Econ 301 - Problem Set 6

Question 5

5. MC curve hits the price line at two output levels. Which is the profit-maximizing output? Why?

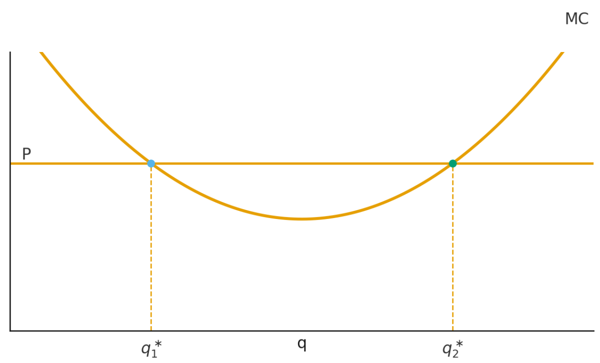

Figure 1: MC curve intersects price line at q_1^* and q_2^*)

$$p = MR(q)$$
$$MR(q^*) = MC(q^*)$$
$$p = MC(q^*)$$
$$\pi(q) = R(q) - C(q)$$
$$\pi'(q) = MR(q) - MC(q)$$
$$\pi'(q) = p - MC(q)$$
$$\pi''(q) = -MC'(q)$$

At q_1^*, the slope of MC is negative, so $\pi''(q_1^*) > 0$ and q_1^* is a profit **minimizer.**
At q_2^*, the slope of MC is positive, so $\pi''(q_2^*) < 0$ and q_2^* is a profit **maximizer.**

Question 6

$C(q) = 100 + 10q - q^2 - \frac{1}{3}q^3$

(a) What is the firm's marginal cost function? What is its profit-maximizing condition if the market price is p? What is its supply curve?

$$MC = \frac{\partial C}{\partial q} = 10 - 2q - q^2$$

Profit-maximizing condition:

$$MR = MC$$

$$p = 10 - 2q - q^2$$

$$q^2 + 2q + (p - 10) = 0$$

$$q = \frac{-2 \pm \sqrt{4 - 4(p-10)}}{2}$$

$$q = \frac{-2 \pm \sqrt{4(1 - (p-10))}}{2}$$

$$q = \frac{-2 \pm 2\sqrt{1 - p + 10}}{2}$$

$$q = -1 \pm \sqrt{11 - p}$$

$$q^* = -1 + \sqrt{11-p} \quad \text{or} \quad q^* = -1 - \sqrt{11-p}$$

$$\pi(q) = R(q) - C(q)$$

$$\pi'(q) = MR(q) - MC(q)$$

$$\begin{aligned} \pi'(q) &= p - MC(q) \\ \pi''(q) &= -MC'(q) \\ &= -(-2 - 2q) \\ &= 2 + 2q \end{aligned}$$

$$2 + 2(-1 + \sqrt{11-p}) \quad \text{and} \quad 2 + 2(-1 - \sqrt{11-p})$$

$$2 - 2 + 2\sqrt{11-p} \quad \text{and} \quad 2 - 2 - 2\sqrt{11-p}$$

$$2\sqrt{11-p} \quad \text{and} \quad -2\sqrt{11-p}$$

Assume $p < 11$

$$\pi''(-1 + \sqrt{11-p}) > 0 \quad \text{and} \quad \pi''(-1 - \sqrt{11-p}) < 0$$

$$q^* = -1 - \sqrt{11-p} \quad \text{(negative, so firm will not produce)}$$

(b) $C(q) = 100 + 20q - q^2 - \frac{1}{3}q^3$

What is the firm's profit-maximizing condition if the market price is p?

$$MR = MC$$

$$p = \frac{\partial C}{\partial q} = 20 - 2q - q^2$$

$$q^2 + 2q + (p - 20) = 0$$

$$q = \frac{-2 \pm \sqrt{4 - 4(p - 20)}}{2}$$

$$q = \frac{-2 \pm \sqrt{4(1 - (p - 20))}}{2}$$

$$q = \frac{-2 \pm 2\sqrt{1 - (p - 20)}}{2}$$

$$q = -1 \pm \sqrt{21 - p}$$

$$q_1^* = -1 + \sqrt{21 - p} \quad \text{or} \quad q_2^* = -1 - \sqrt{21 - p}\,?$$

$$\pi(q) = R(q) - C(q)$$

$$\pi'(q) = MR(q) - MC(q)$$

$$\pi'(q) = p - MC(q)$$

$$\pi''(q) = -MC'(q)$$

$$= -(-2 - 2q)$$

$$= 2 + 2q$$

$$q_1^* : 2 + 2(-1 + \sqrt{21 - p}) = 2 - 2 + 2\sqrt{21 - p} = 2\sqrt{21 - p}$$

$$q_2^* : 2 + 2(-1 - \sqrt{21 - p}) = 2 - 2 - 2\sqrt{21 - p} = -2\sqrt{21 - p}$$

Assume $p < 21$

$$\pi''(q_1^*) > 0 \quad \text{and} \quad \pi''(q_2^*) < 0$$

$$q^* = q_2^* = -1 - \sqrt{21 - p} \quad \text{(negative, so firm will not produce)}$$

Question 7

What is the effect of an ad valorem tax of v (the share of the price that goes to the government) on a competitive firm's profit-maximizing output, given the market price is unaffected?

$$\pi(q, v) = (1 - v)pq - C(q)$$

$$\frac{\partial \pi}{\partial q} = (1 - v)p - MC(q) = 0$$

$$p - \frac{MC(q)}{1 - v} = 0$$

$$\frac{\partial}{\partial v}\left(p - \frac{MC(q^*)}{1 - v}\right) = 0$$

$$\frac{\partial MC}{\partial q}\frac{\partial q^*}{\partial v}(1 - v) - MC(q^*)(-1) = 0$$

$$\frac{\partial MC}{\partial q}\frac{\partial q^*}{\partial v}(1 - v)^{-1} - \frac{MC(q^*)}{(1 - v)^2} = 0$$

$$\frac{\partial MC}{\partial q}\frac{\partial q^*}{\partial v}(1 - v)^{-1} - MC(q^*)(1 - v)^{-2} = 0$$

Rearranging:

$$\boxed{\frac{\partial q^*}{\partial v} = -\frac{MC(q^*)}{\frac{\partial MC}{\partial q}(1 - v)}}$$

Question 8

$$\min_{L,K} \; rK + wL + F \quad \text{s.t.} \quad q = K^{1/3}L^{1/3}$$

F is unavoidable in the short run but avoidable in the long run.

(a) Derive the cost functions for the short run and for the long run.

Short run:

$$\mathcal{L} = rK + wL + F - \lambda(q - K^{1/3}L^{1/3})$$

$$[L]: \quad w + \lambda\tfrac{1}{3}K^{1/3}L^{-2/3} = 0 \quad \Rightarrow \quad \lambda = \frac{-w}{\frac{1}{3}K^{1/3}L^{-2/3}}$$

$$[K]: \quad r + \lambda\tfrac{1}{3}K^{-2/3}L^{1/3} = 0 \quad \Rightarrow \quad \lambda = \frac{-r}{\frac{1}{3}K^{-2/3}L^{1/3}}$$

$$[\lambda]: \quad q = K^{1/3}L^{1/3}$$

$$\frac{w}{\frac{1}{3}K^{1/3}L^{-2/3}} = \frac{r}{\frac{1}{3}K^{-2/3}L^{1/3}}$$

$$wK^{-1/3}L^{2/3} = rK^{1/3}L^{-1/3}$$

$$wK^{-1/3}L = rK^{1/3}$$

$$L = \frac{r}{w}K$$

$$q = K^{1/3}\left(\frac{r}{w}K\right)^{1/3}$$

$$q = K^{1/3}\left(\frac{r}{w}\right)^{1/3}K^{1/3}$$

$$q = \left(\frac{r}{w}\right)^{1/3}K^{2/3}$$

$$q^3 = \left(\frac{r}{w}K^2\right)\frac{w}{r}$$

$$K^2 = \frac{w}{r}q^3 \quad \Rightarrow \quad K = \left(\frac{w}{r}\right)^{1/2}q^{3/2}$$

$$L = \frac{r}{w}\left(\frac{w}{r}\right)^{1/2}q^{3/2} \quad \Rightarrow \quad L = \left(\frac{r}{w}\right)^{1/2}q^{3/2}$$

$$C_S(q) = \begin{cases} r\left(\frac{w}{r}\right)^{1/2}q^{3/2} + w\left(\frac{r}{w}\right)^{1/2}q^{3/2} + F, & \text{if } q > 0, \\ 0, & \text{if } q = 0 \end{cases}$$

Long run:
F can be avoided if $q = 0$, so

$$C_L(q) = \begin{cases} r\left(\frac{w}{r}\right)^{1/2}q^{3/2} + w\left(\frac{r}{w}\right)^{1/2}q^{3/2} + F, & \text{if } q > 0, \\ 0, & \text{if } q = 0 \end{cases}$$

(b) Short run:

$$\max_q \pi(q) \quad \Rightarrow \quad \max_q \left[pq - C_S(q)\right]$$

$$\max_q \left[pq - \left(r\left(\frac{w}{r}\right)^{1/2} + w\left(\frac{r}{w}\right)^{1/2}\right)q^{3/2} - F\right]$$

$$\max_q \; pq - 2(rw)^{1/2}q^{3/2} - F$$

$$\pi'(q) = p - 3(rw)^{1/2}q^{1/2} = 0$$

$$p = 3(rw)^{1/2}q^{1/2} \quad \Rightarrow \quad q^{1/2} = \left(\frac{p}{3(rw)^{1/2}}\right)$$

$$q^* = \left(\frac{p}{3(rw)^{1/2}}\right)^2$$

$$q^* = \begin{cases} \left(\frac{p}{3\sqrt{rw}}\right)^2, & \text{if } p\left(\frac{p}{3\sqrt{rw}}\right)^2 - 2\sqrt{rw}\left(\frac{p}{3\sqrt{rw}}\right)^3 - F > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Long run:

$$q^* = \begin{cases} \left(\frac{p}{3\sqrt{rw}}\right)^2, & \text{if } p\left(\frac{p}{3\sqrt{rw}}\right)^2 - 2\sqrt{rw}\left(\frac{p}{3\sqrt{rw}}\right)^3 - F > 0, \\ 0, & \text{otherwise.} \end{cases}$$

If the firm produces:

$$\pi(q) = p\left(\frac{p^2}{9rw}\right) - 2(rw)^{1/2}\left(\frac{p^2}{9rw}\right)^{3/2} - F$$

If not:

$$\pi(q) = 0$$