

Solutions to Chapter 11 Exercises

SOLVED EXERCISES

If you find a blank space where an equation or figure may appear, please select that area for it to appear.

S1. (a) The number choosing X should decrease to move the population division between X and Y away from the unstable center equilibrium.

(b) Because the line for action X is above the line for action Y when 100 people choose X, and because we expect the number choosing X to decrease from this point—see part (a)—it must be the case that the lines in the diagram represent *costs* associated with each action. Thus, when 100 people are choosing X, the cost of X exceeds the cost of Y, so people will want to switch from X to Y.

S2. (a) This game is a prisoners' dilemma since $s(n)$ is greater than $p(n + 1)$ for all n . Any single player can always raise his payoff by switching from participating to shirking.

(b)
$$T(n) = n \times p(n) + (100 - n) \times s(n) = n^2 + (100 - n)(4 + 3n)$$

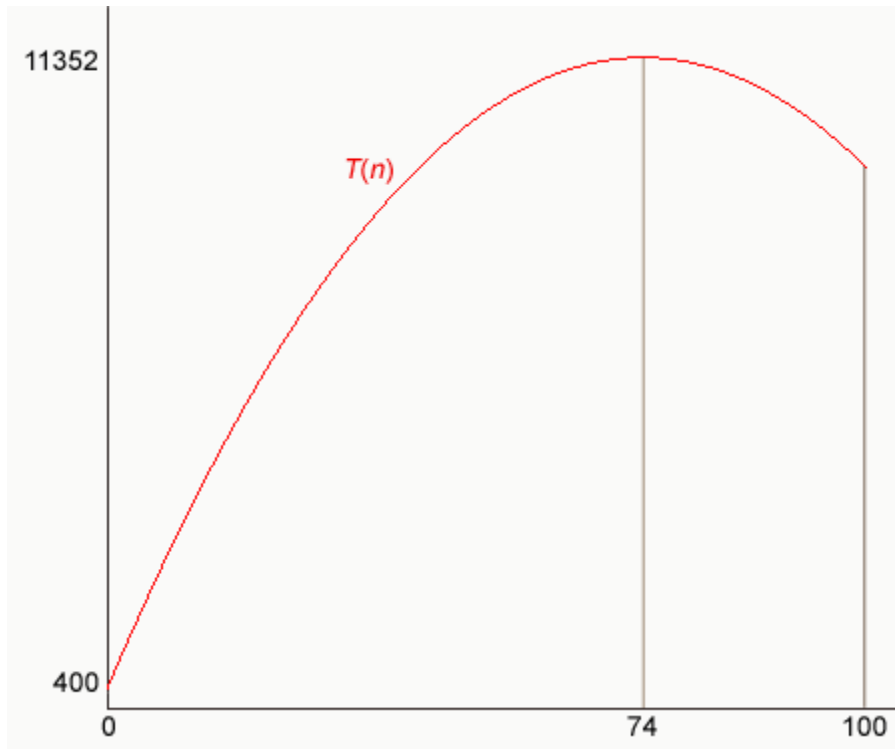
(c) Plugging the appropriate values into the text's Equation (12.1) yields

$$\begin{aligned} T(n + 1) - T(n) &= n + 1 - 4 - 3n + n(n + 1 - n) + (100 - n - 1)(4 + 3n + 3 - 4 - 3n) \\ &= -3 - 2n + n + 3(100 - n - 1) \\ &= 294 - 4n. \end{aligned}$$

This formula is positive for all n up to $n = 73$, but turns negative for $n = 74$ and higher. Thus, the last time the change in total social payoff expression is positive is for $T(74) - T(73)$. Thus T is maximized at $n = 74$.

Using calculus, you can find the same answer by differentiating $T(n)$ from part (b). $T'(n) = 2n + 3(100 - n) - (4 + 3n) = 296 - 4n$. Set this derivative equal to zero to find that $T(n)$ is maximized at $n = 74$.

Or you can find the same answer using a graph of $T(n)$:

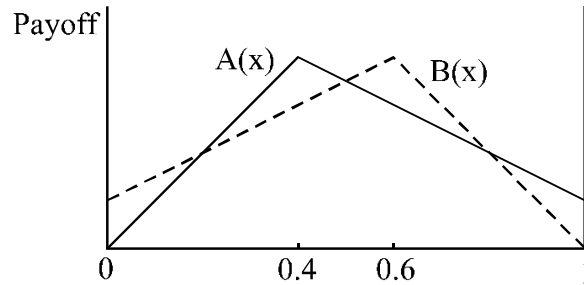


(d) When $n = 74$, each participant receives a benefit $p(74) = 74$. If one participant were to switch to shirking, he would receive a benefit of $s(73) = 4 + 3(73) = 223$. Each participant thus has a private incentive to become a shirker (as is always the case in a prisoners' dilemma).

(e) If the game is repeated, the players can take turns playing the role of shirker (and receiving the higher private benefit). Social norms of the threat of future punishment might keep each player in his assigned role for that game. A similar procedure might work even if the players' future interaction were in a different game.

S3. (a) Let x be the proportion of the population in Alphaville. Then the payoffs to living in the towns are, for Alphaville, $A(x) = x$ if $x < 0.4$ and $0.6 - 0.5x$ if $0.4 < x < 1$. For Betaville, the payoffs are $B(x) = 0.1 + 0.5x$ if $x < 0.6$ and $1 - x$ if $0.6 < x < 1$.

(b) The graph below shows that there are five equilibria. Three exist where $A(x)$ and $B(x)$ cross: at $x = 0.2$ [$A(x) = B(x) = 0.2$], at $x = 0.5$ [$A(x) = B(x) = 0.35$], and at $x = 0.8$ [$A(x) = B(x) = 0.2$]. There are also two equilibria at the end points. At $x = 0$ (when nobody lives in Alphaville), $B(x) = 0.1 > A(x) = 0$, so nobody will move into Alphaville. Similarly, at $x = 1$, $A(x) = 0.1 > B(x) = 0$, so nobody will move into Betaville:



(c) For $0 < x < 0.2$, $B(x) > A(x)$; for $0.2 < x < 0.5$, $A(x) > B(x)$; for $0.5 < x < 0.8$, $B(x) > A(x)$; and for $0.8 < x < 1$, $A(x) > B(x)$. Given these relationships, if x is just below 0.2, people will move to Betaville, and x will fall. If x is just above 0.2, people will move to Alphaville, and x will rise. Thus, the $x = 0.2$ equilibrium is unstable. Similar analysis shows that the $x = 0.8$ equilibrium is also unstable. The equilibria at $x = 0$, $x = 0.5$, and $x = 1$ are stable, because x moves toward those values.

S4. (a) A person will voluntarily contribute if his benefit from doing so exceeds the \$100 cost. Assuming that n other people are already contributing, a person's benefit from beginning to contribute is the difference between his benefit when there are $n + 1$ contributors and when there are n contributors: $(n + 1)^2 - n^2$. Once there are 50 other contributors, this benefit is $51^2 - 50^2 = 101 > 100$. So once the mayor convinces 50 people to contribute, the rest will join in voluntarily.

(b) There are two stable Nash equilibria in this game. When $n = 0$, no one is contributing, and no one wants to. When $n = 100$, everybody is contributing, and each person receives a private gain from so doing. (With 100 discrete individuals, there is no unstable equilibrium in the middle. When 50 other people are contributing, everybody wants to contribute; when 49 others are contributing, nobody gains from becoming contributor number 50. There is no division at which the next possible contributor is indifferent about his choice.)

S5. (a) When H of the 99 residents of San Miniato go truffle hunting, each hunter earns $\$200 \times (\$10 - (H/10)) = \$2,000 - \$20 \times H$ and each non-hunter earns \$50. If no one else were to go hunting, each resident would earn $\$2,000 - \$20 \times 1) = \$1,980$, far more than what they can earn in other work; so, each resident prefers to go hunting if no one else does. On the other hand, if everyone else were to go hunting, each resident would earn $\$2,000 - \$20 \times 99) = \$20$, less than what can be earned in other work; so, each resident prefers not to go hunting if everyone else does.

(b) We proceed by backward induction. Note that hunting is profitable, relative to the outside option of other work, whenever 97 or fewer of the residents decide to hunt. Why? If 97 hunt, then hunting generates earnings of \$60 per hunter, better than the outside option of \$50. Should fewer than 97 hunt, hunting becomes even more attractive.

First, consider the last resident to decide whether to hunt. If all ($H = 98$) or all-but-one ($H = 97$) of the previous 98 residents have chosen to hunt, then this last person will choose not to hunt, since hunting will generate earnings of only \$20 (if $H = 98$) or \$40 (if $H = 97$). On the other hand, if 96 or fewer of the others are hunting, the last resident will prefer to hunt.

Next, consider the second-to-last resident to decide. If all ($H = 97$) of the previous residents have chosen to hunt, then the second-to-last person will choose not to hunt. Otherwise, if 96 or fewer of the previous 97 residents have chosen to hunt, then the second-to-last person will choose to hunt. Why? This is obvious if 95 or fewer of the previous residents are hunting, since then hunting is profitable no matter what the last resident does. So consider the case in which 96 of the previous 97 residents decided to hunt. The second-to-last resident knows that, if he decides to hunt, the last resident will decide *not* to hunt; thus, hunting is profitable for him.

Next, consider the third-to-last resident to decide (and, by extension, all previous deciders). This resident prefers to hunt no matter what previous residents have decided to do. Why? This is obvious if 94 or fewer of the previous 96 residents have chosen to hunt, since then hunting is profitable no matter what the last two residents. So consider the case in which 95 or 96 of them have decided to hunt. If all 96 previous residents are hunting and the third-to-last resident also chooses to hunt, then (as previously shown) the second-to-last resident and the last resident will both decide *not* to hunt, making hunting profitable for the third-to-last resident. Similarly, if 95 previous residents are hunting and the third-to-last resident also chooses to hunt, then (as previously shown) the second-to-last resident will choose to hunt and the last resident will choose *not* to hunt—and hunting will again be profitable since there will again be exactly 97 people hunting.

Note that, behind the apparent complexity here (there are a lot of subgames that need to be considered!), residents' decisions are driven by a very simple logic: no one ever wants to be the 98th person to hunt! Anticipating that, any of the first 97 residents to decide can rest easy in anticipating that there will never be more than 97 hunters and so hunting will be profitable. Thus, in the unique subgame-perfect equilibrium of this game, the first 97 residents to decide (including the third-to-last resident) will all choose to hunt while the last two will choose not to hunt.

More precisely: the first 97 residents will choose to hunt no matter what previous residents have decided to do; the 98th (second-to-last) resident will choose to hunt if 96 or fewer previous residents decided to hunt and not hunt if all 97 of them decided to hunt; and the 99th (last) resident will choose to hunt if 96 or fewer previous residents decided to hunt and not hunt if 97 or 98 of them decided to hunt. The resulting subgame-perfect equilibrium outcome is that the first 97 residents will hunt and the last two residents will not hunt.

Overall, then, 97 residents will earn $\$2,000 - 97 \times \$20 = \$60$ and two residents will earn \$50. Total daily income for the town as a whole is therefore $97 \times \$60 + 2 \times \$50 = \$5,920$.

(c) If H residents go hunting, then daily income for the town as a whole will be $H \times (\$2,000 - \$20 \times H) + (99 - H) \times \$50 = \$4,950 + \$1,950 \times H - \$20 \times H^2$. This function is a downward-curving parabola, maximized at 48.75. Because H has to be an integer, this means that the optimal H (from the town's point of view) must be either $H = 48$ or $H = 49$. Plugging in these values, or graphing the function and examining the graph, we see that total town income equals \$52,470 when $H = 48$ and \$52,480 when $H = 49$; so, total town income is maximized by limiting the number of hunters to 49 per day—far less than the 97 who go hunting each day in the subgame-perfect equilibrium derived earlier, and generating far more income as well (\$52,480 when $H = 49$ versus \$5,920 when $H = 97$).

(d) When $H = 49$, a resident not selected to go hunting can earn $\$2000 - \$20 \times 50 = \$1,000$ dollars when sneaking off to hunt illicitly—far more than their meager income of \$50 from other work. To deter such “sneaking,” the townspeople need to find a way to reduce the benefits of sneaking or increase the benefits of doing other work. There are several ways that the townspeople could do so, and students will likely come up with many creative ideas. Here, for illustration purposes, we discuss two particular approaches focusing especially on some of nuances and challenges associated with implementation.

One idea is that the townspeople could establish an “approved marketplace” where truffles from designated hunters can be sold to chefs and other buyers who come from out of town each day looking to purchase. Residents who have secretly been hunting truffles will have an incentive to trade illicitly with buyers outside of this marketplace. So, for this to work, the townspeople will need to monitor and punish anyone caught attempting to trade outside of the marketplace. This could be challenging, of course, as truffle buyers also have a strong incentive to trade outside of official channels—potentially creating a black market for the town's white truffles, undermining the mayor's goals and ultimately reducing town income.

To reduce the incentive to sneak off to the forest, the town might also provide extra income to those who work in other ways. In particular, suppose that each person who *verifiably* does some other work during the day is paid a bonus of B ! Such a policy transforms work that normally would only earn \$50 into work that now earns $\$(50 + B)$ per day—reducing their incentive to sneak away, while also potentially making it more easily observable when they do attempt to do so. (Imagine if, in order to collect the $\$B$ bonus, residents have to check in with a program officer who circulates around town, saying hello to everyone every couple of hours. Assuming that it takes at least a couple of hours to gather up your pigs and go off and back to the forest, this sort of system could allow residents who sneak off to be caught and then sanctioned.)

Note: If $\$B > 950$, then residents will gladly stay out of the forest, since hunting generates daily income of \$1,000. However, because 50 people have to be incentivized not to work, funding such a generous policy would require paying out $50 \times \$950 = \$47,500$ per day to non-hunters, nearly all of the

daily income generated by the hunters. The town could in principle tax the truffle hunters but such a policy, collecting essentially all of the fruit of their labor in the forest, would leave them with little incentive to exert effort and could ultimately incentivize them to underreport how many truffles they find and sell them on the black market. Consequently, the idea of providing extra income to non-hunters is unlikely *on its own* to address the challenge—some sort of monitoring and sanctioning mechanism is needed.

S6. (a) By assumption, drilling allows a landowner to extract 100,000 barrels of oil and the breakeven price for drilling is \$40; so, it must be that drilling itself costs \$4,000,000 dollars (or, more simply “4M,” using “M” to mean “million dollars”). On the other hand, because the per-barrel price of oil equals $\$(60 - 25 \times X)$, revenue from drilling equals $6M - 2.5M \times X$ when a fraction X of the landowners drill. The overall profitability of drilling is therefore $2M - 2.5M \times X$ for each landowner. Note that drilling generates 2M in profits when no one else drills ($X = 0$) but 0.5M in losses when everyone else drills ($X = 1$); thus, each landowner prefers to drill when no one else does but not when everyone else does.

(b) By part (a), any Nash equilibrium must have the feature that some landowners drill and others do not; that is, any Nash equilibrium must be in mixed strategies. In order for drilling and not drilling each to be a best response, landowners must be indifferent about whether or not to drill. Since not drilling yields zero profit, drilling must therefore also yield zero profit. In particular, X must solve the equation $2M - 2.5M \times X = 0$, or $X = 4/5 = 0.8$. We conclude that, in the Nash equilibrium, exactly 80% of Bakken landowners drill and, because so many are drilling, they all earn zero profit!!

Note: The interpretation of “mixed strategy” in this context is that of a *population mix*, that a fraction of landowners drill while a fraction of them do not. Individual landowners need not make random decisions, as long as the population as a whole has exactly the right fraction of landowners choosing to drill. See Chapter 12’s treatment of evolutionary games for more detail on the interpretation of a population mix.

(c) Because each drilling landowner extracts 100,000 barrels of oil, the drilling tax collected equals 1M per drilling site. Should fraction X of landowners drill, this tax then converts into a carbon dividend of $1M \times X$ for each landowner (whether they drill or not). Overall, then, landowners who drill pay the government *net tax* equal to $1M \times (1 - X)$ while those who do not drill get a subsidy of $1M \times X$.

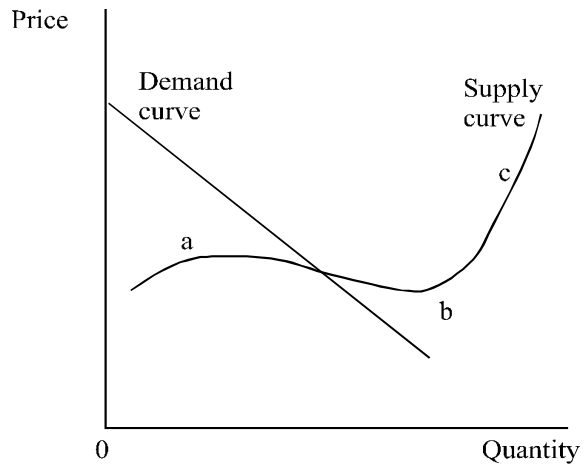
Repeating the analysis in part (a), now with the drilling tax and carbon dividend included: The overall profitability of drilling is now $2M - 2.5M \times X - 1M \times (1 - X) = 1M - 1.5M \times X$ (revenue *minus* cost of drilling *minus* net drilling tax), while the profitability of not drilling is now $1M \times X$ (carbon dividend). Note that drilling generates 1M in profits when no one else drills ($X = 0$) but 0.5M in losses

when everyone else drills ($X = 1$); thus, as in part (a), each landowner prefers to drill when no one else does but not to drill when everyone else does.

(d) As in part (b), any Nash equilibrium must be in mixed strategies. Since not drilling yields profit $1M \times X$, drilling must therefore also yield profit $1M \times X$. In particular, X must solve the equation $1M - 1.5M \times X = 1M \times X$, or $X = 2/5 = 0.4$. We conclude that, in the Nash equilibrium, exactly 40% of Bakken landowners drill and each earns profit equal to $0.4M$.

(e) Bakken landowners are all better off once the drilling tax / carbon dividend is introduced, as their profitability increases from zero to \$400,000—even though less oil is drilled. The reason for this counter-intuitive-seeming result is that Bakken landowners are trapped in a collective-action problem. Each landowner prefers for others *not* to drill, because that keeps the price of oil higher than it otherwise would be. However, absent any way to coordinate their collective actions, landowners have an individual incentive to drill—and do so to such an extent that global oil prices are driven down to the point that they aren't even profiting from the drilling operations. Any mechanism that allows landowners to collectively reduce their production will therefore make them all better off. Viewed that way, it's no surprise that the drilling tax / carbon dividend system examined in this problem benefits landowners. Because this system increases the cost of drilling (via the net tax paid by those who drill) and also increases the benefit of not drilling (via the carbon dividend), landowners will respond by drilling less—allowing for a win-win-win among landowners, the U.S. and Canadian government (which benefit by keeping oil in the ground, as a strategic reserve for future use) and environmental interests seeking to reduce carbon emissions and other harmful environmental impacts associated with drilling.

S7. Here is one possible account. The diagram below shows a possible configuration of industry-level supply and demand. Demand has the normal downward-sloping shape. For supply, over the range from point a to point b there are industry-level scale economies, so quantity rises more than one for one as demand shifts to the right. Beyond point c capacity constraints bite and supply rises rapidly:



On the macroeconomic level, we see a function relating national product to national income (demand) with the shape shown below:

Equilibria occur at the points where this curve crosses the 45° line (the line where national product = national income). Here, point *O* represents a stable collapse equilibrium, point *H* represents a stable good equilibrium, and point *M* represents an unstable middle equilibrium.

S8. Answers should include careful descriptions of the (usually two) strategies available to the group of players and of the benefits derived from the two different choices. Good descriptions of benefits should allow you to sketch a simple graph. From the graph, you should go on to analyze the various possible equilibria in the game and predictions regarding which of the potentially multiple equilibria might be obtained in actual play. You should include comparisons of the predicted outcome to the optimal outcome (that which maximizes the total group benefit).