

Final Practice Questions

1. (*Completion of a Joint Project*) Two researchers, A and B, are at the final stage of a joint project. The project is completed if at least one researcher puts in some additional work, but remains incomplete if both researchers slack off. The cost of the additional work to each researcher is equivalent to $c > 0$ dollars; the cost of slacking off is 0. Denote the dollar value of the completed project to each researcher as u . An incomplete project is worth 0 to the researchers. The project being a joint one, each researcher gets the dollar value u as long as it is completed, regardless whether he has put in the work. The following payoff matrix summarizes the described situation for any given u :

		Researcher B	
		<i>Shirk</i>	<i>Work</i>
Researcher A	<i>Shirk</i>	$0, 0$	$u, u - c$
	<i>Work</i>	$u - c, u$	$u - c, u - c$

For all questions below, the value u of the project can take on 3 values, 0, $\frac{3}{2}c$ and $3c$. Further, only Researcher A knows which value that u takes, while Researcher B knows only that u takes on the 3 values with equal probability (equal to $\frac{1}{3}$).

For questions (a)-(d), we assume that the two researchers decide simultaneously whether to shirk or work. This is therefore a Bayesian game in which Researcher A has three types.

- (a) Find a pure-strategy Bayesian Nash equilibrium where Researcher B chooses *Shirk*.
- (b) Does there exist a pure-strategy Bayesian Nash equilibrium where Researcher B chooses *Work*? Explain your answer.

(c) Show that there is a mixed-strategy Bayesian Nash equilibrium, where Researcher B randomizes between "Shirk" and "Work" and Researcher A randomizes only when the project value is $3c$ (and does not randomize when the project value is either 0 or $\frac{3}{2}c$). [You may want to construct an equilibrium in which types 0 and $\frac{3}{2}c$ of Researcher A choose *Shirk* with probability 1.]

(d) Does there exist any other mixed-strategy Bayesian Nash equilibrium? Explain your answer.

For questions (e)-(g), imagine that Researcher A takes the lead and makes the choice between *Shirk* and *Work*, and Researcher B decides between *Shirk* and *Work* after observing Researcher A's choice. This is therefore a signaling game in which the sender, Researcher A, has three types.

(e) Show that there is a pooling equilibrium where Researcher A chooses *Shirk* regardless of the project value. [You need to apply Bayes' rule to determine Researcher B's belief after observing *Shirk*.]

(f) Does there exist a perfect Bayesian equilibrium where Researcher A chooses *Work* if the project value is $3c$ and *Shirk* otherwise? Explain your answers. [In such a partially pooling, partially separating equilibrium, Bayes' rule requires Researcher B's belief after observing *Shirk* to be that the project value is either 0 or $\frac{3}{2}c$ with equal probability.]

(g) Does there exist a perfect Bayesian equilibrium where Researcher A chooses *Shirk* if the project value is 0 and *Work* if the project value is either $\frac{3}{2}c$ or $3c$? Explain your answers.

For the remaining question, imagine that order of decisions is reversed: Researcher B first makes the choice between *Shirk* and *Work*, and then Researcher A decides between

Shirk and *Work* after observing Researcher B's choice. This situation can be modeled as sequential-move game with a chance move after Researcher B has made the effort choice.

(h) Find all subgame perfect equilibria of the game.

2. (*Exiting an industry*) A declining industry currently has N identical firms that must decide whether or not to quit the industry. For any of the N firms, quitting leads to a payoff of 0. If n firms choose to stay, they each pay an operating cost of 1 and share equally the total revenue R ; that is, the payoff to each firm that stays is $R/n - 1$ where n is the total number of firms that stay in the industry. Note that the total industry profit $R - n$ is maximized when $n = 1$.

For questions (a) and (b) below, we assume that $N = 10$ and $R = 3.6$. This means that, for example, if 2 firms choose to stay, then the payoff to each of these 2 firms is $3.6/2 - 1 = 0.8$. In (a), we consider a simultaneous-move game where all 10 firms choose at the same time whether to exit or stay in the industry.

(a) In a pure-strategy Nash equilibrium, how many firms choose to stay in the industry? [Hint: this is a collective-action game; since all firms are identical, you only need to specify the number of firms that choose to stay in any pure-strategy Nash equilibrium.]

In (b), we consider a two-stage game where in the first stage an industry regulator sets a fee that must be paid by all firms that choose to stay, and in the second stage all 10 firms simultaneously choose whether to exit or stay in the industry. The payoff to the regulator is the total fees collected. For example, if the regulator sets the fee to 2, and 4 firms decide to stay, then the payoff to the regulator is $2 \times 4 = 8$, the payoff to each of the 4 firms is $3.6/4 - 1 - 2 = -2.1$, and the payoff to each exiting firm remains 0.

- (b) In a pure-strategy subgame perfect equilibrium, how many firms choose to stay in the industry? [Hint: use backward induction; for all subgames with some Nash equilibrium number \hat{n} of firms that choose to stay, use a Nash equilibrium condition to find the highest total fee consistent with it; then find the number \hat{n} that maximizes the total fees collected by the regulator.]

For all remaining questions, we assume that $N = 2$ and refer to the two firms as A and B. Further, R can take the value of 3.6, 1.8 or 0, and which value R takes is only known to firm A. Firm B knows only that the three values are equally likely; that is, the probability that R is equal to each of the three values is $1/3$. For each firm, the payoff from exiting the industry remains 0, and the payoff from staying is determined by the actual value of R and by whether the other firm exits or stays. For example, if $R = 1.8$, then the payoff to A and B is $1.8/2 - 1 = -0.1$ if both choose to stay, and the payoff is $1.8 - 1 = 0.8$ to the firm that stays if the other firm exits.

For questions (c) to (f), we consider a simultaneous-move game between A and B. This is a Bayesian game in which firm A has three types, corresponding to the three possible values of R .

- (c) Show that in any Bayesian Nash equilibrium, firm A chooses to exit if $R = 0$ and stay if $R = 3.6$. [Hint: argue that type 0 and type 3.6 both have a dominant action.]
- (d) Does there exist a Bayesian Nash equilibrium in which firm B chooses to exit? Explain your answer. [Hint: use what you know from question (c).]
- (e) Does there exist a Bayesian Nash equilibrium in which firm B chooses to stay? Explain your answer. [Hint: use what you know from question (c).]
- (f) Show that there exists a mixed-strategy Bayesian Nash equilibrium in which firm B randomizes between exiting and staying. [Hint: you get partial credit by using what

you know from questions (c), (d) and (e) to argue that type 1.8 of A must randomize too; to get the remaining points, you need to find the two mixing probabilities.]

For questions (g) to (i), we consider a sequential-move game in which first firm A chooses between exiting and staying in the industry, and then firm B makes the choice between exiting and staying after observing firm A's choice (but without observing the value of R). This is a signaling game where as the sender firm A has three types corresponding to the three possible values of R , and as the receiver firm B can condition its action on firm A's choice.

- (g) Show that in any perfect Bayesian equilibrium, firm B chooses to exit the industry after observing that firm A chooses to exit. [Hint: argue that conclusions from (c) still apply; use Bayes' rule to argue that, after observing firm A exiting, B's most optimistic belief about the value of R is that it is equally likely to be 0 and 1.8; then argue that even under this most optimistic belief B is better off exiting the industry.]
- (h) Show that in any perfect Bayesian equilibrium, firm B chooses to stay in the industry after observing that firm A chooses to stay. [Hint: use the conclusions from (c) and Bayes' rule to determine B's most pessimistic belief about the value of R after observing firm A staying.]
- (i) Use (g) and (h) to show that in any perfect Bayesian equilibrium, firm A chooses to stay in the industry only when $R = 3.6$.

For question (j), we consider a sequential-move game in which first firm B chooses between exiting and staying in the industry, and then firm A makes the choice between exiting and staying after observing firm B's choice.

- (j) Show that in the subgame perfect equilibrium of this game, firm B chooses to stay in the industry. [Hint: use backward induction; for each choice by B, there are three

subgames corresponding the three possible values of R ; from B's perspective, the three subgames occur with equal probabilities.]

3. (*Parking the Bus*) Two football teams, B and C, meet in a group stage game of the Champions League. Each team can choose either an attacking formation (A) or a defensive formation (P). If either team, or both teams, choose P , the outcome of the game is a draw, with each team getting 1 point. If both teams choose A , the outcome depends on the relative strength of their attacking formations. If B has a strong attacking formation and C has a weak one, then B wins, with B getting 3 points and C getting 0 point. If B has a weak attacking formation and C has a strong one, then C wins, with B getting 0 point and C getting 3 points. If both have a strong attacking formation or both have a weak attacking formation, then the outcome is either a win for B, with B getting 3 points and C getting 0 point, or a win for C, with B getting 0 point and C getting 3 points, with equal probabilities. (In this case, each team earn $\frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 0 = 1.5$ points in expectation.) Each team, B or C, knows whether their own attacking formation is strong or weak, but knows only the rival's attacking formation is equally likely to be strong or weak.

In questions (a)-(e), the two teams choose between A and P simultaneously. This is a Bayesian game in which each team, B and C, has two types, strong attacking formation and weak attacking formation, and two actions, A and P .

- (a) Show that there exists a Bayesian Nash equilibrium in which each team chooses A if it has a strong attacking formation and P if it has a weak one.
- (b) Show that there does not exist a Bayesian Nash equilibrium in which each team chooses P if it has a strong attacking formation and A if it has a weak one.
- (c) Show that there does not exist a Bayesian Nash equilibrium in which each team chooses A regardless of whether it has a strong or weak attacking formation.

(d) Show that there exists a Bayesian Nash equilibrium in which each team chooses P regardless of whether it has a strong or weak attacking formation, but the equilibrium strategy is weakly dominated for each player. [Hint: remember that a team gets 1 point by choosing P regardless of the strategy of the rival; show that the equilibrium strategy is weakly dominated by choosing A if it has a strong attacking formation and P if it has a weak formation; that is, show that the latter does equally well as the former for 1 out of 4 pure strategies of the other team, and does strictly better for the rest of 3 strategies.]

For the remaining questions, (e) to (h), consider a sequential-move game where C first chooses between A and P , and after C 's choice, B chooses between A and P . This is a sequential-move game in which B as the second mover observes the choice made by the first mover, C .

(e) Show there exists a perfect Bayesian equilibrium in which C chooses A if its attacking formation is strong and P if it is weak. [Hint: remember that both teams get 1 point if C chooses P ; you need to use Bayes' rule to determine the belief of B regarding the strength of C 's attacking formation after C chooses A ; you need to do the same for questions (g) and (h) below too.]

(f) Show that there does not exist a perfect Bayesian equilibrium in which C chooses P if its attacking formation is strong and A if it is weak.

(g) Show that there does not exist a perfect Bayesian equilibrium in which C chooses A regardless of whether it has a strong or weak attacking formation. [Hint: after using Bayes' rule for B 's belief upon observing C 's choice of A , you may make use of your answer to question (c).]

(h) Does there exist a perfect Bayesian equilibrium in which C chooses P regardless of whether it has a strong or weak attacking formation? Explain your answer. [Hint: can Bayes' rule be used to derive B's belief when C chooses A ? whatever his belief, will B ever not want to choose A if it is strong? will C then want to deviate to A if its attacking formation is strong?]

4. (*Job application*) There are two job-seekers, A and B, and two jobs, H and L. Job H pays three times as L: for each job-seeker, the payoff is 3 if he gets H, 1 if he gets L, and 0 if he gets neither job. Each job-seeker knows whether himself is skilled or unskilled. In either case, he does not know if the other job-seeker is skilled or unskilled, only that the other one is skilled with probability p and unskilled with probability $1 - p$ (so p is known to A and B). If A or B is only one that applies for a given job, then he gets the job regardless of whether he is skilled or unskilled. If A and B both apply for the same job, A gets the job if A is skilled and B is unskilled, B gets the job if B is skilled and A is unskilled, and each gets the job with probability $\frac{1}{2}$ if both are skilled or both are unskilled.

For questions (a)-(e), A and B simultaneously decide which of the two jobs to apply for. This is a Bayesian game between A and B, where both of them have two types. [Hint: Whether or not you can answer (a), in answering (b), (c) and (d) you should focus your analysis on the unskilled type of A and B, assuming that the skilled type applies for H in the equilibrium under consideration.]

(a) Show that in any Bayesian Nash equilibrium, each job-seeker applies for Job H if he is skilled.

(b) Show that if $p \leq \frac{1}{3}$, it is a Bayesian Nash equilibrium for both job-seekers to apply for Job H regardless of whether they are skilled or unskilled.

(c) Show that if $p \geq \frac{5}{7}$, it is a Bayesian Nash equilibrium for A and B to apply for H if

they are skilled and L if they are unskilled.

- (d) Show that if $\frac{1}{3} < p < \frac{5}{7}$, there is a Bayesian Nash equilibrium in which A and B apply for H if they are skilled and randomize between H and L if they are unskilled.

For questions (e) and (f), Job-seeker A first decides which of the two jobs to apply for, and after observing the choice of A, Job-seeker B then decides which job to apply for. This is a sequential-move game where as the second-mover B observes the first-mover A's choice but not A's type. [Hint: The same logic as in (a) establishes that in any perfect Bayesian equilibrium both A and B apply for H if they are skilled, so for both (e) and (f) you only need to analyze the choices of unskilled job-seekers in the equilibrium under consideration.]

- (e) Show that if $p \geq \frac{2}{3}$, there is a perfect Bayesian equilibrium in which A applies for Job H if he is skilled and L if he is unskilled. [Hint: You need to first work out an unskilled B's belief about A's type and his best response after observing that A has applied to H, and then verify unskilled A's best choice between the two jobs.]
- (f) Show that if $p \leq \frac{2}{3}$, there is a perfect Bayesian equilibrium in which A applies for H regardless of whether he is skilled or not. [Hint: You need to first work out an unskilled B's belief about A's type after observing that A has applied to H, and then distinguish two cases, depending on the value of p , for unskilled B's best response to A applying for H, one where the best response is also applying for H, and the other where it is applying for L.]

5. (*Second Opinion*) Asher takes his car to the local mechanic Ed, believing that with probability p (a number between 0 and 1) his car has a major problem that needs to be fixed. If the car has just a minor problem, Ed can fix it at no cost; in this case, both Asher and Ed

receive a payoff of 0. If the car does have a major problem, Ed can offer to fix it at a charge of \$1000. In this case, if Asher rejects the quote, both get a payoff of 0; and if Asher accepts it, Ed gets a payoff of $\$1000 - \$500 = \$500$ (\$500 is the cost of fixing the problem to Ed), and Asher gets a payoff of $\$2000 - \$1000 = \$1000$ (\$2000 is Asher's valuation for having the problem fixed). But Ed might lie and tell Asher to pay \$1000 when the car has only a minor problem; if the offer is accepted Asher's payoff is $-\$1000$ (he has \$0 valuation for fixing a minor problem) and Ed's payoff is \$1000. Aware of this possibility, Asher may take his car elsewhere for a second opinion before deciding whether to accept or reject Ed's quote of \$1000 for fixing a major problem. It costs \$200 to Asher but the second opinion allows Asher to find out whether Ed has been truthful. If Ed is found out to have been truthful, Asher takes his car back to Ed to fix it at price \$1000; in this case Ed's payoff remains \$500 but Asher's payoff becomes $\$1000 - \$200 = \$800$. If Ed is found out to have lied, Asher's payoff is $-\$200$ and Ed's payoff is $-\$10,000$, which is the cost of losing his reputation to Ed. For the three questions below, we look for perfect Bayesian equilibria in which Ed always tells Asher about his car's major problem if it does have one.

- (a) Show that there is no perfect Bayesian equilibrium in which Ed is always truthful regardless of the value of p .
- (b) Show that if $p \geq 0.8$, there is a perfect Bayesian equilibrium in which Ed always lies.
- (c) Show that if $p < 0.8$, there is a perfect Bayesian equilibrium in which Ed mixes between lying and telling the truth and Asher mixes between seeking and not seeking a second opinion.

6. (*Settling out of Court*) A plaintiff (P) is suing a defendant (D) for damage in a civil lawsuit. If P wins the lawsuit in a trial, D pays P \$20 million; if P loses in the trial, there is no transfer between P and D. The cost of going to trial is \$2 million for both P and D,

including fees to lawyers and to the court. In the Canadian system, the loser pays the winner's cost. This means that if P wins in the trial, P gets \$20 million and D loses \$24 million; if D wins in the trial, P loses \$4 million, and D loses nothing.

Who wins in the trial depends on the relative strength of the evidence that P and D have. Each party, P or D, knows whether his own evidence is strong or weak, but the other party knows only the rival's evidence is equally likely to strong or weak (this is the prior belief of the latter about the strength of the former's evidence). If P has strong evidence and D has weak evidence, then P wins, with P getting a payoff of \$20 million and D getting a payoff of $-\$24$ million; if P has weak evidence and D has strong evidence, then D wins, with P getting $-\$4$ million and D getting 0. If both have strong evidence or both have weak evidence, then each wins with probability 0.5, with P getting an expected payoff of $0.5 \cdot 20 + 0.5 \cdot (-4) = 8$ million dollars, and D getting $0.5 \cdot (-24) + 0.5 \cdot 0 = -12$ million dollars.

Instead of going to trial, P and D may agree to settle the lawsuit by D paying P half of the damage, which is \$10 million (so P's payoff is \$10 million and D's payoff is $-\$10$ million). In questions (a)-(e), we consider a simultaneous-move game where P and D choose whether or not to settle. The lawsuit goes to trial unless both P and D choose to settle (that is, the case goes to trial if one or both parties choose not to settle). This is a Bayesian game in which each player, P and D, has two types, strong evidence and weak evidence, and two actions, settling the case out of court (*Settle*) and going to trial (*Trial*).

- (a) Show that there exists a Bayesian Nash equilibrium in which each party chooses *Trial* if he has strong evidence and *Settle* if he has weak evidence.
- (b) Show that there does not exist a Bayesian Nash equilibrium in which each party chooses *Settle* if he has strong evidence and *Trial* if he has weak evidence.

- (c) Show that there does not exist a Bayesian Nash equilibrium in which each party chooses *Settle* regardless of he has strong or weak evidence.
- (d) Show that there exists a Bayesian Nash equilibrium in which each party chooses *Trial* regardless of he has strong or weak evidence, but the equilibrium strategy is weakly dominated for each player. [Hint: remember that the case goes to trial if at least one party chooses *Trial*; show that the equilibrium strategy is weakly dominated by choosing *Trial* if the evidence is strong and *Settle* if the evidence is weak; that is, show that the latter does equally well as the former for 1 out of 4 pure strategies of the other player, and does strictly better for the rest of 3 strategies.]
- (e) Does there exist a Bayesian Nash equilibrium in which at least one type of one player mixes between *Trial* and *Settle*? Explain your answer. [Hint: use your answer to question (d).]

For the remaining questions, (f) to (i), consider a sequential-move game where P first chooses whether or not to propose to D to settle out of court. If P proposes to settle and then D agrees, D pays P \$10 million and the game ends. If P proposes to settle and D rejects the proposal, or if P does not propose to settle, the case goes to trial. This is a sequential-move game in which D as the second mover observes the choice made by the first mover, P.

- (f) Show there exists a perfect Bayesian equilibrium in which P proposes to settle if his evidence is weak and does not propose to settle if his evidence is strong. [Hint: you need to use Bayes' rule to determine the belief of D regarding the strength of P's evidence when P proposes to settle; you need to do the same for questions (g) and (h) below too.]

- (g) Show that there does not exist a perfect Bayesian equilibrium in which P proposes to settle if his evidence is strong and does not propose to settle if his evidence is weak.
- (h) Show that there does not exist a perfect Bayesian equilibrium in which P proposes to settle regardless of he has strong or weak evidence.
- (i) Does there exist a perfect Bayesian equilibrium in which P does not propose to settle regardless of he has strong or weak evidence? Explain your answer. [Hint: can Bayes' rule be used to derive D's belief when P proposes to settle? whatever his belief, will D ever reject the proposal if his evidence is weak? will P ever not want to propose to settle if his evidence is weak?]

7. (*Treasure and Gem*) Two kids, Xavier and Yolanda, try to divide some treasure, which we represent here as a unit interval $[0, 1]$, from left end point 0 to the right end point 1. Xavier cuts the treasure into two pieces, represented by a cut point a on the interval. The piece on the left is represented by $[0, a]$, with size a ; the piece on the right is $[a, 1]$, with size $1 - a$. Yolanda gets to choose the piece she wants first. If Yolanda chooses the piece on the left (*Left*), her payoff is a and Xavier's payoff is $1 - a$; if she chooses the piece on the right (*Right*), her payoff is $1 - a$ and his payoff is a .

In questions (a) and (b), Xavier's cut point a can be any number between 0 and 1, and Yolanda gets to see how Xavier's cut point a before deciding whether she wants the piece on the left or the one on the right.

- (a) Describe all the strategies available to Xavier, and to Yolanda.
- (b) Find two subgame perfect equilibria with the same outcome. [Hint: For full credit, you need to specify the equilibrium, not just the outcome.]

For all remaining questions, Xavier can only choose between $a = \frac{1}{4}$ and $a = \frac{3}{4}$. For

questions (c) and (d), Yolanda has to choose between *Left* and *Right* before seeing Xavier's cut point a . We have the following simultaneous-move game for questions (c) and (d).

		Yolanda	
		<i>Left</i>	<i>Right</i>
Xavier	$\frac{1}{4}$	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{3}{4}$
	$\frac{3}{4}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{3}{4}, \frac{1}{4}$

- (c) Does Xavier or Yolanda have a dominant strategy? Are the two strategies of each kid rationalizable? Is there a pure-strategy Nash equilibrium? Explain.
- (d) Find a mixed-strategy Nash equilibrium.

For all remaining questions, there is a gem hidden in the treasure. Xavier knows where it is, but Yolanda knows only that with probability $\frac{1}{2}$ the gem is at the end point of 0, and with probability $\frac{1}{2}$ it is at the end point of 1. For both of Xavier and Yolanda, getting the gem adds $g > 0$ to their payoff. For question (e), consider the Bayesian game where Xavier chooses between $a = \frac{1}{4}$ and $a = \frac{3}{4}$, and Yolanda simultaneously chooses between *Left* and *Right*, with the following payoff tables (the table on the left is when the gem is at 0, and the table on the right is when the gem is at 1, each happening with probability $\frac{1}{2}$):

		Yolanda	
		<i>Left</i>	<i>Right</i>
Xavier	$\frac{1}{4}$	$\frac{3}{4}, \frac{1}{4} + g$	$\frac{1}{4} + g, \frac{3}{4}$
	$\frac{3}{4}$	$\frac{1}{4}, \frac{3}{4} + g$	$\frac{3}{4} + g, \frac{1}{4}$

		Yolanda	
		<i>Left</i>	<i>Right</i>
Xavier	$\frac{1}{4}$	$\frac{3}{4} + g, \frac{1}{4}$	$\frac{1}{4}, \frac{3}{4} + g$
	$\frac{3}{4}$	$\frac{1}{4} + g, \frac{3}{4}$	$\frac{3}{4}, \frac{1}{4} + g$

- (e) Show that there is no pure-strategy Bayesian Nash equilibrium. [Hint: use a contradiction argument; suppose there is a Bayesian Nash equilibrium in which Yolanda chooses *Left*; work out the best response of Xavier...]

For question (f), consider a signaling game in which Yolanda chooses between *Left* and *Right* after observing Xavier's choice between $a = \frac{1}{4}$ and $a = \frac{3}{4}$.

(f) Show that when $g < \frac{1}{2}$, there is a separating equilibrium. [Hint: when $g < \frac{1}{2}$, Yolanda will go for the longer piece even though it does not contain the gem; to get full credit you need to specify strategies of both kids and beliefs of Yolanda.]

8. (*D-day*) Army A plans an amphibious assault on Army B, by choosing one of two possible landing sites, *North* and *South*. Army B is twice as large as Army A, but must choose how to allocate its forces between the two sites. They only care about their own chance of winning. To be precise, we assume that the size of A's invading force is 1, and the size of B's total defending forces is 2. Let us represent B's allocation as a real number b , which ranges from 0 to 2 and stands for the size of its forces allocated to *North*, with the size allocated to *South* equal to $2 - b$. The chance of winning by A at any site is equal to 1 over 1 plus the size of B's forces allocated to it. Thus, for any b , the payoff to A is $1/(1 + b)$ from choosing *North*, and is $1/(1 + 2 - b) = 1/(3 - b)$ from *South*. For example, if $b = 0$ (all forces to *South*), then by choosing *North* Army A gets a payoff of 1, while by choosing *South* Army A gets $1/3$; if $b = 1$ (equal split of forces), then Army A's payoff is equal to $1/2$ regardless whether it chooses *North* or *South*. The payoff to Army B is always 1 minus the payoff to Army A for any outcome.

For question (a), we assume that Army A chooses between *North* and *South* after observing Army B's allocation b of its forces.

(a) What is the outcome of any subgame perfect equilibrium of this sequential-move game? Which army has a greater chance of winning in the subgame perfect equilibrium? Explain your answers. [Hint: The payoff to A is decreasing in b from choosing *North* and increasing in b from *South*; the payoffs to B have the opposite relationships with b because the two payoffs sum up to 1.]

For question (b), Army A has to choose between *North* and *South* at the same time as

Army B chooses the allocation b of its forces. For this question only, we assume that Army B can only choose between $b = 0$ (allocating all its forces to *South*) and $b = 2$ (all to *North*). This is a simultaneous-move game, with the following payoff table:

		Army B	
		$b = 0$	$b = 2$
Army A	<i>North</i>	1, 0	1/3, 2/3
	<i>South</i>	1/3, 2/3	1, 0

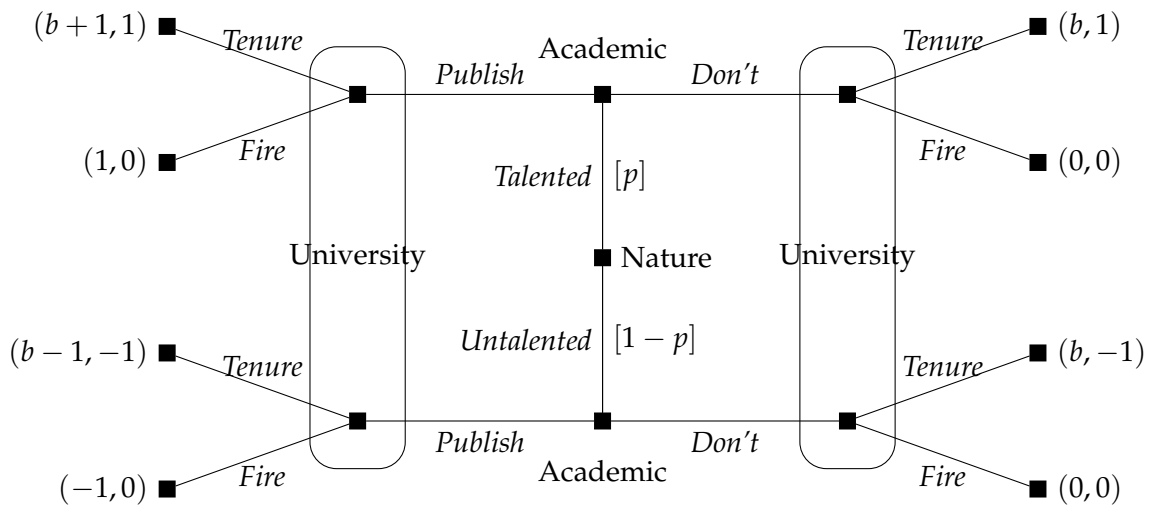
(b) Show that there is no pure-strategy Nash equilibrium, and find a mixed-strategy Nash equilibrium. Which army expects a greater chance of winning in the mixed-strategy Nash equilibrium? Explain.

For question (c), there is a third potential landing site *Central*. As before, for any of the three sites chosen by Army A, the payoff to A is 1 over 1 plus the size of the forces allocated to it by Army B, and the payoff to B is 1 minus the payoff to A. Army B first chooses the size b_N (a number between 0 and 2) of its forces allocated to *North*; after observing b_N Army A chooses one of three sites to attack; if A chooses *North*, the game ends with payoff $1/(1 + b_N)$ to A; if A chooses either *South* or *Central*, B observes the choice and then chooses the allocation of the remaining forces $2 - b_N$ between the two sites; the game ends with payoff $1/(1 + b_S)$ to A if A choose *South* and B allocates b_S to it, and with payoff $1/(1 + b_C)$ if A chooses *Central* and B allocates b_C to it.

(c) How does the subgame perfect equilibrium outcome differ from the equilibrium outcome you found in question (a)? Explain your answer.

9. (*Publish or Perish*) A university's decision on an academic's tenure, which is a lifetime appointment, should be ideally based on the academic's research ability. However, the

university does not observe the academic's research ability, having a prior belief that the academic is either talented in doing research, with probability p , or not talented, with probability, $1 - p$. Publishing yields a payoff of 1 to the academic regardless of whether he is talented or not. The talented type likes research and suffers no payoff loss in writing a paper that leads to a publication. The untalented type dislikes research and suffers a payoff loss of 2 from writing a publishable paper. Not publishing yields a payoff of 0 to the academic regardless of whether he is talented or not. Not being able to observe whether the academic is talented or not, the university makes a tenure decision based upon whether the academic has published or not. By retaining a talented academic, the university gets a payoff of 1; by retaining someone untalented, the university gets a payoff of -1 . If the academic is fired, the university gets a payoff of 0. Regardless of his talent, getting tenure yields a payoff of $b > 0$ to the academic.



As illustrated above, the game proceeds as follows: Nature decides whether Academic is talented or not; Academic chooses between *Publish* and *Don't*; University decides be-

tween *Tenure* and *Fire* after observing whether Academic has chosen *Publish* and *Don't*; and finally, University and Academic receive payoffs depending on whether Academic is talented or not, whether Academic has chosen *Publish* or *Don't*, and whether University has chosen *Tenure* and *Fire*. For example, after the untalented type chooses *Publish*, his payoff is $-2 + 1 + b = b - 1$ and University's payoff is -1 if the latter chooses *Tenure*, and his payoff is $-2 + 1 = -1$ and University's payoff is 0 if the latter chooses *Fire*.

- (a) Suppose $b < 1$. Show that there is a separating equilibrium in which only the talented type of Academic chooses *Publish*. [Hint: to get full credit, you must specify the strategy of Academic, and the beliefs and strategy of University.] Are there other perfect Bayesian equilibria (not just separating equilibria)? Explain your answer. [Hint: use the idea of dominance.]
- (b) Suppose $b > 1$ and $p > \frac{1}{2}$. Show that there is a pooling equilibrium in which both the talented type and the untalented type choose *Publish*. [Hint: to get full credit, you must specify the strategy of Academic, and the beliefs and strategy of University.] Does there exist another pooling equilibrium in which both types choose *Don't*? Explain your answer.
- (c) Suppose $b > 1$ and $p < \frac{1}{2}$. Show that there is no semi-separating equilibrium in which the untalented type randomizes between *Publish* and *Don't*, and the talented type chooses *Don't*. [Hint: use a contradiction argument.] Find a semi-separating equilibrium in which the untalented type randomizes and the talented type chooses *Publish*. [Hint: what is the payoff of the untalented type from *Don't*, and what does this mean about University's best response to *Publish*?]