Econ 420
Fall, 2022
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Lecture 10b. Bellman Equation

1. Infinite horizon problems

- Consider $\max_{\{y_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(y_t, y_{t+1})$, subject to $y_{t+1} \in \Gamma(y_t)$, $t = 0, 1, \ldots, \infty$, with y_0 given.
  - $\beta \in (0, 1)$ is subjective discount factor.
  - $\Gamma(y)$ is set of feasible states for next period when y is state in this period.

- Remarks on reformulation.

  - Solve policy z_t in terms of y_t and y_{t+1} from law of motion $y_{t+1} - y_t = Q(y_t, z_t, t)$.

  - Substitute in immediate function $F(y_t, z_t, t)$ and feasibility constraint $G(y_t, z_t, t) \leq 0$.

- Stationarity: same dynamic optimization problem starting at any period t, parameterized by β, F and Γ.
  - Immediate function depends on t only through discounting.
  - Both law of motion and feasibility constraint are independent of t.

- Example: optimal growth.

  - Original formulation: $\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$ subject to law of motion $k_{t+1} = f(k_t) - \delta k_t - c_t$, and two feasibility constraints $c_t \geq 0$ and $k_{t+1} \geq 0$ for each $t = 0, 1, \ldots$, with k_0 given.

  - Reformulation: $\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - \delta k_t - k_{t+1})$ subject to $k_{t+1} \in \Gamma(k_t) \equiv [0, f(k_t) - \delta k_t]$, $t = 0, 1, \ldots, \infty$, with k_0 given.

2. Bellman equation

- Principle of Optimality in a stationary environment.
  - $V(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V(y)$.
  - Maximum value attained in dynamic programming with the initial state x is equal to maximum value attained in a static problem, where the choice variable is the next period state y, the objective function is the sum of immediate function $F(x, y)$ and discounted maximum value attained in dynamic programming with initial state y, and constraint is $y \in \Gamma(x)$.

- Bellman equation in value function.
  - Same value function V appearing on both sides of Bellman equation, due to stationarity.
  - Same Principle of Optimality, or one-shot deviation principle, adapted to stationary environment, even though backward induction does not apply.

- Bellman equation is necessary for optimality.
  - V attains maximum value for dynamic programming as a function of initial state only if it satisfies Bellman equation, or equivalently, only if there is no profitable one-shot deviation.
  - Proof.

3. Sufficiency of Bellman equation

- Satisfying Bellman equation is sufficient for optimality in dynamic programming with finite horizon.
  - If there is no profitable one-shot deviation, then there is no profitable deviation.
  - Reason: backward induction applies, and any deviation is a finite sequence of one-shot deviations.

- Bellman equation is in general insufficient for optimality under infinite horizon.
  - Backward induction does not apply.
  - No profitable one-shot deviation implies no profitable finite number of deviations, but what about an infinite number of deviations?

- An example of insufficiency of Bellman equation.
  - Consider $\max_{\{c_t, y_{t+1}\}_{t=0}^{\infty}} \beta^t c_t$ subject to $0 \leq c_t \leq y_t - \beta y_{t+1}$, with y_0 given.
  - $V^*(x) = x$ satisfies Bellman equation

$$V(x) = \max_{y \leq x/\beta} x - \beta y + \beta V(y),$$

but is not the value function of dynamic programming.

- V satisfying Bellman equation is sufficient for V to be maximum value function of dynamic programming with initial state y_0, if in addition V satisfies a boundary condition: for all feasible $\{y_t\}_{t=0}^{\infty}$ with $y_0 = 0$,

$$\lim_{T\to\infty} \beta^T V(y_T) = 0.$$

  – Proof.

- Interpretation: above boundary condition on V ensures an infinite number of deviations is not profitable.
- Boundary condition is satisfied if $|F(x, y)| \leq B$.