Solutions to Chapter 15 Exercises

SOLVED EXERCISES

If you find a blank space where an equation or figure may appear, please select that area for it to appear.

S1. Recall that an object has an objective value if all bidders have the same value for it and know the value.

(a) A \$20 gift card at Store X is worth \$20 to someone who is already going to be shopping at Store X, but worth less to others who would not normally shop there. For that reason, gift cards may not have an objective value. However, because Amazon is so ubiquitous, most students in a game-theory class are likely to be using Amazon in the near future to buy something; so, the gift card is worth \$20 to most students, and all of them know this.

(b) Students will naturally differ in how much they are willing to pay to have lunch with the professor; so, the "object" here clearly does not have an objective value. (Whether bidders have private or common values is unclear and will depend on why students vary in their willingness to pay for lunch. For instance, if students vary in how much they *like* the professor, and don't care what others think about her, then the auction will have private values. On the other hand, if students are unsure how *interesting* it will be to have lunch with the professor, they will naturally care about the opinions of others and the auction will not have private values.)

(c) Students' desire for a bottle of water will depend on how thirsty they are; so, the "object" here also clearly does not have an objective value. (Because one's thirst does not depend on how thirsty others are, bidders in an auction for a bottle of water will naturally have private values.)

S2. The painter can compare her estimated cost with a job's true cost only when she does the job. But the painter does a job only when she agrees (through the bidding process) to do it for less than anybody else would charge. The fact that she submitted the lowest bid suggests that this is a job for which the painter has likely underestimated the real cost. This is therefore a winner's curse; the painter only wins contracts that tend to cost more than she expected.

S3. If you turn out to be the lowest bidder and therefore fail to get the object, this must be because all of the others got a higher estimate of the value of the object than you did. Therefore, you have reason to believe that you got an exceptionally low estimate—one with a large and negative error. This is a "loser's curse," just like the winner's curse that occurs when only one object is auctioned among many bidders, and you are the highest bidder only if you get an exceptionally high estimate. You will correct for this loser's curse by bidding somewhat more aggressively than would be justifiable on the basis of your own estimate alone. The precise calculation of course requires more information on the probability distribution of the errors, and so on. (See Wolfgang Pesendorfer and Jeroen Swinkels, "The Loser's Curse and Information Aggregation in Common Value Auctions," *Econometrica*, vol. 65, no. 6 [November 1997], pp. 1247–1281.)

S4. (a) If you offer \$3,000, the current owner sells only if the car's true value to him is less than \$3,000. Since all values between \$1,000 and \$3,000 are equally likely, a car that you'll buy has an expected value of \$2,000. On average, such a car will be worth (4/3) × \$2,000 = \$2,667 to you. If you offer \$3,000, therefore, you can expect the transaction to cause you to lose \$333 in value.

(b) Suppose that your offer equals B. The current owner will sell you his car if its value to him is between \$1,000 and B; the expected value of such a car is $1{,}000 + (B - 1{,}000)/2 = 500 + B/2$. The expected value to you of a car that is sold to you is thus $(4/3)(500 + B/2)$, and your net gain is $(4/3)(500 + B/2) - B$. If you are not going to lose money, this expression must equal 0, and this occurs at $B = 2{,}000$.

S5. (a) Bob wins the auction with probability $v_B/12$ and, when winning, gets value v_B but pays $v_B/2$ on average. Overall, then, Bob's expected surplus $S(v_B) = \frac{v_B}{12} \times \frac{v_B}{2} = \frac{(v_B)^2}{24}$. For the specific values mentioned, $S(12) = 6$; $S(9) = \frac{27}{8} = 3.375$; $S(6) = \frac{3}{2} = 1.5$; and $S(3) = \frac{3}{8} = 0.375$.

(b) By part (a), $S(12) = 6 > 12/3$ and $S(9) = 3.375 > 9/3$; so, Bob prefers having an auction for the whole cupcake rather than getting half for free given value $v_B = \$12$ or $v_B = \$9$. On the other hand, since $S(6) = 1.5 < 6/2$ and $S(3) = 0.375 < 3/3$, Bob prefers getting half for free given value $v_B = \$6$ or $v_B = \$3$.

S6. (a) If your opponent always bids half her value, and her value is uniformly distributed on [0, 1], then her bid will be uniformly distributed on [0, 0.5].

If you bid $b = 0.1$, the probability of winning is $0.1/(0.5 - 0) = 0.2$.

If you bid $b = 0.4$, the probability of winning is $0.4/(0.5 - 0) = 0.8$.

If you bid $b = 0.6$, you cannot be outbid by your opponent, because 0.6 is greater than her maximum possible bid. Your probability of winning is 1.

(b) $\Pr(\text{win}) = 2b$ for $0 \leq b \leq 0.5$

$= 1$ for $b > 0.5$

(c) $\pi = (v - b) \times \Pr(\text{win}) + 0 \times \Pr(\text{lose}) = (v - b) \times 2b = 2bv - 2b^2$ for $0 \leq b \leq 0.5$

$= v - b$ for $b > 0.5$

(d) The first-order condition for π when $0 \leq b \leq 0.5$ is: $\partial\pi/\partial b = 2v - 4b = 0$, which implies that $b^* = v/2$. Because $v \leq 1$, the solution b is ≤ 0.5. Also, the profit π decreases as b increases beyond 0.5. Therefore $b = v/2$ gives the global maximum of π.

(e) We see that the best response to an opponent's bidding half her value is to bid half your value. Likewise, when you bid half your value, your opponent's best response is to bid half her value. The bidding strategy $b(v) = v/2$ is a mutual best response, or Nash equilibrium, for this auction.

S7. Consider the case when $150 < v_H < 200$. Because player L remains in the auction up to price $v_L + 50$, which is always less than 150, player H always wins and pays $v_L + 50$. Since $E[v_L] = 50$, the seller's expected revenue in this case is 50 + 50 = 100. In the terminology stated in the exercise, we have shown that $R_1 = 100$.

What about the case when $100 < v_H < 150$? Depending on who wins, the seller will collect either (i) $v_L + 50$, if v_H exceeds $v_L + 50$ so that player H wins, or (ii) $v_H - 50$, if $v_L + 50$ exceeds v_H so that player L wins. In either case, the amount that the seller receives is always at least 50. In the terminology stated in the exercise, we have shown that $R_2 \geq 50$.

Overall, then, the seller's expected revenue $R_1/2 + R_2/2 \geq 100/2 + 50/2 = 75$, which exceeds 50.

P.S.: It is possible to calculate R_2 exactly without resorting to calculus or a computer but doing so requires some additional "math facts" that are not commonly known. We provide these details next.

In computing R_2, we are focused on the case when $100 < v_H < 150$. Now divide this further into two subcases, depending on whether (A) $0 < v_L < 50$ or (A) $50 < v_L < 100$. Let R_{2A} be the expected revenue in subcase (A) and let R_{2B} be the expected revenue in subcase (B). We will solve separately for R_{2A} and R_{2B}, which will then give us $R_2 = R_{2A}/2 + R_{2B}/2$.

In subcase (A), player L drops out before the price reaches 100, player H always wins, and player H pays $v_L + 50$. Because $0 < v_L < 50$, the average value of v_L in this case is 25; so, the auction generates expected revenue $R_{2A} = 75$ in this subcase.

In subcase (B), player L and player H each drop out at a uniform price between 100 and 150. Consequently, each player wins half of the time with a final price equal to the dropout point of the other player. Here we use a math fact that is simple and very useful to know, but unfamiliar to many people.

Useful Math Fact: Suppose that you have two independent random variables that are uniformly distributed over an interval [X,Y]. The expected value of each one of these random variables is the midpoint of the interval, which is $X/2 + Y/2$; most people know that. But as it turns out, there are also very simple expressions for the expected value of the maximum and the minimum of these two random variables. In particular, the expected value of the maximum is $X/3 + 2Y/3$, which is two-thirds of the way up the interval, and the expected value of the minimum is $2X/3 + Y/3$, which is one-third of the way up the interval. (For instance, if X = 0 and Y = 1, then each random variable has expected value 1/2, the maximum has expected value 2/3, and the minimum has expected value 1/3.

Because the auction ends when the lowest of the players' two dropout points is reached, and these dropout points are independently and uniformly distributed on the interval [100,150], the Useful Math Fact tells us that the price at which the auction ends is on average $116 + 2/3$ (or "116.66"), which is one-third of the way up the interval. However, this price is only paid half of the time, when player H wins; when player L wins, the expected payment is 50 less. Overall then, expected revenue in this subcase is $116.66/2 + 66.66/2$, which is 91.66.

Putting this all together, we have $R_2 = 75/2 + 91.66/2 = 83.33$ and $R = 100/2 + 83.33/2 = 91.66$. So, introducing the subsidy for player L increases expected revenue from 50 to 91.66.

S8. (a) Suppose all the other $n - 1$ bidders are using the given bidding-strategy function $b(V)$, and your value is X. You have to choose your bid y. Let Y be the solution to $y = b(Y)$. So your bid is equivalent to pretending to have a value Y and applying the bidding-strategy function $b(Y)$ to it. You will win if all other $n - 1$ bidders' values are such that their $b(V) < b(Y)$, that is, $V < Y$. Since each V is uniformly and independently distributed in the interval [0,1], the probability of all $n - 1$ values being less than Y is Y^{n-1}. You get X (your true value) if you win. You pay $b(Y)$ (using the pretend value in your bidding strategy) whether or not you win (probability 1). So your expected payoff, denoted by $\Pi(Y)$, is

$$\Pi(Y) = XY^{n-1} - b(Y) = XY^{n-1} - [\frac{n-1}{n}]Y^n.$$

Then

$$\Pi'(Y) = X(n-1)Y^{n-2} - \left[\frac{n-1}{n}\right] nY^{n-1} = (n-1)Y^{n-2}(X-Y).$$

So $\Pi'(Y) > 0$ for $Y < X$, and < 0 for $Y > X$. Therefore $Y = X$ maximizes your expected payoff, that is, using the bidding function *b(V)* for your true value *X* is your optimal strategy. This function is your best response when all the other $n - 1$ players are using it. That is, all *n* players using the strategy *b(V)* is a Nash equilibrium.

(b) $b_3(V) > b_2(V)$ when $\left(\frac{2}{3}\right)V^2 > \left(\frac{1}{2}\right)V$, or $V > \frac{3}{4}$. And:

$b_4(V) > b_3(V)$ when $(\frac{3}{4})V^2 > \left(\frac{2}{3}\right)V$, or $V > \frac{8}{9}$.

For the general case, we have the following:

$$b_{n+1}(V) > b_n(V) \text{ when } \frac{n}{n+1}V^{n+1} > \frac{n-1}{n}V^n \text{ or } V > \frac{(n+1)(n-1)}{n^2} = \frac{n^2-1}{n^2}.$$

That is, with a larger number of players, bids will be lower for most players, but higher only for those who happen to have private values at the very top end of the range.