Econ 420
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Lecture 10a. Dynamic Programming: Backward Induction

1. Basic idea

- Consider $\max_{\{y_t\}_{t=1}^T, \{z_t\}_{t=0}^T} \sum_{t=0}^T F(y_t, z_t, t)$, subject to:
  - Law of motion $y_{t+1} - y_t = Q(y_t, z_t, t)$, for all $t = 0, 1, \ldots, T$, with y_0 and y_{T+1} given.
  - Feasibility $G(y_t, z_t, t) \leq 0$, for all $t = 0, 1, \ldots, T$.

- Dynamic link requires forward looking.
  - Starting from terminal date, backward induction solves for optimal policy for current time period as function of current state, given the maximum value function starting from next period.
  - Made possible because current state is sufficient to determine optimal way of going forward.
  - Also known as recursive method.

2. Value functions and backward induction

- Class of date T problems: for each y_T, consider

$$\max_{z_T} F(y_T, z_T, T)$$

subject to $y_{T+1} - y_T = Q(y_T, z_T, T)$ and $G(y_T, z_T, T) \leq 0$, where y_{T+1} is given.

  - If $\{y_t^*\}_{t=1}^T, \{z_t^*\}_{t=0}^T$ is solution, then z_T^* solves date T problem associated with y_T^*.
  - Denote maximum value function of date T problem as $V(y_T, T)$.

- Class of date $T-1$ problems: for each y_{T-1}, consider

$$\max_{y_T, z_{T-1}} F(y_{T-1}, z_{T-1}, T-1) + V(y_T, T)$$

subject to law of motion $y_T - y_{T-1} = Q(y_{T-1}, z_{T-1}, T-1)$ and feasibility $G(y_{T-1}, z_{T-1}, T-1) \leq 0$.

  – If $\{y_t^*\}_{t=1}^T, \{z_t^*\}_{t=0}^T$ is solution, then (y_T^*, z_{T-1}^*) solves the date $T-1$ problem associated with y_{T-1}^*.

  – Denote maximum value function of date $T-1$ problem as $V(y_{T-1}, T-1)$.

- Backward induction: for each $t = 0, \ldots, T$, starting from $t = T$ with $V(y_{T+1}, T+1) = 0$, define (maximum) value function $V(y_t, t)$ in date t problem:

$$V(y_t, t) = \max_{y_{t+1}, z_t} F(y_t, z_t, t) + V(y_{t+1}, t+1),$$

subject to $y_{t+1} - y_t = Q(y_t, z_t, t)$ and $G(y_t, z_t, t) \leq 0$.

  - If $\{y_t^*\}_{t=1}^T, \{z_t^*\}_{t=0}^T$ is solution, then for each $t = 0, \ldots, T$, (y_{t+1}^*, z_t^*) solves the date t problem associated with y_t^*.
  - Proof.

3. Principle of Optimality

- $\{y_t^*\}_{t=1}^T, \{z_t^*\}_{t=0}^T$ is solution to original problem only if (y_{t+1}^*, z_t^*) solves date t problem associated wth y_t^* for each $t = 0, \ldots, T$.
  - Known as Principle of Optimality in dynamic programming.
  - "Only if" means Principle of Optimality provides necessary conditions for solution.
  - The conditions are also sufficient here.

- Principle of Optimality uses backward induction and constructs value functions to reduce test for optimality of $\{y_t^*\}_{t=1}^T, \{z_t^*\}_{t=0}^T$ among all feasible $\{y_t\}_{t=1}^T, \{z_t\}_{t=0}^T$, to a sequence of optimality tests for each (y_{t+1}^*, z_t^*) in the static date t maximization problem associated with y_t^*.
  - For given y_t^*, a feasible (y_{t+1}, z_t) in date t problem represents "one-shot deviations" form solution.
  - Principle of Optimality: no profitable one-shot deviation is both necessary and sufficient for optimality.

4. Dynamic programming and optimal control

- Rewrite date t problem as

$$\max_{z_t} F(y_t, z_t, t) + V(y_t + Q(y_t, z_t, t), t + 1),$$

subject to $G(y_t, z_t, t) \leq 0$.

  - Law of motion is used to eliminate y_{t+1} as a choice variable.
  - z_t is the only choice variable.
  - Let λ_t be multiplier for feasibility constraint.

- Derive identical first order conditions as optimal control.
  - Use first order condition and envelope theorem in the date t maximization problem, and then set $\pi_t = V_y(y_t, t)$.
  - Proof.

- Relation between dynamic programming and optimal control.
  - Both break down dynamic optimization into a sequence of static optimization problems.
  - Dynamic programming: indirectly with value function and backward induction.
  - Optimal control: directly with shadow value of state and Hamiltonian.