of Bellman equation defines a value function W; if $W = V$, then V solves the Bellman equation.
  - A solution to Bellman equation is a fixed point in the mapping from one valuation to another value function.

- How to solve Bellman equation?
  - Iteration and convergence.
  - Backward induction and convergence.
  - Guess and verify.

- We use optimal growth as an example to illustrate all 3 methods.
  - $\max \sum_{t=0}^{\infty} \beta^t \ln c_t$ subject to $k_{t+1} = k_t^\alpha - c_t$, where $\alpha \in (0,1)$ and $k_0 > 0$ is given.
  - Bellman equation:

$$V(k) = \max_{\tilde{k} \in [0, k^\alpha]} \ln(k^\alpha - \tilde{k}) + \beta V(\tilde{k}).$$

2. Iteration and convergence

- General idea to solve Bellman equation

$$V(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V(y).$$

  - Start with arbitrary V_0 and solve right-hand side for V_1.
  - Plug in V_1 and solve right-hand side for V_2.
  - Keep going until solution converges.
  - Limit solves Bellman equation.

- $V_1(k) = \max_{\tilde{k} \in [0, k^\alpha]} \ln(k^\alpha - \tilde{k}) + \beta V_0(\tilde{k})$.
  - Start with $V_0(k) = 0$.
  - Solve for $V_1(k) = \ln k^\alpha$.

- $V_2(k) = \max_{\tilde{k} \in [0, k^\alpha]} \ln(k^\alpha - \tilde{k}) + \beta V_1(\tilde{k})$.
  - Solve for $V_2(k) = (1 + \alpha\beta) \ln k^\alpha + c_2$, where c_2 is a constant.

- $V_3(k) = \max_{\tilde{k} \in [0, k^\alpha]} \ln(k^\alpha - \tilde{k}) + \beta V_2(\tilde{k})$.
  - Solve for $V_2(k) = (1 + \alpha\beta + (\alpha\beta)^2) \ln k^\alpha + c_3$, where c_3 is a constant.

- Induction:

$$V_n(k) = \frac{1-(\alpha\beta)^n}{1-\alpha\beta}\ln k^{\alpha} + c_n,$$

where $\{c_n\}$ satisfies

$$c_n = \beta c_{n-1} + \frac{\alpha\beta-(\alpha\beta)^n}{1-\alpha\beta}\ln\left(\frac{\alpha\beta-(\alpha\beta)^n}{1-\alpha\beta}\right) - \frac{1-(\alpha\beta)^n}{1-\alpha\beta}\ln\left(\frac{1-(\alpha\beta)^n}{1-\alpha\beta}\right).$$

  - Proof.

- Convergence:

$$\lim_{n\to\infty} V_n(k) = \frac{\ln k^\alpha}{1-\alpha\beta} + c_*,$$

where

$$c_* = \frac{1}{1-\beta}\left(\alpha\beta\frac{\ln(\alpha\beta)}{1-\alpha\beta} + \ln(1-\alpha\beta)\right).$$

  - $\lim_{n\to\infty} V_n(k)$ solves Bellman equation.

3. Backward induction and convergence

- General idea to solve Bellman equation

$$V(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V(y).$$

  - Assume T is last period, with $V_{T+1}(k) = 0$ for all k.
  - Solve for $V_t(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V_{t+1}(y)$ by backward induction, for each $t = 0, \ldots, T$.
  - If $V_0(x)$ converges as T goes to infinity, then the limit solves Bellman equation.

- $V_t(k) = \max_{\tilde{k} \in [0, k^\alpha]} \ln(k^\alpha - \tilde{k}) + \beta V_{t+1}(\tilde{k})$, for each $t = 0, \ldots, T$.

  – From first method,

$$V_0(k) = \frac{1 - (\alpha\beta)^{T+1}}{1 - \alpha\beta} \ln k^\alpha + c_0,$$

  where $\{c_t\}$ satisfies

$$\begin{aligned} c_t = \beta c_{t+1} &+ \frac{\alpha\beta - (\alpha\beta)^{T+1-t}}{1 - \alpha\beta} \ln\left(\frac{\alpha\beta - (\alpha\beta)^{T+1-t}}{1 - \alpha\beta}\right) \\ &- \frac{1 - (\alpha\beta)^{T+1-t}}{1 - \alpha\beta} \ln\left(\frac{1 - (\alpha\beta)^{T+1-t}}{1 - \alpha\beta}\right). \end{aligned}$$

- Convergence: same limit as in first method for $V_0(k)$ as T goes to infinity

$$\lim_{T\to\infty} V_0(k) = \frac{\ln k^\alpha}{1-\alpha\beta} + c_*,$$

where

$$c_* = \frac{1}{1-\beta}\left(\alpha\beta\frac{\ln(\alpha\beta)}{1-\alpha\beta} + \ln(1-\alpha\beta)\right).$$

  - $\lim_{T\to\infty} V_0(k)$ solves Bellman equation.

4. Guess and verify

- General idea to solve Bellman equation

$$V(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V(y).$$

  - Guess functional form for V or for the optimal policy y as function of x.
  - Use first order condition and envelope theorem for right-hand side of Bellman equation to verify that guess is correct.

- Bellman equation

$$V(k) = \max_{\tilde{k} \in [0, k^\alpha]} \ln(k^\alpha - \tilde{k}) + \beta V(\tilde{k}).$$

  – Guess optimal policy is $\tilde{k} = s k^\alpha$, where $s \in (0, 1)$ to be determined.

- Envelope theorem implies

$$V'(k) = \frac{\alpha}{(1-s)k}.$$

  – Integrating,

$$V(k) = \frac{\alpha}{1-s}\ln k + c,$$

  where c is a constant.

- Verify the guess by plugging $V(k)$ and $\tilde{k}$ into Bellman equation.
  - Functional form with respect to k is same on both sides.
  - Solve for s and c by equating coefficients.
  - Solution is

$$V(k) = \frac{\ln k^{\alpha}}{1-\alpha\beta} + c,$$

where

$$c = \frac{1}{1-\beta}\left(\alpha\beta\frac{\ln(\alpha\beta)}{1-\alpha\beta} + \ln(1-\alpha\beta)\right).$$

- Another guess: from backward induction, we guess that the solution V to Bellman equation takes form of

$$V(k) = r \ln k^{\alpha} + c.$$

where r and c are constants to be determined.

  – Using first order condition we find optimal $\tilde{k}$

$$\tilde{k} = \frac{\alpha\beta r}{1 + \alpha\beta r} k^{\alpha}.$$

- Verify the guess by plugging $V(k)$ and $\tilde{k}$ into Bellman equation.
  - Functional form with respect to k is same on both sides.
  - Solve for r and c by equating coefficients.
  - Solution is

$$V(k) = \frac{\ln k^{\alpha}}{1 - \alpha\beta} + c,$$

where

$$c = \frac{1}{1-\beta} \left(\alpha\beta \frac{\ln(\alpha\beta)}{1 - \alpha\beta} + \ln(1 - \alpha\beta) \right).$$