Econ 221
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Chapter 7. Simultaneous-Move Games: Mixed Strategies

- Now we return to simultaneous-move games.
- We resolve the issue of non-existence of Nash equilibrium in pure strategies through intentional mixing.
- Real life examples of mixing: sports, free riding, tax audit.

7.1 What is a mixed strategy

- A penalty taker with a powerful left-sided shot.

Keeper

		Left	*Right*
Kicker	*Left*	1, 0	0.4, 0.6
	Right	0, 1	1, 0

  - This is a zero-sum game, but unlike Matching Pennies, it is asymmetric.

– There is no Nash equilibrium in pure strategies.

– Each strategy of each player is rationalizable.

– The only way to win, or equivalently not to lose, is to keep the opponent guessing by mixing between your own pure strategies.

– Mixing is therefore making your own choice of strategy intentionally random from perspective of opponent.

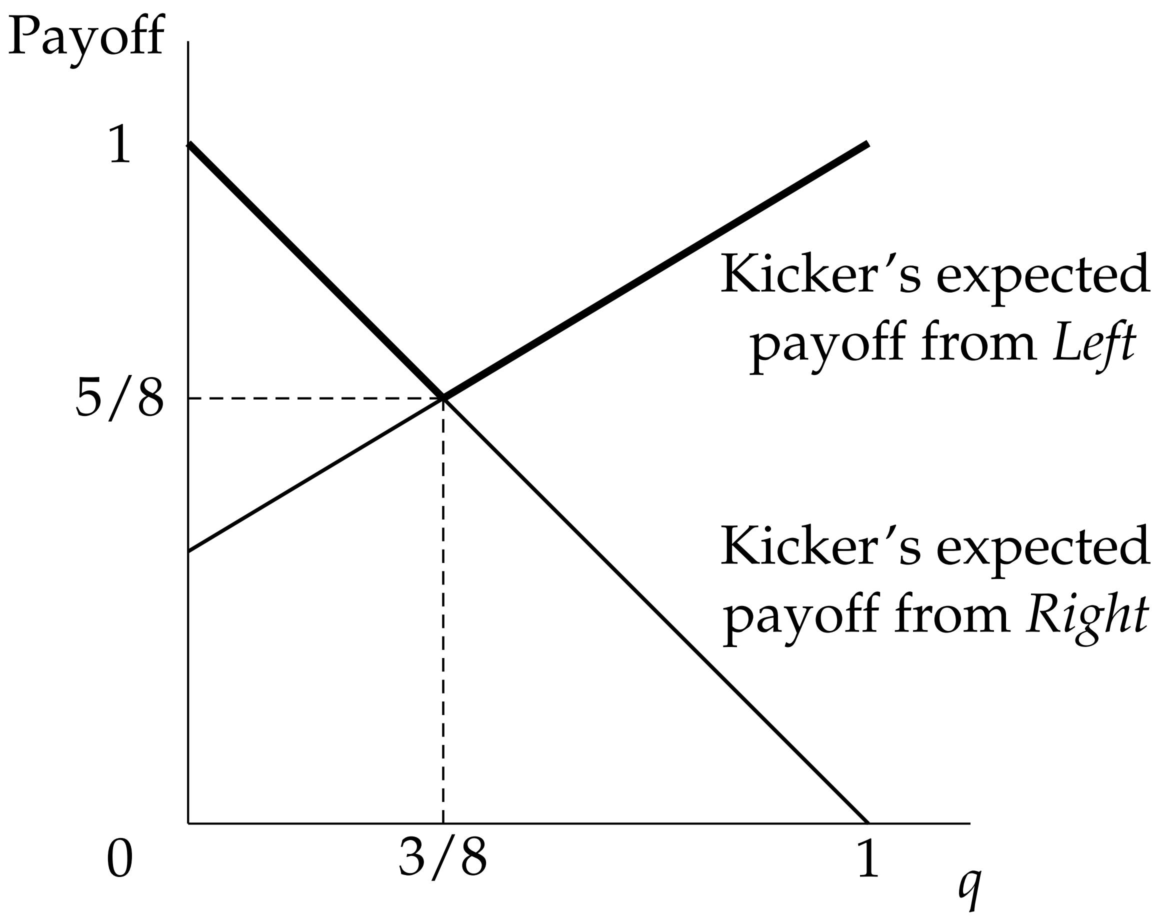
- A mixed strategy of a player assigns a probability of playing each pure strategy, such that sum of assigned probabilities is equal to 1 over all the strategies of the player.
  - Mixing doesn't mean assigning same probability to each pure strategy.
  - With only two pure strategies, a mixed strategy can be represented by one number between 0 and 1.
  - A pure strategy is a degenerate mixed strategy, with 1 assigned to the pure strategy and 0 to all other pure strategies.

- Expected payoff in the presence of mixing.
  - Expected payoff to a player is probability-weighted sum of the player's payoffs, with probabilities determined by corresponding mixing.
  - Mixing is independent across players.
  - Example: against Keeper's mixed strategy of *Left* with probability q and *Right* with probability $1-q$, expected payoff is $q \cdot 1 + (1-q) \cdot 0.4$ to Kicker from *Left* , and $q \cdot 0 + (1-q) \cdot 1$ from *Right*.

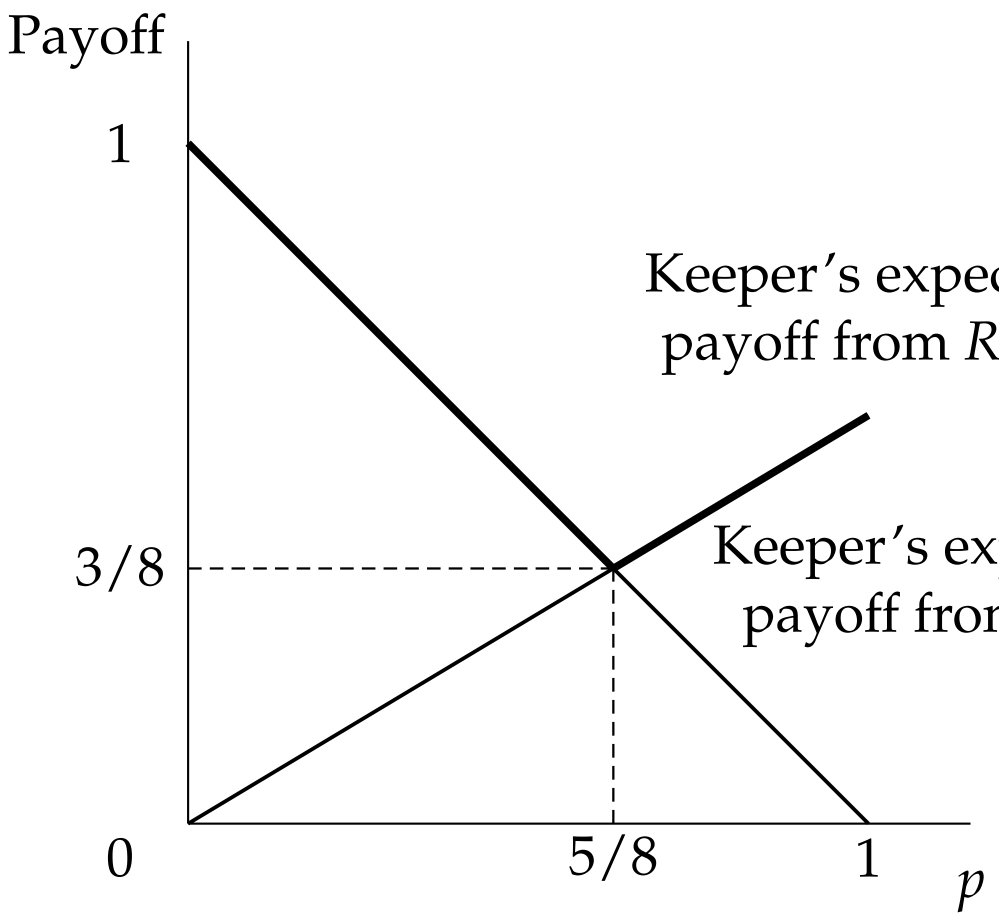
7.2 Mixing moves

- Benefit of mixing.
  - Suppose that Keeper plays a mixed strategy of *Left* with probability q and *Right* with probability $1 - q$.
  - Expected payoff to Kicker is $q \cdot 1 + (1 - q) \cdot 0.4$ from *Left*, $q \cdot 0 + (1 - q) \cdot 1$ from *Right*.

- Kicker's best response is $p = 1$ if $q > 3/8$ and $p = 0$ if $q < 3/8$.
- If $q = 3/8$, Kicker is indifferent between *Left* and *Right*, and expected payoff from best response is minimized.
- Benefit of mixing to Keeper: since this is a zero-sum game, Keeper should mix between *Left* and *Right* with exactly $q = 3/8$, and thus increases guaranteed payoff from 0 to 3/8.



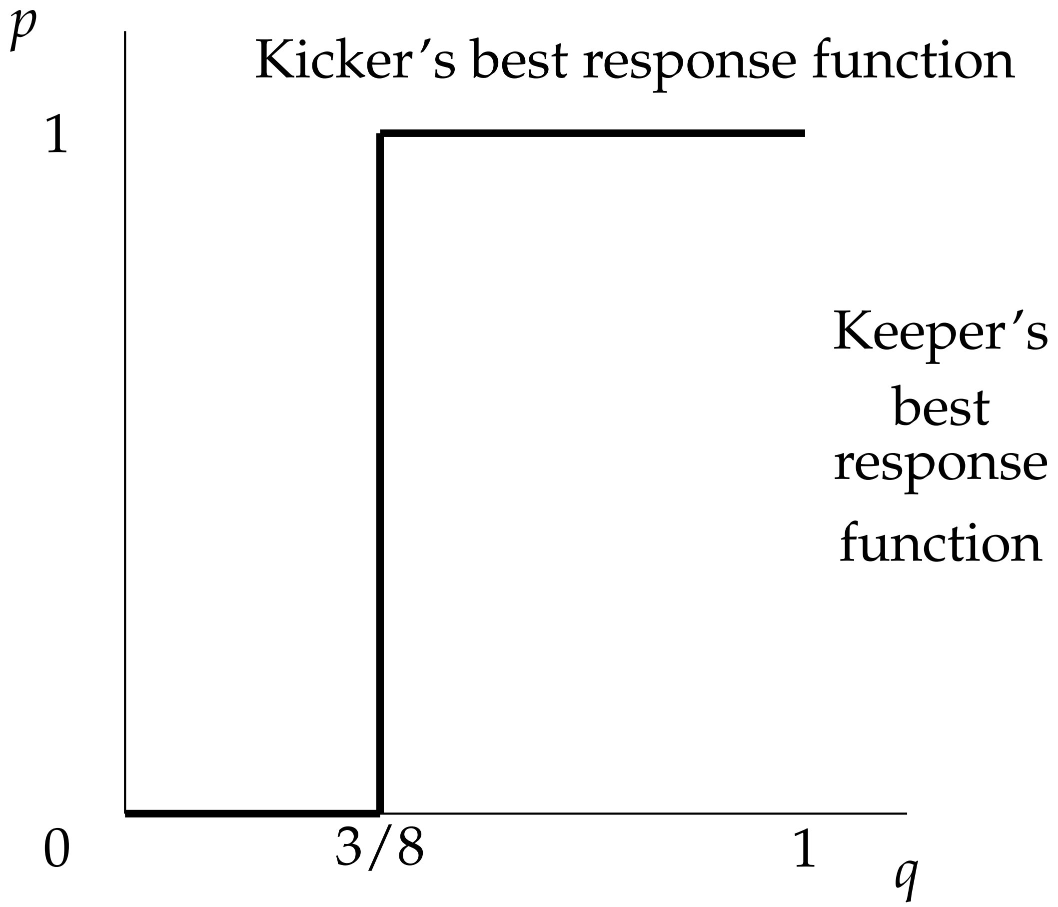
Benefit of mixing to Keeper.



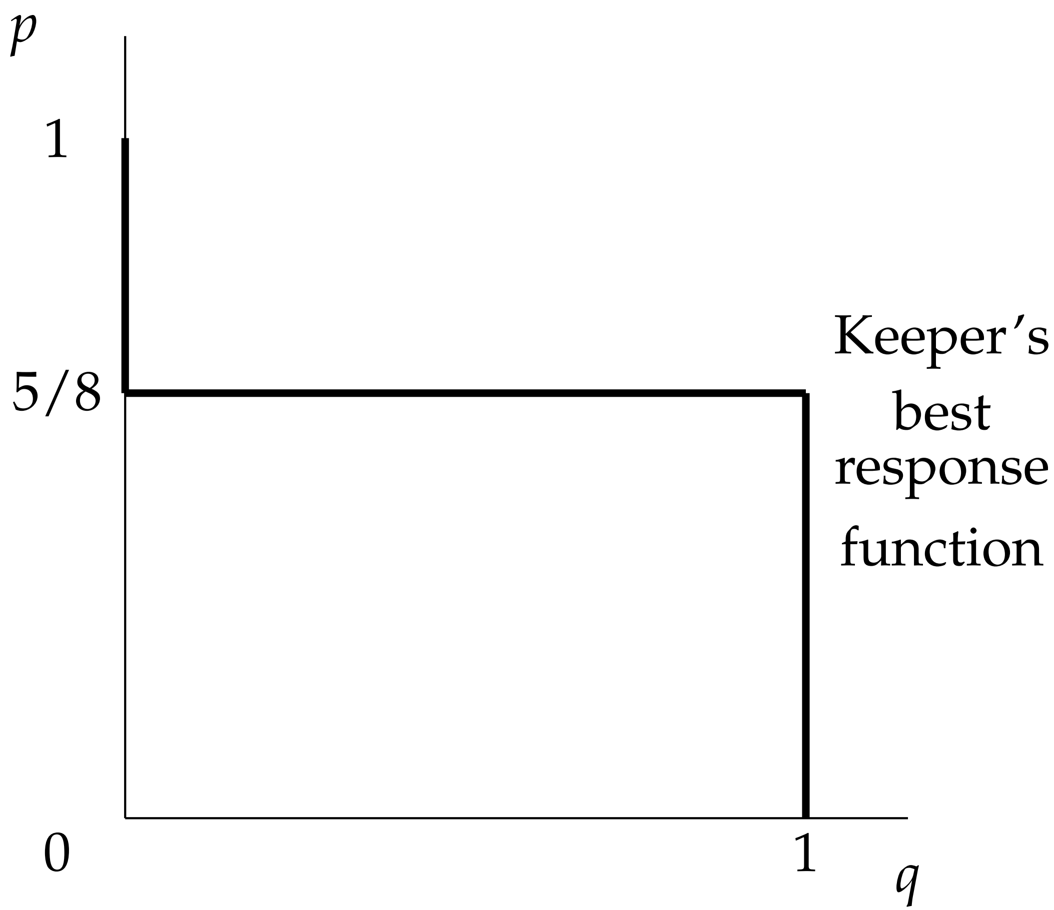
Benefit of mixing to Kicker.

- Kicker should mix with p of *Left* and $1-p$ of *Right* in such a way that Keeper is indifferent between *Left* and *Right*.
  - Equating expected payoff of Keeper $p \cdot 0 + (1-p) \cdot 1$ from *Left* to expected payoff $p \cdot 0.6 + (1-p) \cdot 0$ from *Right* gives $p = 5/8$.
  - Benefit of mixing to Kicker: by choosing $p = 5/8$, Kicker increases guaranteed payoff from 0.4 (from choosing $p = 1$) to 5/8.

- Nash equilibrium in mixed strategies.
  - With only two pure strategies, a mixed strategy can be represented by one number between 0 and 1, so mixed strategies are special case of continuous strategies.
  - Nash equilibrium can be found by intersection of best response functions.



Kicker's best response function in Penalty Kick.



Keeper's best response in Penalty Kick.

7.3 Nash equilibrium as a system of beliefs and responses

- In a Nash equilibrium, each player plays a best response against the equilibrium strategies of other players.
  - Recall two features of Nash equilibrium: non-cooperative and correct beliefs.
  - Nash equilibrium in mixed strategies requires players to have correct beliefs in that they know equilibrium mixtures of all players.

7.4 Mixing in non-zero-sum games

- Free-rider problem in crime-reporting.
  - Two witnesses of crime decide whether to report it or not; reporting it costs 2 individually; each receives 3 if at least one of them reports it and 0 if neither reports it.

Witness 2

		Report	*Don't*
Witness 1	*Report*	$1, 1$	$1, 3$
	Don't	$3, 1$	$0, 0$

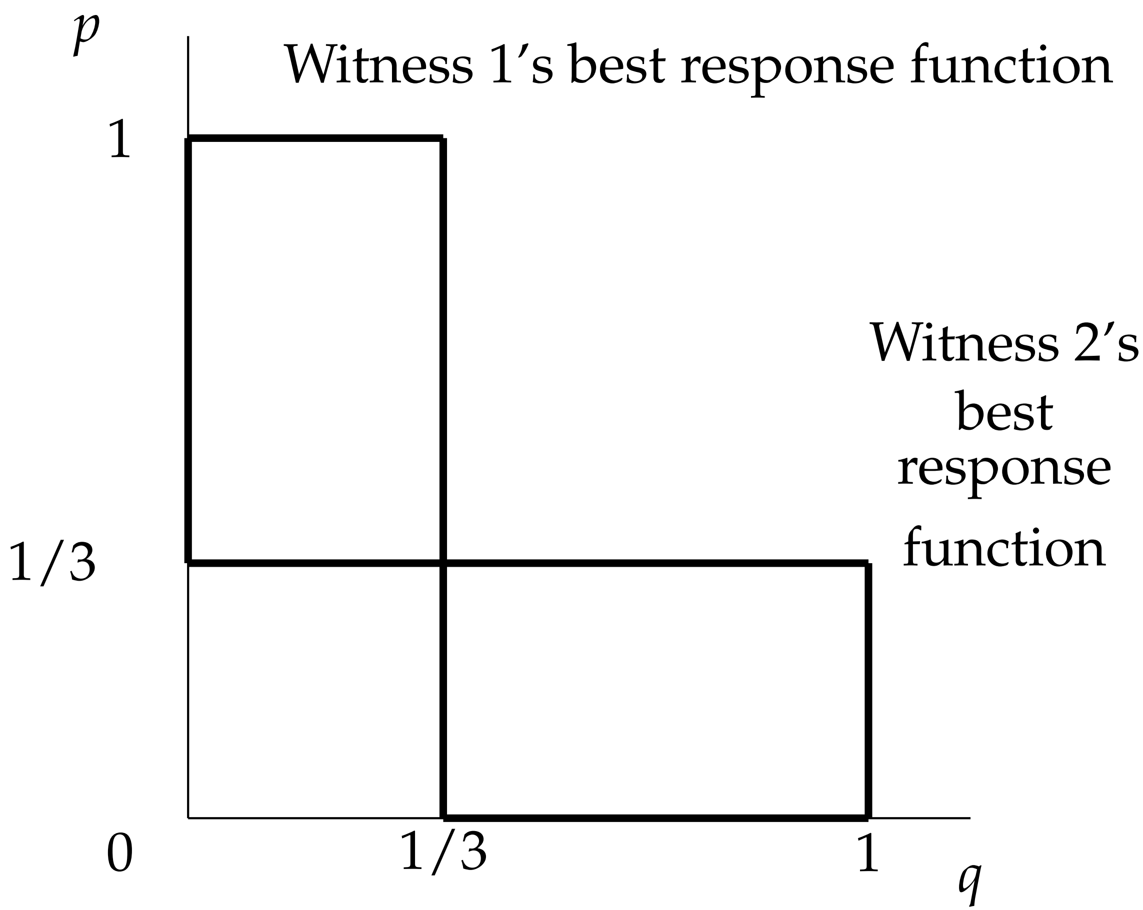
– This is the same as the Hawk-Dove game, with two Nash equilibria in pure strategies.

		Witness 2	
		Report	*Don't*
Witness 1	*Report*	$1, 1$	*1, 3*
	Don't	*3, 1*	$0, 0$

– There is a mixed strategy equilibrium.

- Principle of making your opponent indifferent: your opponent is willing to mix between two pure strategies only when you mix your own two pure strategies in a way such that your opponent is indifferent.

- Apply the principle: if Witness 2 chooses *Report* with probability q (and *Don't* with probability $1 - q$), then expected payoff of Witness 1 is 1 from *Report*, and $3q$ from *Don't*, so $q = 1/3$; symmetrically, Witness 1 has to choose *Report* with probability $p = 1/3$ to make Witness 2 indifferent.



Three Nash equilibria in Reporting a Crime.

7.5 General discussion of mixed-strategy equilibria

- Weak sense of equilibrium.
  - Nash equilibrium in pure strategies is typically strict in that each player plays the unique best response to equilibrium strategies of opponents.
  - Principle of making your opponent indifferent means that each player's equilibrium mix is not a strict best response to opponent's equilibrium mix.

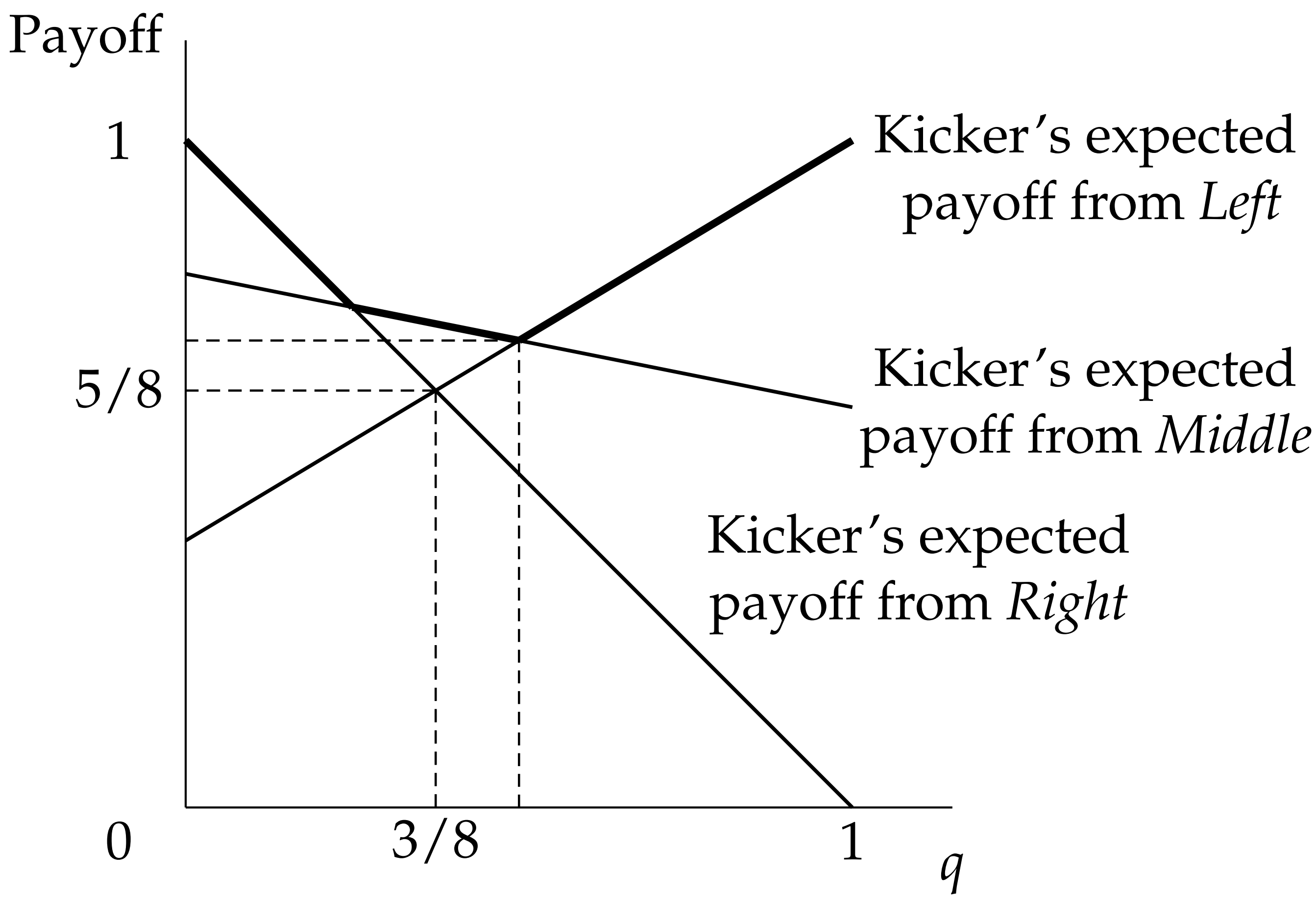
7.6 Mixing when one player has three or more pure strategies

- Penalty Kick with Panenka.

Keeper

		Left	*Right*
	Left	1, 0	0.4, 0.6
Kicker	*Middle*	0.6, 0.4	0.8, 0.2
	Right	0, 1	1, 0

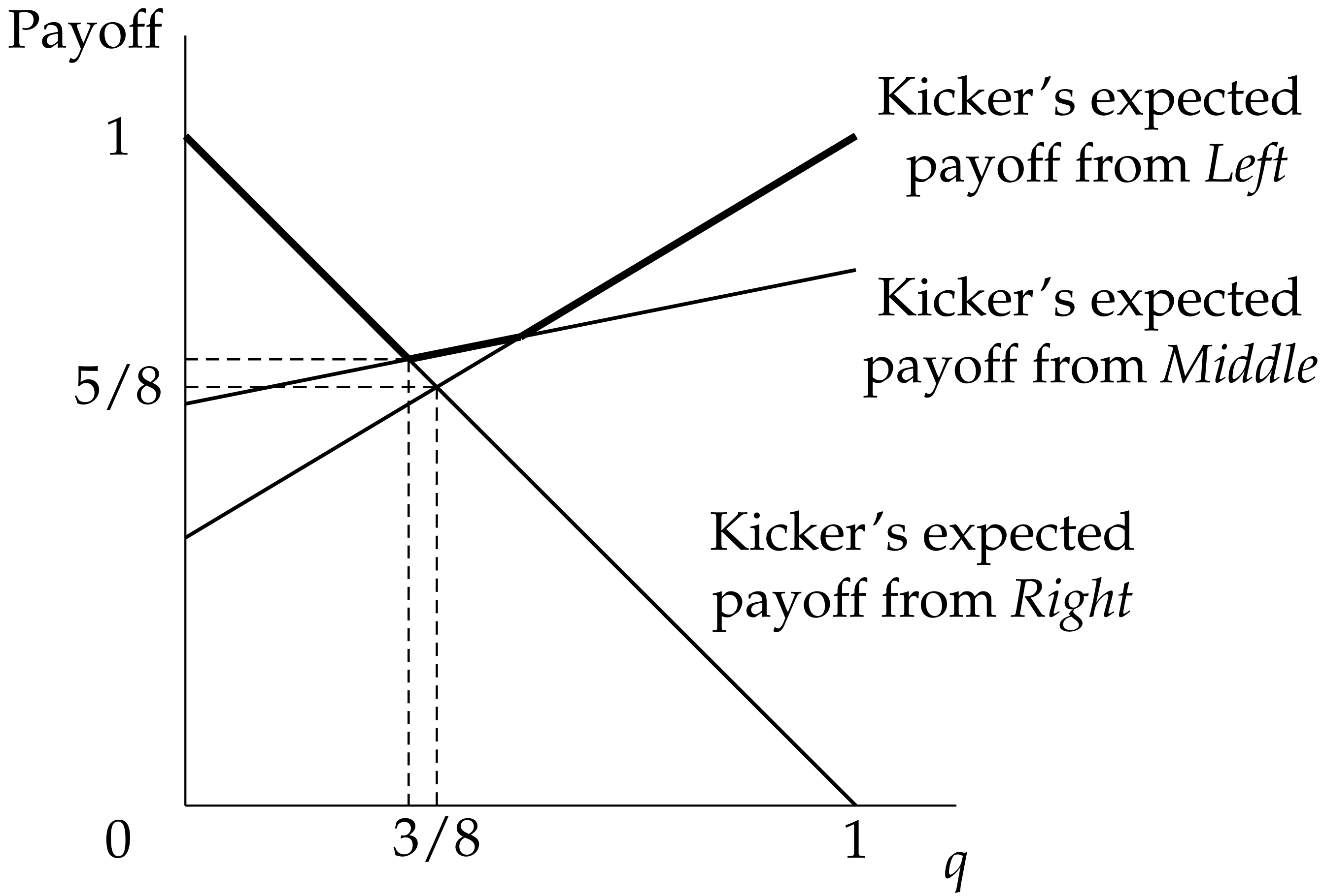
  – Keeper still has only two pure strategies so we continue to represent any mixed strategy with probability q that Keeper chooses *Left*.



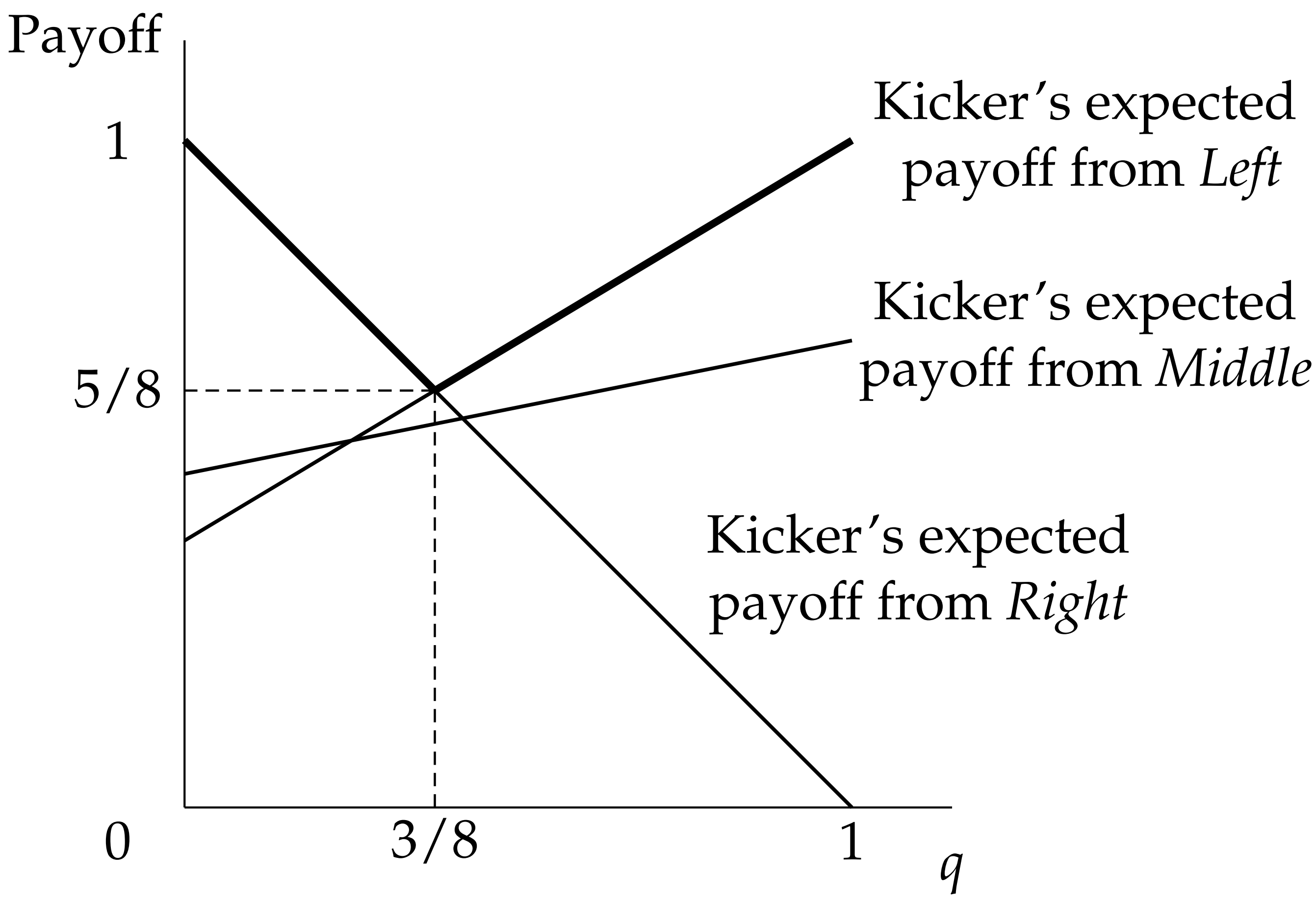
Penalty Kick with Panenka.

- To minimize Kicker's payoff from best response, Keeper has to keep Kicker indifferent between *Left* and *Middle*: equating $q \cdot 1 + (1-q) \cdot 0.4$ to $q \cdot 0.6 + (1-q) \cdot 0.8$ gives Keeper's equilibrium mix $q = 0.5$.
- Kicker does not use *Right* in equilibrium.
- In order for Keeper to mix, Kicker needs to choose *Left* with probability p_l and *Middle* with probability $1 - p_l$ to make Keeper indifferent: equating $p_l \cdot 0 + (1 - p_l) \cdot 0.4$ to $p_l \cdot 0.6 + (1 - p_l) \cdot 0.2$ gives Kicker's equilibrium mix $p_l = 0.25$.

- Modified principle of making your opponent indifferent when at least one player has more than two pure strategies: in any mixed-strategy equilibrium, a player has to be indifferent among all pure strategies used in equilibrium, and prefers any of them to the ones unused in equilibrium.
  - When only one player has three or more strategies in a zero-sum game, only two will be used in equilibrium, and which two is determined by the opponent mixing to minimize the former's best payoff.



Penalty Kick: Panenka is used with *Right*.



Penalty Kick: Panenka is never used.

7.7 Mixing when both players have three or more strategies

- Only strategies that survive iterated elimination of never best responses can be used in equilibrium
- So we apply modified principle of making your opponent indifferent to reduced game.
- Again, some pure strategies may not be used in equilibrium, but mixing among three or even more pure strategies can occur in equilibrium.
- Verifying equilibrium is easier than finding equilibrium.

- 3-by-3 Penalty Kick.

		Keeper		
		Left	*Middle*	*Right*
Kicker	*Left*	1, 0	1, 0	0, 1
	Middle	1, 0	0,1	1, 0
	Right	0, 1	1, 0	1, 0

- Kicker and Keeper both choosing each strategy with equal probability is a Nash equilibrium.